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On Cyclic Hypergroups with Period

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In this paper we find out a large class of P-cyclic hypergroups with period introduced in [4].

V článku sestrojíme velkou třídu P-cyklických hypergrup s periodou.

В докладе мы построим большой клас Р-циклигеских гипергрупп с периодом.

In this paper we use the definition of hypergroup introduced by F. Marty [3] in 1934.

Definition [1], [2]. Let H be a non empty set equipped with a hyperoperation,

$$*: H \times H \to \mathscr{P}(H): (x, y) \mapsto x * y \subset H, \quad x * y \neq \emptyset$$

(We set $A * B = \bigcup_{\substack{a \in A \\ b \in B}} a * b$ and $a * B = \{a\} * B$, $A * b = A * \{b\}$)

which is associative $x * (y * z) = (x * y) * z$, $\forall x, y, z \in H$,
and the condition $x * H = H * x = H$, $\forall x \in H$ is valid, then
the hyperstructure $\langle H, * \rangle$ is called a hypergroup.

We will study cyclic hypergroups as they are introduced by Wall in [5] i.e. hypergroups $\langle H, * \rangle$ that have an element $h \in H$, called generator, such that

$$H = h \cup h^2 \cup \ldots \cup h^n \cup \ldots .$$

If there exists an integer $n > 0$ such that

$$H = h \cup h^2 \cup \ldots \cup h^n \tag{1}$$

then the hypergroup $\langle H, * \rangle$ is called cyclic with finite period. If n is the minimal number for which the relation (1) is valid then we say that h has period n. The cyclic hypergroup $\langle H, * \rangle$ is called "cyclic with period" [4] if all the generators have the

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same period. The cyclic hypergroups are the ones, that have been called P-cyclic hypergroups [4] and defined as follows:

Let $(H_n, \cdot)$ be a cyclic group with n-elements and $P \subset H$. If we consider the hyperoperation

$$*^P: H \times H \to \mathscr{P}(H): (x, y) \mapsto x *^P y = x \,.\, y(\{e\} \cup P)$$

(where e is the unit element of $(H, \cdot)$) then the $\langle H_n, *^P \rangle$ is a cyclic hypergroup with period $\leqq n$.

In the following we deal with singletons for P i.e. $P = \{a^x\}$ where a is a generator of $(H_n, \cdot)$. A large class of P-cyclic hypergroups with period is obtained. We write z^μ for the powers of z in the group, and $z^{[\mu]}$ for the powers in hypergroups, and we write $z^{v[\mu]}$ instead of $(z^v)^{[\mu]}$.

First we prove the following:

Theorem 1. If $(n, x) = 1$, $n > 2$, then the P-cyclic hypergroup $\langle H_n, *^{a^x} \rangle$ is not cyclic with period.

Proof. Since $(n, x) = 1$ we have $(n, x, n) = 1$, hence [4], the element $a^n = e$ is a generator of period n. On the other hand from Thm. 2 [4] we obtain that the element a^x is a generator with period $[n/2] + 1$ where $[n/2] = z$ when $n = 2z$ or $n = 2z + 1$.

Therefore $\langle H_n, *^{a^x} \rangle$ is not cyclic with period.

In order to prove our main theorem we first prove the following Lemmas.

Lemma 1. Let $(H_n, \cdot)$, $n > 2$, be a finite cyclic group. We suppose that $n = \varkappa\lambda$ and $\varkappa \leqq \lambda$. Then for the P-cyclic hypergroups $\langle H_n, *^{a^{\varphi\lambda}} \rangle$ where $(\varphi, \varkappa) = 1$ and for all $\mu \in \mathbb{N}_0$ such that $(\mu, \varphi\lambda, n) = 1$ the following is valid:

The set $a^{\mu[v]}$, where $1 \leqq v \leqq \varkappa$, has exactly v elements different from the elements of the set

$$a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[v-1]}.$$

Proof. The set

$$a^{\mu[v]} = \{a^{\mu v}, a^{\mu v + \varphi\lambda}, \ldots, a^{\mu v + (v-1)\varphi\lambda}\} = \{a^{\mu v + x\varphi\lambda}: 0 \leqq x < v\}$$

has at the most n elements.

We can also write

$$a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[v-1]} = \{a^{\mu s + t\varphi\lambda}: 1 \leqq s \leqq v \text{ and } 0 \leqq t < s\}$$

First we prove that

$$a^{\mu[v]} \cap (a^{\mu[1]} \cup \ldots \cup a^{\mu[v-1]}) = \emptyset\,.$$

Suppose the contrary. Then we can write

$$a^{\mu v + x\varphi\lambda} = a^{\mu s + t\varphi\lambda} \Rightarrow \mu v + x\varphi\lambda \equiv (\mu s + t\varphi\lambda) \bmod n \Rightarrow$$

$$\Rightarrow \mu(v - s) + \varphi\lambda(x - t) \equiv 0 \bmod n \Rightarrow \lambda \mid \mu(v - s)$$

but $(\mu, \lambda) = 1$ (since $(\mu, \varphi\lambda, n) = 1$) hence $\lambda \mid v - s$ which is a contradiction. It remains to prove that the set $a^{\mu[v]}$ has v different elements. Supposing the contrary, we can find $x \neq y$ with

$$0 \leqq x, y < v\,, \quad \text{such that} \quad a^{\mu v + x\varphi\lambda} = a^{\mu v + y\varphi\lambda} \Rightarrow (x - y)\,\varphi\lambda \equiv 0 \bmod n \Rightarrow$$
$$\Rightarrow (x - y)\,\varphi \equiv 0 \bmod \varkappa \Rightarrow \varkappa \mid x - y$$

which is a contradiction.

Lemma 2. With the assumptions of Lemma 1 we have the following:

The set $a^{\mu[v+1]}$ where $\varkappa - 1 \leqq v \leqq \lambda - 1$, has exactly $\varkappa$ elements different from the elements of the set

$$a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[v]}\,.$$

Proof. We observe that in the set

$$a^{\mu[v+1]} = \{a^{\mu(v+1)}, a^{\mu(v+1)+\varphi\lambda}, \ldots, a^{\mu(v+1)+v\varphi\lambda}\}$$

we have

$$a^{\mu(v+1)+\varkappa\varphi\lambda} = a^{\mu(v+1)},\ a^{\mu(v+1)+(\varkappa+1)\varphi\lambda} = a^{\mu(v+1)+\varphi\lambda}, \ldots\,.$$

Therefore

$$a^{\mu[v+1]} = \{a^{\mu(v+1)}, a^{\mu(v+1)+\varphi\lambda}, \ldots, a^{\mu(v+1)+(\varkappa-1)\varphi\lambda}\} = \{a^{\mu(v+1)+x\varphi\lambda}: 0 \leqq x < \varkappa\}\,.$$

We shall prove that

$$a^{\mu[v+1]} \cap (a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[v]}) = \emptyset\,.$$

Indeed, suppose that

$$a^{\mu(v+1)+x\varphi\lambda} = a^{\mu s+t\varphi\lambda} \quad \text{where} \quad 1 \leqq s \leqq v \quad \text{and} \quad 0 \leqq t < s\,.$$

Then $\mu(v + 1 - s) + \varphi\lambda(x - t) \equiv 0 \bmod n$ and so $\lambda \mid v + 1 - s$ which is a contradiction.

Finally we observe, as in Lemma 1, that the set

$$a^{\mu[v+1]} = \{a^{\mu(v+1)+x\varphi\lambda}: 0 \leqq x < \varkappa\}$$

has $\varkappa$ different elements.

Lemma 3. With the same assumptions as in Lemma 1 we have the following:

The set $a^{\mu[\lambda+\varrho]}$ where $1 \leqq \varrho < \varkappa$, has exactly $\varkappa - \varrho$ elements different from the elements of the set

$$a^{\mu[1]} \cup \ldots \cup a^{\mu[\lambda+\varrho-1]}\,.$$

Proof. We observe that in the set

$$a^{\mu[\lambda+\varrho]} = \{a^{\mu(\lambda+\varrho)}, a^{\mu(\lambda+\varrho)+\varphi\lambda}, \ldots, a^{\mu(\lambda+\varrho)+(\lambda+\varrho-1)\varphi\lambda}\}$$

we have

$$a^{\mu(\lambda+\varrho)+\varkappa\varphi\lambda} = a^{\mu(\lambda+\varrho)},\ a^{\mu(\lambda+\varrho)+(\varkappa+1)\varphi\lambda} = a^{\mu(\lambda+\varrho)+\varphi\lambda}, \quad \text{e.t.c.}$$

Therefore

$$a^{\mu[\lambda+\varrho]} = \{a^{\mu(\lambda+\varrho)}, \ldots, a^{\mu(\lambda+\varrho)+(\varkappa-1)\varphi\lambda}\} = \{a^{\mu(\lambda+\varrho)+\omega\varphi\lambda} : 0 \leqq \omega < \varkappa\}$$

and we deduce, as in Lemma 1, that the set $a^{\mu[\lambda+\varrho]}$ has exactly $\varkappa$ elements.

Now we want to find which of the elements of the set $a^{\mu[\lambda+\varrho]}$ belong to the set

$$a^{\mu[1]} \cup \ldots \cup a^{\mu[\lambda+\varrho-1]} = \{a^{\mu s+t\varphi\lambda} : 1 \leqq s \leqq \lambda + \varrho - 1,\ 0 \leqq t < s\}$$

i.e. we want to find out ω's such that

$$a^{\mu(\lambda+\varrho)+\omega\varphi\lambda} = a^{\mu s+t\varphi\lambda} \Leftrightarrow \mu(\lambda + \varrho - s) + \varphi\lambda(\omega - t) \equiv 0 \bmod n \Leftrightarrow$$

$$\Leftrightarrow \varrho - s \equiv 0 \bmod \lambda \Leftrightarrow \varrho = s\,.$$

In this case we have

$$\mu\lambda + \varphi\lambda(\omega - t) \equiv 0 \bmod n \quad \text{or} \quad \mu + \varphi(\omega - t) \equiv 0 \bmod \varkappa\,.$$

But $(\varphi, \varkappa) = 1$ so there exist $\lambda_1, \lambda_2 \in \mathbb{Z}$ such that $\lambda_1\varphi + \lambda_2\varkappa = 1$, therefore $\lambda_1\varphi\mu + \lambda_2\varkappa\mu + \varphi(\omega - t) \equiv 0 \bmod \varkappa$ or $\varphi(\lambda_1\mu + \omega - t) \equiv 0 \bmod \varkappa$ or $\lambda_1\mu + \omega - t \equiv 0 \bmod \varkappa$.

Finally $\omega \equiv (t - \lambda_1\mu) \bmod \varkappa$ and since $\varrho = s$ and $0 \leqq t < s$ we can take $t = 0, 1, 2, \ldots, \varrho - 1$. This means that we have ϱ different values for ω, therefore ϱ elements of the set $a^{\mu[\lambda+\varrho]}$ belong to the set

$$a^{\mu[1]} \cup \ldots \cup a^{\mu[\lambda+\varrho-1]} \quad \text{Q.E.D.}$$

Theorem 2. Let $(H_n, \cdot)$ be a finite cyclic group, where $n > 2$, $n = \varkappa\lambda$, $\varkappa \leqq \lambda$. Then the P-cyclic hypergroups $\langle H_n, *^{a^{\varphi\lambda}}\rangle$ where a is a generator of $(H_n, \cdot)$ and $(\varphi, \varkappa) = 1$, are cyclic with period $\varkappa + \lambda - 1$.

Proof. We know ([4], Th. 1) that an element a^μ is a generator of $\langle H_n, *^{a^{\varphi\lambda}}\rangle$ iff $(\mu, \varphi\lambda, n) = 1$.

Let a^μ be a generator of $\langle H_n, *^{a^{\varphi\lambda}}\rangle$. Then, according to the Lemmas 1, 2, 3 the set

$$a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[\varkappa+\lambda-1]}$$

contains exactly n different elements, i.e. every generator a^μ has period $\varkappa + \lambda - 1$.

Precisely,

the set $a^{\mu[1]} \cup \ldots \cup a^{\mu[\varkappa-1]}$ has exactly $\dfrac{(\varkappa - 1)\,\varkappa}{2}$ elements (Lemma 1)

the set $a^{\mu[\varkappa]} \cup \ldots \cup a^{\mu[\lambda]}$ has exactly $(\lambda - \varkappa + 1)\,\varkappa$ elements (Lemma 2) and

the set $a^{\mu[\lambda+1]} \cup \ldots \cup a^{\mu[\varkappa+\lambda-1]}$ has exactly $\dfrac{(\varkappa - 1)\,\varkappa}{2}$ elements (Lemma 3)

Therefore the set $a^{\mu[1]} \cup \ldots \cup a^{\mu[\varkappa+\lambda-1]}$ contains n elements.

Theorem 3. Let $(H_n, \cdot)$ be a finite cyclic group where $n = \varkappa(\varkappa + 1) > 2$, and let a be a generator. Then the P-cyclic hypergroups $\langle H_n, *^{a^{\varphi\varkappa}}\rangle$, where $(\varphi, \varkappa + 1) = 1$, are cyclic with period $2\varkappa$.

In order to prove this theorem we shall prove the following Lemmas.

Lemma 4. The set $a^{\mu[\nu+1]}$, $\forall \mu \in \mathbb{N}o$ with $(\mu, \varphi\varkappa, n) = 1$ and $1 \leqq \nu < \varkappa$, has exactly $\nu + 1$ elements different from the elements of the set $a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[\nu]}$.

Proof. We follow the same procedure as in Lemma 1, and deduce that

$$a^{\mu[\nu+1]} \cap \left(a^{\mu[1]} \cup \ldots \cup a^{\mu[\nu]}\right) = \emptyset .$$

Moreover, since $(\varphi, \varkappa + 1) = 1$, it is clear that the $\nu + 1$ elements of $a^{\mu[\nu+1]}$ are different from each other.

Lemma 5. The set $a^{\mu[\varkappa+\varrho]}$, $\forall \mu \in \mathbb{N}o$ with $(\mu, \varphi\varkappa, n) = 1$ and $1 \leqq \varrho \leqq \varkappa$, has exactly $\varkappa - \varrho + 1$ elements different from the elements of the set $a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \ldots \cup a^{\mu[\varkappa+\varrho-1]}$.

Proof. The proof of this Lemma goes as in Lemma 3.

Proof of theorem 3. The element a^{μ} is a generator of $\langle H_n, *^{a^{\varphi\varkappa}} \rangle$ iff $(\mu, \varphi\varkappa, \varkappa(\varkappa + 1)) = 1$. Therefore according to the Lemma 4 the set $a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \ldots \cup a^{\mu[\varkappa]}$ has $\varkappa(\varkappa + 1)/2$ different elements and from Lemma 5 we see that the set $a^{\mu[\varkappa+1]} \cup \ldots \cup a^{\mu[2\varkappa]}$ has $\varkappa(\varkappa + 1)/2$ new different elements. This means that the set $a^{\mu[1]} \cup a^{\mu[2]} \cup \ldots \cup a^{\mu[2\varkappa]}$ has exactly $\varkappa(\varkappa + 1) = n$ elements and so $a^{\mu[1]} \cup \ldots \ldots \cup a^{\mu[2\varkappa]} = H_n$.

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