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M. Deza; Ivo Rosenberg

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General Convolutions Motivated by Designs

M.-M. DEZA*) AND I. G. ROSENBERG**)

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The paper is motivated by the algebraic treatment of generalized designs and it extends certain ideas of Graver and Jurkat. The main part is devoted to the study of convolution in fairly general setting extending semigroup rings.

Článek přináší algebraický přístup k jistým kombinatorickým strukturám a zobecňuje práce Gravera a Jurkata. Hlavní část je věnována studiu konvolucí v dostatečně obecném pojetí rozšiřujícím pologrupové okruhy.

Статья предлагает некоторый алгебраический подход к комбинаторным структурам определённого типа и является обобщением работ Гравера и Юрката. Её главная часть посвящена изучению конволюций, в достаточно общем понимании, расширяющим полугрупповые кольца.

Introduction

This paper was inspired by Graver and Jurkat's study [5] and its ideas come both from the first author's papers [1, 2], the joint paper [3] and the unpublished paper [9] by the author. The motivation comes from generalized design (§ 4) and the paper studies convolution as a natural tool which for binary designs was introduced in [5, 6]. It became soon clear that the technique extends to non-binary designs and, in fact, the constructions hold for semigroup rings. In [9] semigroup rings were extended to structures with convolution in which the semigroup D is replaced by a partial groupoid (binary operation), the ring $\boldsymbol{R}$ is considerably weakened and the set $\mathscr{E}$ of maps $D \to R$ with finite support is supposed to have only a weaker finiteness property. In this paper we go even further and replace the semigroup D by a multigroupoid (hypergroupoid) $\langle D, \circ \rangle$ which is a map from D^2 into the subsets of D (equivalent to a ternary relation on D). Within this framework the above assumptions seem to be as general as possible. Although the presentation is inevitably cumbersome, this approach keeps up with the algebraic tradition to start with the highest meaningful level of generality and add further restrictions only when they become necessary. For example, we often have to assume that $\boldsymbol{R}$ is distributive

*) CNRS Paris, 17, passage de l'Industrie, 750 10 Paris, France

**) Math. et Stat., Université de Montréal, C.P. 6128, Succ. A, Montréal, Québec H3C 3J7, Canada

(Propositions 1.9, 2.2), is a ring (Proposition 2.5), satisfies the medial law (Proposition 1.9) and $R^2 \neq \{0\}$ (Proposition 2.2 and 2.5). We concentrate on three main topics: 1) a homomorphism of the convolution on the direct power, 2) a similar homomorphism induced by a family $\{A(d): d \in D\}$ of subsets of D, in particular, for $A(d)$ induced by an order on D and 3) extensions of the above to direct sums and products. In fact this algebraic part became so large and of independent interest that we have decided to restrict generalized designs to a mere introduction. For this reason it is not shown that they form convolutive R-structures (for binary designs this is indicated in [5]).

In § 1 we define the general convolutive R-structure on a subset $\mathscr{E}$ of D^R, bring forth a few common examples and define a selfmap ψ of $\mathscr{E}$. For R distributive and satisfying the medial law we give a sufficient condition for ψ to be a homomorphism of the convolution into the direct power. This map is a sort of linear transformation of $\mathscr{E}$. With a certain cancellation property the above condition is also necessary.

In § 2 we study a similar selfmap χ based on a map A from D into the subsets of D. For $\boldsymbol{R}$ distributive and $R^2 \neq \{0\}$ we give a necessary and sufficient condition making χ a homomorphism of the convolution into the direct power. In particular, if $\boldsymbol{R}$ is a non-zero ring, $\leqq$ is a locally finite order on D and $A(d) = \{x \in D: x \geqq d\}$ for all $d \in D$, we give a necessary and sufficient condition for χ to be an isomorphism of the convolution onto the direct power. In particular, if the multigroupoid is a partial groupoid with domain F (i.e. $|x \circ y| = 1$ for $(x, y) \in F$ and $x \circ y = \emptyset$ otherwise) the condition is 1) $\leqq$ is a partial meet-semilattice in which x and y have a common lower bound (and hence the meet $x \wedge y$ exists) iff $(x, y) \in F$ and 2) the operation of the partial groupoid is the partial meet.

In § 3 we introduce direct sums and products of convolutive $\boldsymbol{R}$-structures and show that the maps and conditions from §§ 1–2 naturally extend to direct sums and product. This part may be seen as a natural extension of direct sums and Kronecker products of $n \times n$ matrices (over a ring).

Following [5] and [2] in § 4 we briefly outline some designs and generalized designs from our point of view. The latter are based on the semilattice order $\preceq$ on $\boldsymbol{s} = \{0, 1, \ldots, s - 1\}$ in which $0 \prec i$ $(i = 1, \ldots, s - 1)$ i.e. 0 is the least element and all the others are maximal elements $(s \geqq 2)$. We birefly mention the connection between $(\boldsymbol{3}^m, \preceq)$ and the cross-polytopes. The generalized designs are presented as $f: \boldsymbol{s}^m \to \mathbb{N}$ such that the map $f^{\preceq}_m$ is constant on certain subsets of $\boldsymbol{s}^m$. We give the corresponding combinatorial matrices. For $s > 2$ this is essentially unique in contrast to the case $s = 2$ where $\preceq$ is the order of a boolean algebra an and so admits also $\vee$ and $\wedge$ [5].

Since the paper involves unusual structures, we have tried to make it self-contained and give most of the proofs, which albeit routine, may help the reader. A reader not interested in infinite D or outlandish $\boldsymbol{R}$ may think about a semigroup ring over a finite semigroup D and an associative and commutative ring $\boldsymbol{R}$ with identity and ignore all the cumbersome assumptions.

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1. Convolutions and Homomorphisms

1.1 Let R be a nonempty set with two binary operations $+$ and $\cdot$ and an element 0 such that $\langle R; +, 0\rangle$ is an abelian monoid and $x\,0 = 0\,x = 0$ for all $x \in R$ (as usual, we often abbreviate $a \cdot b$ by ab). In other words, $\boldsymbol{R} = \langle R; +, \cdot, 0\rangle$ is a universal algebra with two binary operations $+$ and $\cdot$ and a constant 0 such that

$$(1) \qquad x + 0 = x\,, \quad x\,0 = 0x = 0\,, \quad x + y = y + x\,,$$
$$x + (y + z) = (x + y) + z$$

holds for all $x, y, z \in R$.

1.2 Let D be a set and $\boldsymbol{D} = \langle D; \circ\rangle$ a multigroupoid (or hypergroupoid) on D, i.e. $(x, y) \to x \circ y$ is a map from D^2 into the set $\mathscr{P}(D)$ of the subsets of D. Note that fot $x, y \in D$ the subset $x \circ y$ of D may also be empty. Equivalently, the multigroupoid $\circ$ may be conceived as the ternary relation $\{(x, y, z)\colon x, y \in D,\ z \in x \circ y\}$.

Next let $\boldsymbol{C}$ be a family of subsets of D such that

(i) $\boldsymbol{C}$ is an ideal of $(\mathscr{P}(D), \subseteq)$, i.e. $\boldsymbol{C}$ is closed under union ($X \cup Y \in \boldsymbol{C}$ whenever $X, Y \in \boldsymbol{C}$ and $\boldsymbol{C}$ is hereditary ($Y \in \boldsymbol{C}$ whenever $Y \subseteq X \in \boldsymbol{C}$). Further $\boldsymbol{C}$ contains all singletons $\{d\}$ ($d \in D$).

(i) For all $X, Y \in \boldsymbol{C}$ and $d \in D$ the set

$$(2) \qquad \{(x, y) \in X \times Y\colon d \in x \circ y\}$$

is finite and

$$(3) \qquad X \circ Y := \bigcup_{Y \in X} \bigcup_{y \in Y} x \circ y \in \boldsymbol{C}\,.$$

Note that for D finite the conditions are equivalent to $\boldsymbol{C} = \mathscr{P}(D)$.

1.3 Let R^D denote the set of maps from D into R. For $f \in R^D$ put $\operatorname{supp} f :=$ $:= \{d \in D\colon f(d) \neq 0\}$ and set

$$\mathscr{E} := \mathscr{E}_{\boldsymbol{RDC}} = \{f \in R^D\colon \operatorname{supp} f \in \boldsymbol{C}\}\,.$$

Later we shall need the following maps. For $r \in R$ and $d \in D$ let $[r_d]$ be defined by $[r_d]\,(d) = r$ and $[r]_d\,(x) = 0$ otherwise (thus $[r]_d$ is a "spike" or Dirac function). As $\{d\} \in \boldsymbol{C}$ we have $[r]_d \in \mathscr{E}$.

On $\mathscr{E}$ we can define pointwise the operations $+$ and $\circ$. For $f, g \in \mathscr{E}$ let $f + g$ and fg be defined by setting $(f + g)\,(d) := f(d) + g(d)$, and $(fg)\,(d) := f(d)\,g(d)$ for all $d \in D$. Since $\operatorname{supp}(f + g) \subseteq \operatorname{supp} f \cup \operatorname{supp} g$, by (i) we have $f + g \in \mathscr{E}$. Similarly $fg \in \mathscr{E}$ because $\operatorname{supp} fg \subseteq \operatorname{supp} f \in \boldsymbol{C}$.

For each $x \in R$ and $f \in \mathscr{E}$ let $x_l f$ and $f x_r$ be defined by setting

$$(x_l f)(d) := x f(d), \quad (f x_r)(d) := f(d)\, x$$

for all $d \in D$. Clearly both supp $x_l f$ and supp $f x_r$ are included in supp f and so $x_l f$, $f x_r \in \mathscr{E}$.

1.4 We introduce another operation $*$ on $\mathscr{E}$. For $f, g \in \mathscr{E}$ define $f * g$ by setting

$$(f * g)(d := \sum_{d \in x \circ y} f(x)\, g(y) \tag{4}$$

for all $d \in D$. The sum is over the set of all (x, y) such that $d \in x \circ y$ which, according to (2) is finite and so the value $(f * g)(d)$ is well defined. Put $X :=$ supp f and $Y :=$ $:=$ supp g. By definition $X, Y \in \boldsymbol{C}$. The summation in (4) may be restricted to $(x, y) \in X \times Y$, so supp $(f * g) \subseteq X \circ Y$ and thus, according to (3), it belongs to $\boldsymbol{C}$. Thus $\mathscr{E}$ is closed under $*$. We call $f * g$ the *convolution* of f and g. The algebra

$$\boldsymbol{E} := \langle \mathscr{E};\ +, \cdot, *, \{y_l : y \in R\}, \{y_r : y \in R\} \rangle$$

is called a *convolutive **R**-structure*. We consider a few examples.

1.5 Examples. 1. Let n be a positive integer, $D := \{1, \ldots, n\}^2$ and put $(i, j) \circ$ $\circ (k, l) := \{(i, l)\}$ if $j = k$ and $(i, j) \circ (k, l) = \emptyset$ otherwise. Then $\boldsymbol{E}$ is isomorphic to the set of $n \times n$ matrices over R with the standard sum and product $*$, r_l and r_d are the left and right scalar multiples and $\cdot$ is the Hadamard (or Schur) product. Usually $\boldsymbol{R}$ is assumed to be a ring but it may be also a lattice with zero (i.e. $x + y$ is the join or sup and xy is the meet or inf).

2. Let $\langle D; \circ \rangle$ be such that $x \circ y = \{x \cdot y\}$ where $\langle D; \cdot \rangle$, is a groupoid. Then in (4) we sum over all $(x, y) \in D^2$ such that $x \cdot y = d$. In particular, if $\langle D, \cdot \rangle$ is a semigroup, and $\boldsymbol{R}$ a commutative and associative ring and $\boldsymbol{C}$ consists of the finite subsets of D, then $\boldsymbol{E}$ is isomorphic to the semigroup ring of $\boldsymbol{D}$ over $\boldsymbol{R}$. If, moreover, $\boldsymbol{D}$ is a group, it is isomorphic to the group ring of D over R. For $\boldsymbol{D} = \{\mathbb{N}; +\}$ where $\mathbb{N} := \{0, 1, \ldots\}$ is the set of non-negative integers the corresponding semigroup ring is isomorphic to the ring $R[x]$ of the polynomials in one indeterminate.

3. Let $D := \langle \mathbb{N}, + \rangle$, $\boldsymbol{C} = P(\mathbb{N})$ (all subsets of $\mathbb{N}$) and R a ring. Then $\langle \mathscr{E}; +, * \rangle$ is isomorphic to the ring $\boldsymbol{R}[[x]]$ of formal power series in one indeterminate x. This example can be modified to yield the ring of formal power series in several indeterminates.

1.6 The following transformation of $\mathscr{E}$ is an extension of a linear transformation of a finite-dimensional vector space. Let $\lambda: D^2 \to R$. For $d \in D$ define the *d-row* $\lambda_d: D \to R$ of λ by setting $\lambda_d(x) := \lambda(d, x)$ for all $x \in D$. Suppose that for each $X \in \boldsymbol{C}$ the map λ satisfies:

(1) $X \cap$ supp λ_d is finite for all $d \in D$, and

(2) the set $\{d \in D: X \cap \text{supp}\, \lambda_d \neq \emptyset\}$ belongs to $\boldsymbol{C}$.

Given $f \in \mathscr{E}$ put

$$(5) \qquad f^\lambda(d) := \sum_{x \in D} \lambda_d(x) f(x)$$

for all $d \in D$. Put $X := \operatorname{supp} f$. If $f^\lambda(d) \neq 0$ then $\lambda_d(x) f(x) \neq 0$ for some $x \in D$ when $x \in X \cap \operatorname{supp} \lambda_d$. Thus $X \cap \operatorname{supp} \lambda_d \neq \emptyset$ and therefore by (2)

$$\operatorname{supp} f^\lambda \subseteq \{d \in D \colon X \cap \operatorname{supp} \lambda_d \neq \emptyset\} \in \boldsymbol{C}$$

proving that $f^\lambda \in \mathscr{E}$.

Let $\langle G; \cdot \rangle$ and $\langle G'; \cdot' \rangle$ be groupoids. A map $\varphi \colon G \to G'$ is a *homomorphism* of $\langle G; \cdot \rangle$ into $\langle G'; \cdot' \rangle$ if $\varphi(x \circ y) = \varphi(x) \cdot' \varphi(y)$ for all $x, y \in G$. Similarly for $r \colon G \to G$ and $r' \colon G' \to G'$ the map φ is a *homomorphism* of $\langle G; r \rangle$ into $\langle G'; r' \rangle$ if $\varphi(r(x)) = r'(\varphi(x))$ for all $x \in G$. For constants $0 \in G$ and $0' \in G'$, the map φ is a *homomorphism* of $\langle G; 0 \rangle$ into $\langle G; 0' \rangle$ if $\varphi(0) = 0'$. Finally for universal algebras $\boldsymbol{G}$ and $\boldsymbol{G}'$ of the same type with at most binary operations the map φ is a *homomorphism* if it is a homomorphism for each operation separately. A homomorphism of $\boldsymbol{G}$ into itself is an *endomorphism*. We have the following immediate lemma:

1.7 Lemma *If the left distributive law*

$$x(y + z) = xy + xz$$

holds for all $x \in \operatorname{im} \lambda$ *and* $y, z \in R$, *then* $\psi \colon f \to f^\lambda$ *is an endomorphism of* $\langle \mathscr{E}; + \rangle$. *If*

$$(x_1 + \ldots + x_n)\, y = x_1 y + \ldots + x_n y$$

holds for all positive integers n *and for all* $x_1, \ldots, x_n \in (\operatorname{im} \lambda)\, R$, *then* ψ *is an endomorphism of* $\langle \mathscr{E}; y_r \rangle$. *If*

$$y(x_1 + \ldots + x_n) = y x_1 + \ldots + y x_n$$

holds for all positive integers n, *all* $x_1, \ldots, x_n \in (\operatorname{im} \lambda)\, R$ *and* $y(ux) = u(yx)$ *is true for all* $u \in \operatorname{im} \lambda$ *and,* $x \in R$, *then* ψ *is an endomorphism of* $\langle \mathscr{E}; y_l \rangle$.

1.8 Put

$$S := \{y_r \colon y \in R\}\ X\ \{y_l \colon y \in R\,,\quad y(ux) = u(yx) \quad \forall u \in \operatorname{im} \lambda,\, x \in R\}\,.$$

We shall see that certain cases $\psi \colon f \to f^\lambda$ is a homomorphism of $\boldsymbol{E} := \langle \mathscr{E}; +, *, S \rangle$ into $\boldsymbol{E}' := \langle \mathscr{E}; +, \cdot, S \rangle$. (Although the universe is the same for both $\boldsymbol{E}$ and $\boldsymbol{E}'$, the map ψ is not an endomorphism because $*$ is replaced by $\cdot$.) If ψ is a permutation of $\mathscr{E}$, this permits to describe $\boldsymbol{E}$ in terms of the more transparent structure $\boldsymbol{E}'$. In other words, the convolution may be then replaced by the pointwise product. As we shall see, this is guaranteed if we have enough of "independent characters". First we need the following concept. Let $A \subseteq R$. We say that $r \in R$ is *A-cancellative* if $ar = a'r$ implies $a = a'$ whenever a is a sum of elements from A and $a' \in A^2$ (i.e. $a' = a_1 a_2$ where $a_1, a_2 \in A$). We have:

1.9 Proposition. *Let* R *be distributive and satisfy the medial law*

$$(xy)(zt) = (xz)(yt) . \tag{6}$$

Further let λ: $D^2 \to R$ *be such that for each* $X \in C$ *all the sets* $X \cap \operatorname{supp} \lambda_d$ $(d \in D)$ *are finite and* $\{d \in D: X \cap \operatorname{supp} \lambda_d \neq \emptyset\} \in C$.; *Then:*

(i) *if for all* $d, x, y \in D$

$$\lambda_d(x) \lambda_d(y) = \sum_{z \in x \circ y} \lambda_d(z) , \tag{7}$$

then $\psi: f \to f^\lambda$ *is a homomorphism of* $\boldsymbol{E} := \langle \mathscr{E}; +, *, S \rangle$ *into* $\boldsymbol{E}' := \langle \mathscr{E}; +, \cdot, S \rangle$

(ii) *Conversely, if for each* $d \in D$ *there is an* $\operatorname{im} \lambda_d$*-cancellative element* $r \in R^2$ *and* ψ *is a homomorphism of* $\langle \mathscr{E}; * \rangle$ *into* $\langle \mathscr{E}, \cdot \rangle$, *then* (7) *holds.*

Proof. Let $f, g \in \mathscr{E}$ and $d \in D$. Using (5), (4) and the distributive laws we get

$$(f * g)^\lambda (d) = \sum_z \lambda_d(z) (f * g)(z) = \sum_z \lambda_d(z) \sum_{z \in x \circ y} f(x) g(x) = \tag{8}$$
$$= \sum_{x,y} (\sum_{z \in x \circ y} \lambda_d(z)) (f(x) g(y)) .$$

On the other hand, from (5), the distributive laws, and the medial law (6) we have

$$f^\lambda(d) g^\lambda(d) = (\sum_x \lambda_d(x) f(x)) (\sum_y \lambda_d(y) g(y)) = \tag{9}$$
$$= \sum_{x,y} (\lambda_d(x) f(x)) (\lambda_d(y) g(y))$$
$$= \sum_{x,y} (\lambda_d(x) \lambda_d(y)) (f(x) g(y)) .$$

Suppose that (7) holds. From (8) and (9) we obtain that $(f * g)^\lambda (d) = f^\lambda(d) . g^\lambda(d)$ for all $d \in D$ providing (i).

We prove (ii). Let $d, x, y \in D$ and let $r = r'r'' \in R^2$ be $\operatorname{im} \lambda_d$-cancellative. Choosing $f := [r']_x$ and $g := [r'']_y$ (see 1.3) from (8) and (9) we obtain

$$\sum_{z \in x \circ y} \lambda_d(z) (r'r'') = (\lambda_d(x) \lambda_d(y)) (r'r'')$$

(because the other terms vanish). Using the right distributive law we get

$$(\sum_{z \in x \circ y} \lambda_d(z)) r = (\lambda_d(x) \lambda_d(y)) r$$

which amounts to (7) as r is $\operatorname{im} \lambda_d$-cancellative. □

1.10 Remark. Suppose $\langle D; \cdot \rangle$ is a partial groupoid (i.e. $(x, y) \to x \cdot y$ is a map from a subset F of D^2 into D such that $x \circ y = \{x \cdot y\}$ for $(x, y) \in F$ and $x \circ y = \emptyset$ otherwise). Then (7) becomes

$$\lambda_d(x \cdot y) = \lambda_d(x) \lambda_d(y) \tag{10}$$

for all $(x, y) \in F$ and $\lambda_d(x) \lambda_d(y) = 0$ otherwise. Thus the d-th row of λ is a "partial character" i.e. a homomorphism of $\langle D; \cdot \rangle$ into $\langle R; \cdot \rangle$ such that $\lambda_d(x) \lambda_d(y) = 0$ for all $(x, y) \in D^2 \setminus F$.

1.11 Remark. There is a natural question whether ψ is injective, surjective or bijective. Suppose that $\boldsymbol{R}$ is a ring. Then ψ is injective iff $h \in \boldsymbol{R}^{\boldsymbol{D}}$, supp $h \in \boldsymbol{C}$ and

$$\sum_{x \in D} \lambda_d(x)\, h(x) = 0 \quad (\forall d \in D)$$

implies $h(x) = 0$ for all $x \in D$. Let Ldenote the matrix $(\lambda_i(j))$ whose rows and columns are indexed by D. Cleatly the above questions are questions about the linear transformation of E defined by L. Suppose that D is finite. Then the injectivity of ψ is equivalent to the independence of $\{\lambda_d : d \in D\}$ over $\boldsymbol{R}$ (considered as vectors over $\boldsymbol{R}$). Assume that $\boldsymbol{R}$ is a commutative and associative ring with identity. We can form determinant $d :=$ det L(defined in a standard fashion). By Cramer's rule we obtain: *If L is invertible, then ψ is a bijection. In particular, if $\boldsymbol{R}$ is a field, then the following conditioms are equivalent:* (i) *ψ is injective,* (ii) *ψ is surjective,* (iii) *ψ is bijective,* (iv) *L is non-singular, and* (v) *$\{\lambda_d : d \in D\}$ is a basis of the vector space R^D (over R).*

2. Homomorphisms based on set systems and orders

2.1 We take a slightly different approach. Let $A: D \to \mathscr{P}(D)$ be such that for all $X \in \boldsymbol{C}$

(1) $X \cap A(d)$ is finite for all $d \in D$, and

(2) $\{d \in D : X \cap A(d) \neq \emptyset\} \in \boldsymbol{C}$.

To each $f \in E$ assign f^A defined by setting

$$f^A(d) := \sum_{x \in A(d)} f(x) \tag{11}$$

for each $d \in D$. The condition (1) guarantees that the sum in (11) is finite while (2) is sufficient for supp $f^A \in \boldsymbol{C}$. Thus f^A is well defined and $f^A \in \mathscr{E}$. For $r \in R$ and $n \in \mathbb{N}$ put $0r = 0$ and for $n > 0$ let nr stand for $r + \ldots + r$ (n times). For $r \in R \setminus \{0\}$ the *order* $\varrho(r)$ is the least positive integer n such that $nr = 0$ provided such n exists $\varrho(r) = \infty$ otherwise (i.e. if $nr \neq 0$ for all $n = 1, 2, \ldots$). Let $A \subseteq R$. If the set $0 := := \{\varrho(a) : a \in A \setminus \{0\})$ contains ∞ or is infinite put $\varrho(A) = \infty$, otherwise let $o(A)$ be the least common multiple of the numbers from 0. For integers x and y put $x \equiv y \pmod{\infty}$ iff $x = y$, Let S, $\boldsymbol{E}$ and $\boldsymbol{E}'$ be as in 1.8. We have:

2.2 Proposition. *Let $\boldsymbol{R}$ be distributive, let $R^2 \neq \{0\}$ and let $A: D \to \mathscr{P}(D)$ satisfy the assumptions of* 2.1. *Then $\chi: f \to f^A$ is a homomorphism from E into E' if and only if*

(3) *For all $x, y, d \in D$ the cardinality m of the set $(x \circ y) \cap A(d)$ satisfies $m \equiv 1 \pmod{\varrho(R^2)}$ if $x, y \in A(d)$ and $m \equiv 0 \pmod{\varrho(R^2)}$ otherwise.*

Proof. The proof of Lemma 1.7 can be easily modified to show that χ is an endomorphism of $\langle \mathscr{E}; +, S \rangle$. Thus it suffices to consider only the operations $*$ and $\cdot$. Let $f, g \in \mathscr{E}$ and $d \in D$. Then by (11) and (4)

$$(f * g)^A(d) = \sum_{z \in A(d)} \sum_{z \in x \circ y} f(x)\, g(y) = \sum_{x,y} |(x \circ y) \cap A(d)|\, f(x)\, g(y)\,. \tag{12}$$

On the other hand,

$$f^A(d)\, g^A(d) = \Big(\sum_{x \in A(d)} f(x)\Big)\Big(\sum_{y \in A(d)} g(y)\Big) = \sum_{x,y \in A(d)} f(x)\, g(y)\,. \tag{13}$$

Necessity: Let $x, y, d \in D$ and let $b = rr' \in R^2$. In (12) and (13) put $f := [r]_x$ and $g := [r']_y$ (see 1.3). Further put $m := |(x \circ y) \cap A(d)|$. Note that by 2.1 the cardinality m is finite. Now the right side of (12) reduces to mb and similarly the right side of (13) is b if $x, y \in A(d)$ and 0 otherwise. We have the following cases:

a) Let $x, y \in A(d)$. Then by the above $mb = b$. We show that

$$m \equiv 1 \pmod{\varrho(b)}\,. \tag{14}$$

Indeed, (14) is obvious if $o(b) = \infty$ as it just means $m = 1$. Thus let $\varrho(b) =: k$ be finite. Using the fact that kb is the least multiple of b which is equal 0 it is easy to prove (14).

Suppose that $\varrho(R^2) = \infty$. If there is $b \in R^2 \setminus \{0\}$ with $\varrho(b) = \infty$, then by (14) we have $m \equiv 1 \pmod{\infty}$. Thus assume that the set $\sigma := \{\varrho(b)\colon b \in R^2 \setminus \{0\}\}$ is an infinite subset of $\{1, 2, \ldots\}$. Then again by (14) we have $m = 1$ and so $m \equiv 1 \pmod{\infty}$.

Suppose that $\varrho(R^2) = k$ is finite. By the Chinese remainder theorem the congruences (14) are equivalent to $m \equiv 1 \pmod{k}$ proving the claim in this case.

b) Let $(x, y) \notin A^2(d)$. Then from (12) and (13) we get $mb = 0$. Proceeding as in the case a) above we obtain that $m \equiv 0 \pmod{\varrho(R^2)}$. This concludes the proof of the necessity.

Sufficiency. Let $d \in D$. 1) First suppose that $\varrho(R^2) = \infty$. Let $x, y \in D$ and $m := |(x \circ y) \cap A(d)|$. From (3) we have $m = 1$ if $(x, y) \in A^2(d)$ and $m = 0$ otherwise. From (12) and (13) we get

$$(f * g)^A(d) = \sum_{x,y \in A(d)} f(x)\, g(y) = f^A(d)\, g^A(d) \tag{15}$$

proving that $(f * g)^A = f^A g^A$.

2) Finally let $\varrho(R^2) =: k$ be finite. Let $x, y \in D$ be such that $b := f(x)\, g(y) \neq 0$. Then $b \in R^2 \setminus \{0\}$ and so by the definition $\varrho(b)$ divides k. Put $m := |(x \circ y) \cap A(d)|$. We have two cases: a) First suppose that $x, y \in A(d)$. From $m \equiv 1 \pmod{k}$ we get $m \equiv 1 \pmod{\varrho(b)}$ and so $mb = b$. b) Let $(x, y) \notin A^2(d)$. By the same token we have $m \equiv 0 \pmod{\varrho(b)}$. Applying a) and b) to (12) and (13) we get (15) which proves the required $(f * g)^A = f^A g^A$. □

2.3 Remark. Suppose that R is distributive and has a left identity 1 (i.e. a left neutral element for the multiplication satisfying $1r = r$). Let $k := \varrho(1)$. For $r \in R$ from

$$kr = k(1r) = 1r + \ldots + 1r = (k\,1)\, r = 0r = 0$$

we see that $\varrho(r)$ divides k. Since $1 = 1.\,1 \in R^2 \setminus \langle 0\rangle$ we see that $\varrho(R^2) = k$. Now put $\lambda_d(x) := 1$ if $x \in A(d)$ and $\lambda_d(x) = 0$ otherwise. Let $x, y, d \in D$ and let $m := |(x \circ y) \cap A(d)|$. the condition (7) from Proposition 1.8 is $1 = m\,1$ if $x, y \in A(d)$

and $0 = m\,1$ otherwise. These conditions are in turn equivalent to the condition (iii) from Proposition 2.2. However, Proposition 2.2 is not a corollary of Proposition 1.8 because it holds even if $\boldsymbol{R}$ has no left identity or does not satisfy the medial law (6).

2.4 We turn to a particular but important case. Let $\leqq$ be an *order* on D (i.e. a reflexive, transitive and antisymmetric relation which is often called a partial order, an ordering on D or a poset). For $d \in D$ we put

$$[d) := \{x \in D\colon x \geqq d\}\ ; \quad (d] := \{x \in D\colon x \leqq d\}\ .$$

For $d \leqq d'$ the set $[d) \cap (d'] = \{x \in D\colon d \leqq x \leqq d'\}$ is called an *interval*. The order is *locally* finite if all intervals are finite. Put $A(d) = [d)$ for all $r \in D$ and write $f^{\leqq}$ instead of f^A i.e. put

$$f^{\leqq}(d) := \sum_{x \geqq d} f(x) \tag{16}$$

for all $d \in D$.

Let $(D, \leqq)$ be a locally finite and $x, y \in D$. Put $h(x, y) := -1$ if $x \nless y$, $h'x, y) = 0$ if $x = y$ and for $x < y$ let $h(x, y)$ denote the greatest integer n such that

$$x = z_0 < z_1 < \ldots < z_n = y$$

(such an n exists in a locally finite order). We define the "Möbius inversion" $\mu\colon D^2 \to \mathbb{Z}$ (cf [4]): by induction on $n := h(x, y)$ as follows:

1) If $n := h(x, y) \leqq 0$ put $\mu_x(y) := h(x, y) + 1$.

2) If $n := h(x, y) > 0$ put

$$\mu_x(y) := -\sum_{x < z \leqq y} \mu_z(y)\ . \tag{17}$$

Suppose that $(D, \leqq)$ also satisfies the following conditions. For all $X \in \boldsymbol{C}$ we have: (1) $[d) \cap X$ is finite for all $d \in D$ and (2) the set $\{d \in D\colon [d) \cap X \neq \emptyset\}$ belongs to $\boldsymbol{C}$. For $f \in E$ define $f^0\colon D \to R$ by setting

$$f^o(d) := \sum_x \mu_d(x) f(x) \tag{18}$$

for all $d \in D$. Note that in view of $\mu(d, x) = 0$ for $x \notin [d)$ and (i), the sum in (18) is a finite one. Moreover, due to (ii) we have $\operatorname{supp} f^o \in \boldsymbol{C}$ and so $f^o \in E$; hence $\varphi\colon f \to f^o$ maps $\mathscr{E}$ into itself, We have:

2.5 Corollary. *Let $\boldsymbol{R}$ be a ring with $R^2 \neq \{0\}$ and let $(D, \leqq)$ be a locally finite order such that for all $X \in \boldsymbol{C}$ we have:* (1) *$[d) \cap X$ is finite for all $d \in D$ and* (2) *$\{d \in D\colon [d) \cap X \neq \emptyset\} \in \boldsymbol{C}$. Then the following conditions are equivalent for $\chi\colon f \to f^{\leqq}$ and $\varphi\colon f \to f^o$.*

1) *χ is a homomorphism from $\boldsymbol{E}$ into $\boldsymbol{E}'$.*

2) *χ is an isomorfihism from $\boldsymbol{E}$ onto $\boldsymbol{E}'$ whose inverse is φ.*

3) *For all $d, x, y \in D$ the cardinality m of the set $(x \circ y) \cap [d)$ satisfies $m \equiv 1 \pmod{\varrho(R^2)}$ if $x \geqq d$, $y \geqq d$ and $m \equiv 1 \pmod{\varrho(R^2)}$ otherwise.*

Proof. (1) $\Leftrightarrow$ (3). Proposition 2.2. (2) $\Rightarrow$ (2). Evident. (1) $\Rightarrow$ (2). It suffices to show that $\psi = \chi^{-1}$ i.e. that $f^{\leqq o} = f^{o\leqq} = f$ for all $f \in \mathscr{E}$. Let $f \in \mathscr{E}$ and $d \in D$. Using (18) (16), the distributivity and $\mu_d(x) = 0$ for $d \nleqq x$ we obtain

$$f^{\leqq o}(d) = \sum_x \mu_d(x)\, f^{\leqq}(x) = \sum_x \mu_d(x) \sum_{x \leqq y} f(y) = \sum_{x \leqq y} \mu_d(x) f(y) =$$
$$= \sum_{d \leqq x \leqq y} \mu_d(x) f(y) = \sum_{d \leqq y} \Big(\sum_{d \leqq x \leqq y} \mu_d(x) \Big) f(y) \,,$$

It follows from (17) that $\sum_{d \leqq x \leqq y} \mu_d(x)$ vanishes for $d < y$ and equals 1 for $d = y$; whence $f^{\leqq o}(d) = f(d)$. Similarly

$$f^{o\leqq}(d) = \sum_{d \leqq x} f^o(x) = \sum_{d \leqq x} \sum_y \mu_x(y) f(y) =$$
$$= \sum_{d \leqq x \leqq y} \mu_x(y) f(y) = \sum_{d \leqq y} \Big(\sum_{d \leqq x \leqq y} \mu_x(y) \Big) f(y) = f(d) \,. \qquad \square$$

2.6 We apply Corollary 2.5 to the case that $x \circ y = \{x \cdot y\}$ for $(x, y) \in F$ and $x \circ y = \emptyset$ otherwise where $\langle D, \cdot \rangle$ is a partial groupoid with domain F (cf. Example 1.5, and 1.10. We say that $(d, \leqq)$ is a *partial meet-semilattice* if for all $y, z \in D$ the set $(y] \cap (z]$ is either empty or has a greatest element which we denote by $y \wedge z$ (in other words, if y and z have a lower bound, then there is a lower bound $y \wedge z$ such that $y \wedge z \geqq d$ for all $d \leqq y, d \leqq z$). Note that the partial operation $\wedge$ in a partial meet-semilattice is *idempotent* (i.e. $(x, x) \in F$ and $x \wedge x = x$ for all $x \in D$), *commutative* (if $(x, y) \in F$, then $(y, x) \in F$ and $x \wedge y = y \wedge x$) and *associative* (if $(x, y) \in F$ and $(x \wedge y, z) \in F$, then $(y, z) \in F$, $(x, y \wedge z) \in F$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$). Conversely if a partial groupoid on D satisfies these laws, then the order $\leqq$ on D defined by setting $x \leqq y$ if $x = x \wedge y$ is a partial meet-semilattice. We have:

2.7 Corollary. *Let $\langle D; \cdot \rangle$ be a partial groupoid with domain F and let $x \circ y = \{x \cdot y\}$ if $(x, y) \in F$ and $x \circ y = \emptyset$ otherwise. Then the condition* 3 *of Corollary* 2.5 *holds if and only if*

$$F := \{(y, z) \in D : (y] \cap (z] \neq \emptyset\} \,,$$

$(D; \leqq)$ is a partial meet-semilattice and $y \cdot z$ is the meet $y \wedge z$ of y and z for all $(y, z) \in F$.

Proof. Necessity. Let $k := \varrho(R^2)$ and let

$$G := \{(y, z) \in D^2 : (y] \cap (T] \neq \emptyset\}$$

consist of pairs having a common lower bound. We prove that $F = G$. 1) Let $(y, z) \in F$ and $d := y \cdot z$. Then $|(y \circ z) \cap [d)| = |\{d\}| = 1$ and $1 \not\equiv 0 \pmod k$ and so by condition 3 we have $y \geqq d$, $z \geqq d$, i.e. $d \in (y] \cap (z]$ proving $(y, z) \in G$ and $F \subseteq G$. 2) Conversely let $(y, z) \in G$. Choose $d \in (y] \cap (z]$ and put $A := (y \circ z) \cap [d)$. Now from the condition 3 we have $|A| \equiv 1 \pmod k$. Taking into account $|A| \leqq 1$ we get $|A| = 1$ proving $y \circ z \neq \emptyset$, hence $y \circ z = \{y \cdot z\}$ and $(y, z) \in F$. Thus $G \subseteq F$ and so $G = F$. Moreover, $x := y \cdot z \geqq d$ for all $d \in B := (y] \cap (z]$.

Let $(y, z) \in F$, $x := y \cdot z$ and $B := (y] \cap (z]$. By the last statement $B \subseteq (x]$. On the other hand, for each $t \in D \setminus B$ applying the condition 3 we get $t \nleqq x$. In particular, $x \in B$ i.e. x is the greatest element of B. It follows that $(D, \leqq)$ is a partial meet-semilattice and $y \cdot z = y \wedge z$ for all $(y, z) \in F$.

Sufficiency. Let $(y, z) \in F$ and let $x := y \wedge z$. Clearly $x \geqq d$ iff $y \geqq d$ and $z \geqq d$ proving 3. □

3. Direct sums and products

3.1 We introduce two constructions for convolutive $\boldsymbol{R}$-structures which are natural extensions of direct sum and Kronecker product of matrices.

Let R be as in 1.1, let $\langle D_j; \circ_j \rangle$ be a multigroupoid, let $\boldsymbol{C}_j \subseteq \mathscr{P}(D_j)$ satisfy the assumptions of 1.2 $(j = 1, 2)$ and let E_j be the corresponding convolutive $\boldsymbol{R}$-structures (cf. 1.4) $j = 1, 2$.

For simplicity we assume that D_1 and D_2 are disjoint and put $D := D_1 \cup D_2$ (if not, replace D by the disjoint union of D_1 and D_2). Define a multigroupoid $\langle D; \circ \rangle$ by setting $x \circ y := x \circ_j y$ if both x and y belong to D_j and $x \circ y := \emptyset$ otherwise. Similarly put $\boldsymbol{C} := \{X_1 \cup X_2 : X_i \in \boldsymbol{C}_i (i = 1, 2)\}$. It is easy see to that $\boldsymbol{C}$ satisfies the assumption of 1.2. The convolutive $\boldsymbol{R}$-structure $\boldsymbol{E}$ corresponding to $\langle D; \circ \rangle$ and $\boldsymbol{C}$ is denoted by $\boldsymbol{E}_1 \otimes \boldsymbol{E}_2$ and called the *direct sum* of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. Put $F_j := \{f \in \mathscr{E} : f(D_{3-j}) = \{0\}\}$ $(j = 1, 2)$ It is easy to verify that F_j is the subuniverse (i.e. carrier of a substructure) of $\boldsymbol{E}$ isomorphic to $\boldsymbol{E}_j$ $(j = 1, 2)$. It is easy to verify that the direct sum induces and abelian monoid on the class of isomorphism types of convolutive $\boldsymbol{R}$-structures.

The transformation $\psi: f \to f^\lambda$ from 1.6–1.11 extends quite naturally to direct sums.

3.2 Proposition. *Let* $\boldsymbol{E} = \boldsymbol{E}_1 \otimes \boldsymbol{E}_2$ *let* $\lambda^j: D_j^2 \to R (j = 1, 2)$ *let* $D := D_1 \cup D_2$ *and let* $\lambda: D^2 \to R$ *be defined by setting* $\lambda(d, x) := \lambda^j(d, x)$ *if both* d *and* x *belong to the same* D_j *and* $\lambda(d, x) := 0$ *otherwise. Then* λ *satisfies one of* 1.6 (i), (1.6 (ii) *and* (7) *if and only if both* λ^1 *and* λ^2 *have the same property.*

Proof. Direct check. For example, suppose that λ^1 and λ^2 satisfy (7). Let $d \in D_j$, $x \in D_k$ and $y \in D_l$. If $j = k = l$ then by the definition of λ and $\circ$ the equation (7) becomes

$$\lambda_d^j(x)\, \lambda_d^j(y) = \sum_{z \in x_j \circ y} \lambda_d^j(z)$$

which holds by assumption. Thus suppose that j, k and l are not equal, then the left side of (7) vanishes. If $k \neq l$, then $x \circ y = \emptyset$ and (7) holds. Thus let $k = l$, then $j \neq k$ and from $d \in D_j$ and $z \in x \circ y \subseteq D_k$ we get $\lambda_d(z) = 0$ and so (7) holds.

Conversely, let (7) hold for λ. Restricting x, d and y to D_j we see at once that (7) holds for λ^j. □

Let $A_j: D_j \to \mathscr{P}(D_j)$ $(j = 1, 2)$ and let $D = D_1 \cup D_2$. The map $A_1 \cup A_2: D \to \mathscr{P}(D)$ is de;ned by $(A_1 \cup A_2)(d) := A_j(d)$ for $d \in D_j (j = 1, 2)$. As usual, the cardinal sum of orders $(D_j, \leqq_j)$ $(j = 1, 2)$ is the order $(D, \leqq)$ with $x \leqq y$ if $x, y \in D_j$, $x \leqq_j y$ for some $j \in \{1, 2\}$ (we assume that $D_1 \cap D_2 = \emptyset$). We have:

3.3 Proposition. *Let* $i \in \{1, 2, 3\}$. *The map* $A_1 \cap A_2$ *satisfies the condition* (i)' *from* 2.1 *and* 2.2 *if and only if both* A_1 *and* A_2 *satisfy* (i). *Similarly the cardinal sum* $(D, \leqq)$ *of orders* $(D_j, \leqq_j)$ $(j = 1, 2)$ *satisfies* 2.5 (i) *if and only if both* $(D_1, \leqq_1)$ *and* $(D_2, \leqq_2)$ *satisfy* 2.5 (i) (i = 1, 2).

Proof. Direct check. We only prove it for $i = 3$. Let A_1 and A_2 satisfy (3) and let $d \in D_j$, $x \in D_k$ and $y \in D_l$. If $j = k = l$ then $(x \circ y) \cap A(d) = (x_j \circ y) \cap A_j(d)$ and the property is inherited from A_j. Suppose $j \neq k = l$. Then $x \circ y \subseteq D_k$ and $A(d) \subseteq D_j$ show $(x \circ y) \cap A(d) = \emptyset$. Since $x \in D_k$ does not belong to $A(d) \subseteq D_j$ we see that (3) holds. Finally let $k \neq l$. Then $x \circ y = \emptyset$ and either $x \notin A(d)$ or $y \notin A(d)$ so that again (3) holds.

Conversely, let (3) hold for $A_1 \cup A_2$. Letting d, x and y range over D_j we see that A_j satisfies (3). □

3.4 We turn to direct products of convolutive R-structures $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. Naturally the *direct product* of the multigroupoids $\langle D_j; \circ_j\rangle$ $(j = 1, 2)$ is the multigroupoid $\langle D, \circ\rangle$ on $D := D_1 \times D_2$ defined by setting

$$(x_1, x_2) \circ (y_1, y_2) := (x_1 \circ y_1) \times (x_2 \circ y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in D$. Put

$$\boldsymbol{C} := \{Z\colon Z \subseteq X_1 \times X_2 \quad \text{for some} \quad X_j \in C_j (j = 1, 2)\} .$$

We have:

3.5 Fact. $\boldsymbol{C}$ satisfies 1.2 (i) and 1.2 (ii).

Proof. Let $X, Y \in \boldsymbol{C}$. Then $X \subseteq X_1 \times X_2$ and $Y_1 \times Y_2$ for some $X_j, Y_j \in \boldsymbol{C}_j$ $(j = 1, 2)$. By the assumption 1.2 (i) we have $Z_j := X_j \cup Y_j \in \boldsymbol{C}_j$ $(j = 1, 2)$ and thus from $X \cup Y \subseteq Z_1 \times Z_2 \in \boldsymbol{C}$ we obtain $X \cup Y \in \boldsymbol{C}$. Clearly $\boldsymbol{C}$ is hereditary and contains all singletons and so it satisfies 1.2 (i). We prove the first statements of (ii). Let $d = (d_1, d_2)$ and for $j = 1, 2$ put $Z_j := X_j \times Y k_j := |\{(x_j, y_j) \in Z_j\colon d_j \in x_j \circ_j y_j\}|$ Note that k_j is finite (by the assumption (2)). Clearly the number of pairs (x, y) with $x = (x_1, x_2) \in X$ and $y = (y_1, y_2) \in Y$ such that $d \in x \circ y$ does not exceed $k_1 k_2$ and hence is finite. We prove (3). For $j = 1, 2$ put $W_j := X_j \circ_j Y_j$. By assumption $W_j \in \boldsymbol{C}_j$. We have

$$X \circ Y = \bigcup_{(x_1,x_2)\in X} \bigcup_{(y_1,y_2)\in Y} (x_1 \circ_1 y_1) \times (x_2 \circ_2 y_2) \subseteq$$

$$\subseteq \Big(\bigcup_{(x_1,y_1)\in Z_1} (x_1 \circ_1 y_1)\Big) \times \Big(\bigcup_{(x_2,y_2)\in Z_2} (x_2 \circ_2 y_2)\Big) \subseteq W_1 \times W_2 \subseteq \boldsymbol{C} .$$

3.6 In view of Fact 3.5 the structure $\mathscr{E}_{\boldsymbol{RDC}}$ is a convolutive $\boldsymbol{R}$-structure. It will be denoted by $\boldsymbol{E}_1 \times \boldsymbol{E}_2$ and called the *direct product* of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. There is a natural embedding of $\boldsymbol{E}_j$ into $\boldsymbol{E}_1 \times \boldsymbol{E}_2$. Also it is easy to see that the direct product defines an abelian monoid on the class of isomorphism types of convolutive $\boldsymbol{R}$-structures.

It can be shown easily that the direct product distributes over the direct sum.

The map ψ: $f \to f^{\lambda}$ from 1.6–1.11 naturally extends to the direct product (for D finite it is the Kronrcker product of the matrices L from 1.11).

3.7 Proposition. *Let $E = E_1 \times E_2$ and let λ^j: $D_j^2 \to R$ $(j = 1, 2)$. Put $D = D_1 \times D_2$ and define λ: $D^2 \to R$ by setting*

$$(19) \qquad \lambda(d, x) := \lambda^1(d_1, x_1)\, \lambda^2(d_2, x_2)$$

for all $d = (d_1, d_2) \in D$ and $x = (x_1, x_2) \in D$. Then for $i = 1, 2$ if at last two of λ, λ^1 and λ^2 satisfy 1.6 (i), *then all three satisfy* 1.6 (i).

Let R be distributive and satisfy the medial law (6). *If* (7) *holds for both λ^1 and λ^2 then* (7) *is true for λ. If* (7) *holds for λ and for some $d_2, x_2, y_2 \in D_2$*

$$(20) \qquad \lambda^2_{d_2}(x_2)\, \lambda^2_{d_2}(y_2) = \sum_{z_2 \in x_2 \circ_2 y_2} \lambda^2_{d_2}(z_2) = r$$

where r is $\operatorname{im} \lambda^2_1$*-cancellative (see* 1.8), *then* (7) *holds for λ_1. A similar statement holds for λ_2.*

Proof. Let $X_j \in C_j$ $(j = 1, 2)$ and $X := X_1 \times X_2$. Let $x = (x_1, x_2) \in X \cap \operatorname{supp} \lambda_d$. Then by (19) we have $\lambda^1_{d_1}(x_1) \neq 0 \neq \lambda^2_{d_2}(x_2)$ proving $x_j \in X_j \cap \operatorname{supp} \lambda^j_{1_j}$ $(j = 1, 2)$. The converse also holds and so $X \cap \operatorname{supp} \lambda_d = (X_1 \cap \operatorname{supp} \lambda^1_{d_1}) \times (X_2 \cap \operatorname{supp} \lambda^2_{d_2})$. From this the validity of 1.6 (1)–(2) follows quite easily. To prove the second statement let $d = (d_1, d_2)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of D. Applying (19), the distributive and medial laws and (7) we get

$$\lambda_d(x)\, \lambda_d(y) = (\lambda^1_{d_1}(x_1)\, \lambda^2_{d_2}(x_2))\, (\lambda^1_{d_1}(y_1)\, \lambda^2_{d_2}(y_2)) =$$
$$= (\lambda^1_{d_1}(x_1)\, \lambda^1_{d_1}(y_1))\, (\lambda^2_{d_2}(x_2)\, \lambda^2_{d_2}(y_2)) = \Big(\sum_{z_1 \in x_1 \circ_1 y_1} \lambda^1_{d_1}(z_1)\Big)\Big(\sum_{z_2 \in x_2 \circ_2 y_2} \lambda^2_{d_2}(z_2)\Big) =$$
$$= \sum_{z_1 \in x_1 \circ_1 y_1} \sum_{z_2 \in x_2 \circ_2 y_2} \lambda^1_{d_1}(z_1)\, \lambda^2_{d_2}(z_2) = \sum_{z \in x \circ y} \lambda_d(z)\,.$$

We prove the last statement. Let λ satisfy (7), let d_2, x_2, y_2 and t be the elements from (20) and let $d_1, x_1, y_1 \in D_1$ be arbitrary. Put $d = (d_1, d_2)$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Proceeding as above and applying (7) to λ we get

$$\Big(\sum_{z_1 \in x_1 \circ_1 y_1} \lambda^1_{d_1}(z_1)\Big)\, r = \Big(\sum_{z_1 \in x_1 \circ_1 y_1} \lambda^1_{d_1}(z_1)\Big)\Big(\sum_{z_2 \in x_2 \circ_2 y_2} \lambda^2_{d_2}(z_2)\Big) = \sum_{z \in x \circ y} \lambda_d(z) = \lambda_d(x)\, \lambda_d(y) =$$
$$= (\lambda^1_{d_1}(x_1)\, \lambda^1_{d_1}(y_1))\, (\lambda^2_{d_2}(x_2)\, \lambda^2_{d_2}(y_2)) = (\lambda^1_{d_1}(x_1)\, \lambda^1_{d_1}(y_1))\, r\,.$$

Since r is $\operatorname{im} \lambda^1$-cancellative, we obtain that (7) holds λ^1. □

We turn to the maps $\chi: f \to f^A$ from § 2. Let A_j: $D_j \to \mathscr{P}(D_j)$ $(j = 1, 2)$ and let $D := D_1 \times D_2$. It is natural to define $A_1 \times A_2$: $D \to \mathscr{P}(D)$ by setting $(A_1 \times A_2) \cdot (d_1, d_2) := A_1(d_1) \times A_2(d_2)$. We have:

3.8 Proposition. *Let $E = E_1 \times E_2$, let A_j: $D_j \to \mathscr{P}(D_j)$ satisfy $A_j(d_j) \neq \emptyset$ for some $d_j \in D_j$ $(j = 1, 2)$ and let $A = A_1 \times A_2$. If two of A_1, A_2 and A satisfy the condition* (i) *from* 2.1 *and* 2.2, *then the third one has the same property,* (i = 1, 2, 3).

Proof. Let $X_j \in C_j$ and $d_j \in D_j$ $(j = 1, 2)$, $X = X_1 \times X_2$ and $d = (d_1, d_2)$. It is immediate that

$$X \cap A(d) = (X_1 \cap A_1(d_1)) \times (X_2 \cap A_2(d_2))$$

which proves the case 2.1 (i). For 2.1 (ii) it suffices to note that

$$\{d \in D: X \cap A(d) \neq \emptyset\} = \{d_1 \in D_1: X_1 \cap A_1(d_1) \neq \emptyset\} \times \{d_2 \in D_2: X_2 \cap A_2(d_2)\}$$

holds for all $d = (d_1, d_2) \in D$. For (3) we have

$$(21) \qquad (x \circ y) \cap A(d) = ((x_1 \circ y_1) \cap A_1(d_1)) \times ((x_2 \circ y_2) \cap A_2(d_2))$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $d = (d_1, d_2)$ are arbitrary elements of D. Let m, and m_i $(i = 1, 2)$ denote the cardinalities of the sets on the left side and on the right side of (21). Clearly $m = m_1 m_2$.

First suppose that both A_1 and A_2 satisfy (3). If $x, y \in A(d)$, then $x_1, y_1 \in A_1(d_1)$ and $x_2, y_2 \in A_2(d_2)$ and by (3) we have $m_1 \equiv m_2 \equiv 1 \pmod k$ where $k := \varrho\, R^2)$. Then $m = m_1 m_2 \equiv 1 \pmod k$. If $(x, y) \notin A^2(d)$, then $(x_j, y_j) \notin A_j^2(d)$ for some $j \in \{1, 2\}$ and so by (3) we have $m_j \equiv 0 \pmod k$ leading to $m \equiv m_1 m_2 \equiv 0 \pmod k$ and we are done in this case.

Next let A and A_1 satisfy (3). Fix $d_1, x_1, y_1 \in D_1$ so that $x_1, y_1 \in A(d_1)$ and let $d_2, x_2, y_2 \in D_2$ be arbitrary. Now $x_2, y_2 \in A_2(d_2)$ iff $x, y \in A(d)$ and $m_2 \equiv m_1 m_2 \equiv m \pmod k$ proves (3) for A_1. $\square$

As usual, the *direct product* of orders $(D_j, \leqq_j)$ $(j = 1, 2)$ is the order $(D, \leqq)$ where $D := D_1 \times D_2$ and $(x_1, x_2) \leqq (y_1, y_2)$ if $x_1 \leqq_1 y_1$ and $x_2 \leqq_2 y_1$. We have:

3.9 Corollary. *Let* $1 \leqq \mathrm{i} \leqq 3$. *If two of the orders* $(D_1, \leqq_1)$, $(D_2, \leqq_2)$ *and* $(D, \leqq)$ *satisfy* 2.5 (i) *then all three satisfy* 2.5 (i).

4. General balanced arrays

In this section we look at a concrete example of a convolutive R-structure which came up in the study of generalized block designs and together with [5] motivated this paper.

4.1 Let $s > 1$ be an integer and $\boldsymbol{s} := \{0, \ldots, s-1\}$. We order $\boldsymbol{s}$ by setting $0 \prec i$ for $i = 1, \ldots, s-1$ (so that 0 is the least element of $\prec$ while all the other elements are maximal). Clearly $(\boldsymbol{s}; \preceq)$ is a meet semilattice.

Let m be a positive integer. Given $a = (a_1, \ldots, a_m) \in \boldsymbol{s}^m$ and $0 \leqq i < s$ put $[a]_i := \{j \in M: a_j = i\}$. We may think that $j \in [a_i]$ is colored by color i and hence a may be viewed as a partition of $M := \{1, \ldots, m\}$ into s pairwise disjoint color blocks (some of which may be empty). Let $\preceq$ denote the componentwise order on s^m. Thus for $a = (a_1, \ldots, a_m) \in s^m$ and $b = (b_1, = (b_1, \ldots, b_m) \in s^m$ we have $a \leqq b$ iff $a_j \leqq b_j$ for $j = 1, \ldots, m$. Expressed differently, $a \leqq b$ iff $[a]_i \subseteq [b]_i$ for $i = 1, \ldots, s - 1$ i.e. b is obtained from a by enlarging some color blocks at the expense of the 0-th color block.

4.2 The poset $(3^m, \leqq)$ also appears in geometry. Denote by $e_1 = (1, 0, \ldots, 0, \ldots$ $\ldots, e_m = (0, \ldots, 0, 1)$ the unit coordinate m-vectors and let K_m denote the convex hull of $\{e_1, \ldots, e_m, -e_1, \ldots, -e_m\}$ (say in $\mathbb{R}^m$ or $\mathbb{Q}^m$). Thus K_2 is a square and K_3 is a octahedron (bipyramid). Let F_m denote the set consisting of the faces of K_m and $\emptyset$ (e.g. F_3 consists of $\emptyset$, 6 vertices, 12 edges and 8 triangles (see Fig. 1)).

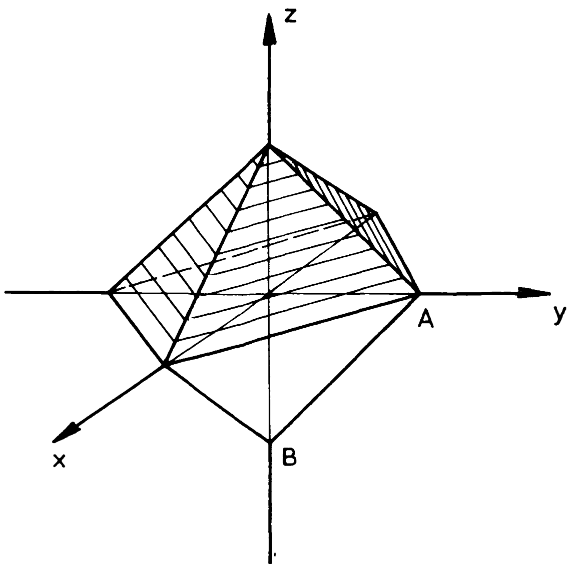

Fig. 1

There is a bijection ζ between 3^m and F_m. Indeed, given $a = (a_1, \ldots, a_m) \in 3^m$ put $b = (b_1, \ldots, b_m)$ where $b_i := -1$ if $a_i = 2$ and $b_i = a_i$ otherwise and let $\zeta(a)$ be the face of C_m whose vertices are the non-zero vectors among $b_1e_1, \ldots, b_me_m$ (e.g. to $(0, 1, 2)$ we associate the edge joining $A = (0, 1, 0)$ and $B = (0, 0, -1)$ (Fig. 1). The set F_m (of facer of K_m), ordered by inclusion, is called the *m-dimensional cross polytope* (cf. [10]). It is easy to see that ζ is an order isomorphism of $(3^m, \leqq)$ onto $(F_m, \subseteq)$. The poset $(3^m, \leqq)$ also plays a role in logic, universal algebra and switching theory. The homomorphisms $(3^m, \leqq) \to (3, \leqq)$, called *regular ternary logic functions*, are suitable for treating ambiguity (cf. [7]).

4.3 Let A and B be two subsets of s^m. Consider a map $f\colon s^m \to \mathbb{N}$ such that $\operatorname{supp} f \subseteq A$ and let $f^{\preceq}$ be defined by (16) in section 2.2. The restriction β of $f^{\preceq}$ to B is called an *A-supported B-covariance pattern* (shortly $A-B$-pattern). A matrix T over s with m rows and exactly $f(x)$ columns x^T for all $x \in s^m$ is a *realization* of $f^{\preceq}[2]$. For $s = 2$ the binary $A-B$-patterns were considered in [1, 3]. For $x \in \boldsymbol{s}^m$ let $\|x\| := m - |[x]_0|$ denote the number of non-zero coordinates of x (i.e. the Hamming distance of x from $(0, \ldots, 0)$) and let $V_t := \{x \in \boldsymbol{s}^m: \|x\| = t\}$ $(t = 0, \ldots, m)$. Clearly V_t consists of m-vectors over $\boldsymbol{s}$ with exactly t non-zero coordinates.

4.4 Examples. Let $s = 2$ and $A = \boldsymbol{2}^m$.

1. Let $B = V_1$. A map $f: \boldsymbol{2}^m \to \mathbb{N}$ is an *exact cover* if $f^{\preceq} \mid V_1$ is constant i.e. $f^{\preceq}(b) = \lambda_1$ for all $b \in V_1$. Now $x = (x_1, \ldots, x_m) \in \boldsymbol{2}^m$ may be identified with $[x]_1 := \{i: x_i = 1\}$ and to f corresponds the set system F consisting of $f(x)$ sets $[x]_1$. $(x \in \boldsymbol{2}^m)$. Clearly F corresponds to an exact cover if each $i \in M := \{1, \ldots, m\}$ belongs to exactly λ_1 members of F.

2. Put $B := V_1 \cup V_2$. A map $f: \boldsymbol{2}^m \to \mathbb{N}$ is an *balanced incomplete block design* if there are $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $f^{\preceq}(b) = \lambda_1$ for all $b \in V_1$ and $f^{\preceq}(b) = \lambda_2$ for all $b \in V_2$ (i.e. $f^{\preceq}(e_i) = \lambda_1$ for $i = 1, \ldots, m$) and $f^{\preceq}(e_i + e_j) = \lambda_2$ for $1 \leqq i \leqq j \leqq m$). The corresponding set system F is thus described by the fact that each $i \in M = \{1, \ldots, m\}$ belongs to exactly λ_1 members of F and each pair $\{i, j\}$ $\{1 \leqq i < j \leqq m\}$ belongs to exactly λ_2 members of F.

3. Let $B = V_1 \cup V_2$, f an $A-B$-pattern, and T is realization. Then $m \times n$ matrix T may be also interpreted as follows. To the i-th row of T associate the set $R_i := \{j: t_{ij} = 1\}$ $(i = 1, \ldots, n)$. It is easy to see that $f^{\preceq}(e_i) = |R_i|$ $(1 \leqq i \leqq n)$ and $f^{\preceq}(e_i + e_j) = |R_i \cap R_j|$ $(1 \leqq i < j \leqq m)$ and thus $f^{\preceq}$ registers the sizes of of $|R_i \cap R_j|$ $(1 \leqq i, j \leqq m)$. Instead of this *covariance or intersection pattern* [1, 3] we could also consider the intersection matrix $(|R_i \cap R_j|)$.

4. Put $B = V_1 \cup \ldots \cup V_t$. A map $f: \boldsymbol{2}^m \to \mathbb{N}$ is a *t-design* if there are $\lambda_1, \ldots, \lambda_t \in \mathbb{N}$ such that $f^{\preceq}(b) = \lambda_h$ for all $h = 1, \ldots, t$ and $b \in V_h$ [5].

4.5 For $a \in \boldsymbol{s}^m$ put $z(a) := (v_1, \ldots, v_{s-1})$ where $v_i := |[a]_i|$ is the number of coordinates of a equal to i $(i = 1, \ldots, s - 1)$, i.e. $z(s)$ is the color frequency vector of a. The set

$$\operatorname{supp} a := [a]_1 \cup \ldots \cup [a]_{s-1}$$

is called the *support* of a.

Let τ be a subset of $M := \{1, \ldots, m\}$ A map $f: \boldsymbol{s}^m \to \mathbb{N}$ is *balanced with respect to* τ if $f^{\preceq}(b) = f^{\preceq}(b')$ whenever $\operatorname{supp} b = \operatorname{supp} b' = \tau$ and $z(b) = z(b)'$ (i.e. in a realization T the number of columns c such that $c^T \leqq b$ is constant for all $b \in B$ with $\operatorname{supp} b = \tau$ and the same $z(b)$ [8]).

We say that $f: \boldsymbol{s}^m \to \mathbb{N}$ is a *balanced array* $(B. A.)$ *of strength* t if $B \subseteq V_r$ and there is a map μ from $Z := \{z(b): b \in B\}$ into $\mathbb{N}$ such that $f^{\preceq}(b) = \mu(z(b))$ for all $b \in B$.

4.6 We apply the results of §§ 1–3 to $\langle D, \circ\rangle = \langle \boldsymbol{s}^m; \wedge\rangle$ where $\wedge$ is the meet semilattice operation (i.e. for $a, b \in \boldsymbol{s}^m$ the coordinates of $c := a \wedge b$ satisfy $c_i \neq 0 \Rightarrow a_i = b_i = c_i$ for $i = 1, \ldots, m$). We start with the case $m = 1$. Let $\boldsymbol{R}$ be an integral domain. It is easy to describe all homomorphism $\varepsilon: \langle \boldsymbol{s}; \wedge\rangle \to \langle \boldsymbol{R}, \cdot\rangle$.

First of all for $0 \leqq i < s$ we have $\varepsilon(i) = \varepsilon(i \wedge i) = \varepsilon^2(i)$ and so $\varepsilon(i)(\varepsilon(i) - 1) = 0$ shows that $\varepsilon(i) \in \{0, 1\}$ for $0 \leqq i < s$. Moreover, we have

$$\varepsilon(0) = \varepsilon(i \wedge j) = \varepsilon(i)\,\varepsilon(j) \tag{22}$$

for all $0 \leqq i < j < s$. We have two cases: a) Suppose $\varepsilon(0) = 1$. Then by (22) we have $\varepsilon(j) = 1$ for all $0 \leqq j < s$. This map λ_0 with $\lambda_0(0) = \ldots = \lambda_0(s - 1) = 1$ is obviously a homomorphism. b) Let $\varepsilon(0) = 0$. Then (22) shows that $\varepsilon(i) = 1$ holds for at most one i. The map σ with $\sigma(0) = \ldots = \sigma(s - 1) = 0$ is obviously a homomorphism but has little use in the sequel. The remaining maps λ_i $(i = 1, \ldots, s - 1)$ satisfy $\lambda_i(i) = 1$ and $\lambda_i(x) = 0$ otherwise. The maps λ_i $(i = 0, \ldots, s - 1)$ define $\lambda\colon \boldsymbol{s}^2 \to R$. The corresponding matrix $L := (\lambda_i(j))$ has rows $(1, 1, \ldots, 1)$ $e_2, \ldots, e_s$ where $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_s = (0, \ldots, 0, 1)$. Clearly λ exists if $\boldsymbol{R}$ has 1 and is unique (up to the permutation of rows) if $\boldsymbol{R}$ is an integral domain. It is easy to verify that the inverse L^{-1} has rows $(1, -1, -1, \ldots, -1)$, $e_2, \ldots, e_s$. Thus, in our case $\boldsymbol{E}$ is isomorphic to $\boldsymbol{E}'$ i.e. $\langle \mathscr{E}; * \rangle \simeq \langle \mathscr{E}; \wedge \rangle$. In particular the subalgebras of the first may be determined from those of the second (cf. [5]). It is easy to see that $f^{\preceq} = f^{\lambda}$ for all $f \in \mathscr{E}$.

4.7 Using 3.6–3.9 we extend λ to $\langle \boldsymbol{s}^m, \wedge \rangle$. For $x = (x_1, \ldots, x_m) \in \boldsymbol{s}^m$ and $y = (y_1, \ldots, y_m) \in \boldsymbol{s}^m$ we put $\lambda(x, y) = 1$ if $x_i = 0$ or $x_i = y_i$ for $i = 1, \ldots, m$ and $\lambda(x, y) = 0$ otherwise. Expressed differently, $L = (\lambda(x, y))$ is the incidence matrix of $(\boldsymbol{s}^m, \preceq)$ i.e. $\lambda(x, y) = 1$ if $x \preceq y$ and $\lambda(x, y) = 0$ otherwise. The inverse matrix L^{-1} is the matrix $(\mu(x, y))$ with (i) $\mu(x, y) = (-1)^{\nu}$ if for all $i = 1, \ldots, m$ we have $x_i = 0$

Table 1.

1	−1	−1	−1	1	1	−1	1	1
.	1	.	.	−1	.	.	−1	.
.	.	1	.	.	−1	.	.	−1
.	.	.	1	−1	−1	.	.	.
.	.	.	.	1	.	.	.	.
.	.	.	.	.	1	.	.	.
.	.	.	.	.	.	1	−1	−1
.	.	.	.	.	.	.	1	.
.	.	.	.	.	.	.	.	1

or $x_i = y_i$ and $\nu := |\{i\colon x_i = 0,\ y_i \neq 0\}|$, and (ii) $\mu(x, y) = 0$ otherwise. The matrix L^{-1} corresponding to $s = 2$ and $m = 3$ is on Table 1. Here $\boldsymbol{3}^2$ is listed lexicographically as $(0, 0), (0, 1), \ldots, (2, 2))$ and dots stand for 0's.

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