Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 25 (1984), No. 1, 75--87

Persistent URL: http://dml.cz/dmlcz/142532

1984 ACTA UNIVERSITATIS CAROLINAE — MATHEMATICA ET PHYSICA VOL. 25. NO. 1

Prime Ideals in Universal Algebras

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Received 30 March 1983

After recalling the notion of ideal in algebras with a constant 0, we introduce a notion of product of ideals and consequently a notion of prime ideal, and of the radical of an ideal. Some results about these notions are shown and in particular we prove a generalization of I. S. Cohen's theorem, after which if all prime ideals are finitely generated then all ideals are finitely generated, that holds in commutative rings with identity.

Poté, co připomeneme pojem ideálu v algebrách s konstantou 0, zavedeme pojem součinu ideálů, pojem prvoideálu a pojem radikálu. Ukážeme některé výsledky, vztahující se k těmto pojmům a zvláště dokážeme zobecnění věty I. S. Cohena, podle níž jestliže všechny prvoideály v komutativním okruhu s jednotkou jsou konečně generované pak všechny ideály jsou konečně generované.

Мы напомним понятие идеала в алгебрах с константой 0 и потом введем понятия произведения идеалов, прямого идеала и радикала. Мы покажем некоторые результаты, относящиеся к этим понятиям и особенно окажем обобщение тоеремы И. С. Когена утверждающей, что если все прямые идеалы в коммутативном кольце с единицей конечно порождены, то все идеалы конечно порождены.

In order to speak meaningfully of "prime ideals" in Universal Altebra, we need

1) a concept of "ideal";
2) a notion of "product" of ideals.

In order to do so in a useful way we need in addition at least the following:

3) to generalize some classical results in Ideal theory.

This paper shows that this threefold aim can be accomplished in a certain sense.

1. Ideals

We will fix once and for all a class K of algebras of a fixed type, and assume that there is a distinguished nullary operation or else a constant, equationally definable in all algebras of K, which we denote by 0.

Surely we want that if $A \in K$ and R is a congruence of A, the equivalence class $[0]\, R$ is to be an "ideal": that is why we introduce our first definition (see [10], [11]):

If p is any term in the variables $\vec{x} = x_1, \ldots, x_n$ and $\vec{y} = y_1, \ldots, y_k$, and $p(\vec{x}, 0, \ldots \ldots, 0) = 0$ holds identically in K, then we say that p is an *ideal term in* $\vec{y}$.

If $A \in K$ and I is a subset of A such that $p(\vec{a}, \vec{b})\, I$ whenever $p(\vec{x}, \vec{y})$ is an ideal term in $\vec{y}$, $\vec{a} \in A^n$, $\vec{b} \in I^k$, then we say that I is an *ideal of* A.

The set $i(A)$ of all ideals of A naturally becomes an algebraic lattice. (For more precision, one should add some qualification, like "K; 0-ideal" or the like, which we let implicit. We will also write "$\vec{a} \in A$" instead of "$\vec{a} \in A^n$", etc). Normal subloops, normal subgroups, invariant subgroups of (multi)operator groups, filters and ideals in (pseudo)boolean algebras, and many more similar notions fall within the scope of our concept of ideal. However, words bear no trade mark, and ideal in lattices or semigroups do not in general coincide with ideals in our sense.

For $A \in K$, and R a congruence of A, we have that $[0]\, R$ is an ideal of A. Sometimes the converse is true, and moreover the ideal completely determines the congruence: we say that K is *ideal determined* if for all $A \in K$, any ideal of A is the class of 0 for exactly one congruence; in this case we let I^δ be the congruence corresponding to an ideal I, $a + I$ be the congruence class of $a \in A$, and A/I be the quotient algebra. Ideal determined equational classes have been nicely characterized by a Mal'cev condition:

1.1. For a variety K, K is ideal determined iff for some positive integer m, there are binary terms $d_1, \ldots, d_m, d_{m+1}$ such that:

$$K \models (d_1(y, z) = 0 \wedge \ldots \wedge d_m(y, z) = 0) \leftrightarrow y = z\,,$$

$$K \models d_{m+1}(y, y) = 0 \wedge d_{m+1}(0, y) = y\,.$$

For a proof see [5]. (It is well known how to transform into a Mal'cev condition the property in 1.1: see also the proof of 1.2 below).

One more important property of ideal determined varieties is that the lattices of congruences of their algebras are modular.

What about "one-sided" ideals?

We let P_0 be the set of all ideal terms. If we restrict ourselves to a subset P of P_0, we have a corresponding notion of *P-ideal* of A, K.

For $S \subseteq A$, we denote by $\bar{S}^{id}$ the *ideal generated by* S, i.e.:

$$S^{id} = \bigcap(I \in i(A) \mid S \subseteq I) =$$
$$= \{p(a, s) \mid p(x, y) \text{ an ideal term in } y,\, a \in A,\ s \in S\}\,.$$

Since an intersection of P-ideals of A is a P-ideal, we can similarly define the *P-ideal generated by* S, and we denote it by $\bar{S}^P$. Some assumption will be needed if one wants to have a closure operation. For instance if $0 \in P$, and P is closed under composition (in the sense that if $p(\vec{x}, \vec{y}) \in P$ and $p_1(\vec{x}^1, \vec{y}^1), \ldots, p_n(\vec{x}^n, \vec{y}^n) \in P$, then also $p(\vec{x}, p_1, \ldots, p_n) \in P$), then "$^{-P}$" is an algebraic closure operation, and the set $i_P(A)$ of all P-ideals of A is an algebraic lattice.

We say that $P \subseteq P_0$ is a *base* of ideal terms for K if $\bar{S}^P = \bar{S}^{id}$ for all $A \in K$ and $S \subseteq A$, i.e. if every P-ideal is an ideal.

1.2. Let K be an ideal determined equational class of finite type. Then there is a finite base of ideal terms.

Proof. Assume K has binary terms $d_1, \ldots, d_m$ satisfying 1.1; then one easily sees that for $A \in K$ and $I \in i(A)$,

$$aI^\delta b \quad \text{iff} \quad d_i(a, b) \in I \quad \text{for} \quad i = 1, \ldots, m\,.$$

If f is an n-ary operation in the type of K, we consider the algebra freely generated in K by $x_1, \ldots, x_n, y_1, \ldots, y_n$; since $f(\vec{x})\, I^\delta f(\vec{y})$, if I is the ideal generated by $\{d_i(x_u, y_u) \mid i = 1, \ldots, m;\ u = 1, \ldots, n\}$, then there must be $2n(m+1)$-ary terms $r_{i,f}$ such that:

$$r_{i,f}(\vec{x}, \vec{y}, 0, 0, \ldots, 0) = 0\,;$$
$$r_{i,f}(\vec{x}, \vec{y}, d_1(x_1, y_1), \ldots, d_m(x_1, y_1), \ldots, d_1(x_n, y_n), \ldots, d_m(x_n, y_n)) = d_i(f(\vec{x}), f(\vec{y}))\,,$$

for $i = 1, \ldots, m$, hold identically in this algebra, hence in K.

Similarly, by translating the first condition of 1.1 into equations, we get quaternary terms $g_1, \ldots, g_m$ such that:

$$g_1(x, y, d_1(x, y), 0) = x\,;$$
$$g_i(x, y, 0, d_i(x, y)) = g_{i+1}(x, y, d_{i+1}(x, y), 0)\,, \quad (i = 1, \ldots, m-1)\,,$$
$$g_m(x, y, 0, d_m(x, y)) = y\,;$$

hold identically in K (see [3]). Since congruences are transitive and symmetric relations, we also get $(2m+3)$-ary terms $q_1, \ldots, q_m$ such that:

$$q_i(x, y, z, d_1(x, y), \ldots, d_m(x, y), d_1(z, y), \ldots, d_m(z, y)) = d_i(x, z)\,;$$
$$q_i(x, y, z, 0, \ldots, 0) = 0\,,$$

hold identically in K for $i = 1, \ldots, m$.

Finally, since x belongs to the ideal generated by $\{y, d_1(x, y), \ldots, d_m(x, y)\}$, there must be a $(m+2)$-ary term p_0 such that in K:

$$x = p_0(x, y, y, d_1(x, y), \ldots, d_m(x, y))\,,$$
$$0 = p_0(x, y, 0, 0, \ldots, 0)\,.$$

Therefore the set

$$\{0, d_i, g_i, q_i, p_0, r_{i,f} \mid i = 1, \ldots, m;\ f \text{ an operation}\}$$

is a base of ideal terms for K. If the type is finite, we get our claim.

The question of the existence of an equational class which is ideal determined, has a finite base for ideal terms but has an infinite number of essentially distinct operations in the type seems to be open.

Remark. Propositin 1.2 should explain why in a number of classical cases ideals admit a simple algebraic definition. For instance if K^+ has an ideal determined reduct variety K, and if terms $r_{i,f}$ with the property in the proof of 1.2 exist for any *new* operation f, then ideals relative to K^+ are simply ideals relative to K which are also $\{r_{i,f} \mid i = 1, \ldots, m; f \text{ a new operation}\}$ – idelas.

2. Multiplication of ideals

An operation of multiplication of ideals will be considered useful for our purposes if it induces a useful concept of primeness.

If $A \in K$, R, R' are congruences of A, we surely want that if x is in the "product" of the ideals $[0]\,R$, $[0]\,R'$, then $x(R \cap R')\,0$ and moreover that x be expressed in terms of a suitable polynomial function involving elements from $[0]R$, $[0]\,R'$.

This leads to our second basic definition (see [11], [5]):

A term $t(\vec{x}, \vec{y}, \vec{z})$ is called a *commutator term in* $\vec{y}, \vec{z}$ if it is an ideal term in $\vec{y}$ and an ideal term in $\vec{z}$.

If $A \in K$ and X, Y are non empty subsets of A, we set:

$$[X, Y] = \{t(\vec{a}, \vec{u}, \vec{v}) \mid t(\vec{x}, \vec{y}, \vec{z}) \text{ a commutator term in } \vec{y}, \vec{z}, \vec{a} \in A,\ \vec{u} \in X,\ \vec{v} \in Y\}\ ;$$

If $I, J \in i(A)$, $[I, J]$ is the *commutator of* I, J.

We observe that $[X, Y]$ is always an ideal. If K is an ideal determined variety, and I, J are ideals of $A \in K$, then $[I, J]^\delta$ is the commutator of the congruences I^δ, J^δ ([4]).

If T_0 is the set of all commutator terms, we can relativize the previous notions to a subset T of T_0:

$$[X, Y]_T = \{(\vec{a}, \vec{u}, \vec{v}) \mid t \in T,\ \vec{a} \in A,\ \vec{u} \in X,\ \vec{v} \in Y\}\ ,$$

and we call it the *T-product of* X, Y. Observe that $[\ ,\ ]_T$ is increasing in both arguments, and that $[X, Y] \subseteq X \cap Y$ if X, Y are ideals.

The notion of T-product captures: product of (one-sided) ideals in rings, the the commutator of subgroups, the meet (= intersection) of ideals in boolean algebras, etc.

We say that the T-product is *finitary in* $A \in K$ if the T-product of two finitely generated ideals of A is finitely generated as an idela.

A *pure* commutator term in $\vec{y}, \vec{z}$ is a commutator term $t(\vec{x}, \vec{y}, \vec{z})$ in which $\vec{x}$ is empty. A set $T \subseteq T_0$ is a *base for* T_0 if for any $t \in T_0$ there are $p(\vec{x}, \vec{y}) \in P_0$ and

$t_1, \ldots, t_k \in T$ such that

$$K \models t = p(\vec{x}, t_1, \ldots, t_k).$$

2.1. If K has a finite base for T_0 composed of pure commutator terms, then the commutator is finitary in every algebra $A \in K$.

Proof. It is clearly enough to show that the commutator $[I, J]$ of two principal ideals $I = (a)$, $J = (b)$, is finitely generated. We show that

$$(*) \qquad [I, J] = \overline{\{t(a, \ldots, a, b, \ldots, b) \mid t \text{ a pure term of the base}\}}^{id}.$$

Let $\vec{\imath} \in I$, $\vec{\jmath} \in J$, $\vec{u} \in A$ and $z = t(\vec{u}, \vec{\imath}, \vec{\jmath})$ be an element of $[I, J]$. Then for some $t_1, \ldots, t_k$ in the base for T, and $p \in P_0$, we have

$$z = p(\vec{u}, t_1(\vec{\imath}^1, \vec{\jmath}^1), \ldots, t_k(\vec{\imath}^k, \vec{\jmath}^k)), \quad \text{where} \quad \vec{\imath}^r \subseteq \vec{\imath}, \quad \vec{\jmath}^r \subseteq \vec{\jmath}.$$

Therefore for some $\vec{u}_h^r, \vec{v}_h^r \in A$, ideal terms p_h, p'_h, we have:

$$t_r(\vec{\imath}^r, \vec{\jmath}^r) = t_r(\overrightarrow{p_h(\vec{u}_h^r, a, \ldots, a)}, \overrightarrow{p'_h(\vec{v}_h^r, b, \ldots, b)}).$$

But $t_r(\overrightarrow{p_h(\vec{x}, y_1^h, \ldots, y_{n_h}^h)}, \overrightarrow{p'_h(\vec{x}', z_1^h, \ldots, z_n^h)})$ is a commutator term in $\vec{y}$, $\vec{z}$, and can be expressed as a composition of terms in the base by an ideal term. Hence z belongs to the second member of $(*)$. The other inclusion is trivial.

Next, some notations. For principal ideals $(a), (b)$, we write $[a, b]$ instead of $[(a), (b)]$. Also, the theory of residuated grupoids (see [1]) can be applied to the structure $\langle i(A), [\ ,\ ]_T \rangle$. In particular we set

$$(I : J)_T = \bigvee (H \in i(A) \mid [H, J]_T \subseteq I),$$

for $I, J \in i(A)$; $(I : J)_T$ is called a *T-residual of* I; a *proper* T-residual if $J \nsubseteq I$. For $I \in i(A)$ we also define by induction:

$$I^{(1)} = I^1 = I;$$

$$I^{(n+1)} = [I^{(n)}, I^{(n)}]_T; \quad I^{n+1} = [I^n, I]_T.$$

I will be called *T-nilpotent* (resp. *T-solvable*) if for some n, $I^n = (0)$ (resp. $I^{(n)} = (0)$). We drop the decoration "T-" everywhere in case $T = T_0$.

3. T-prime ideals

Let K be an ideal determined equational variety. In this section T is a subset of T_0 *such that the following assumptions hold*:

(A 1) For every algebra $A \in K$, $I, J \in i(A)$, one has: $[I, J]_T \in i(A)$.

(A 2) T-product $[\ ,\ ]_T$ is distributive in both arguments over arbitrary joins in $i(A)$.

If $T = T_0$, then A 1, A 2 hold automatically, as a general consequence of commutator theory.

$P \in I(A)$ is *T-prime* (*c-prime* in case $T = T_0$) if: for all $I, J \in i(A)$, if $[I, J]_T \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

$Q \in i(A)$ is *T-semiprime* (*c-semiprime* if $T = T_0$) if: for all $I \in (A)$, if $[I, I]_T \subseteq \subseteq P$, then $I \subseteq P$.

A subset M of A is a *T, m-system* (resp. *T, n-system*) if: for all $a, b \in M$, $[a, b]_T \cap \cap M \neq \emptyset$, (resp. for all $a \in M$, $[a, a]_T \cap M \neq \emptyset$).

The proof of the following is more or less routine:

3.1. (i) For $P \in i(A)$, the following are equivalent:

1. P is T-prime;
2. for all $a, b \in A$, if $[a, b]_T \subseteq P$, then $a \in P$ or $b \in P$;
3. $A \setminus P$ is a T, m-system;
4. (In case $T = T_0$) $D(A/P) = (0)$,

(where for $B \in K$, $D(B) \equiv \bigcup_{x \neq 0} ((0) : (x))$, "zero divisors" of B).

(ii) For $Q \in i(A)$ the following are equivalent:

1. Q is T-semiprime;
2. For all $a \in A$, $[a, a]_T \subseteq Q$ implies $a \in Q$;
3. $A \setminus Q$ is a T, n-system;
4. A/Q has no non zero nilpotent ideals.

As to the existence of prime proper (i.e. $\neq A$) ideals, we quote the following:

3.2. Let S be a T, m-system, $I \in i(A)$, $I \cap S = \emptyset$. If P is maximal in $\{J \in i(A) \mid I \subseteq J$ and $J \cap S = \emptyset\}$, then P is T-prime.

Proof. Let $a, b \notin P$ and $[a, b]_T \subseteq P$. Take $s \in (P \vee (a)) \cap S$, $r \in (P \vee (b)) \cap S$. Then

$$[s, r]_T \subseteq [P \vee (a),\ P \vee (b)]_T = [P, P]_T \vee [(a), P]_T \vee [P, (b)]_T \vee [a, b]_T \subseteq P,$$

hence $[s, r]_T \cap S = \emptyset$, and S would not be a T, m-system.

3.3. Assume $[A, A]_T = A$; then any maximal proper ideal is T-prime.

The proof is similar to that of 3.2.

Observe that A is finitely generated as an ideal iff there is a finitely generated ideal F such that $[F, A] = A$. The condition is satisfied if there is a formal unit, i.e. an element $u \in A$ such that $[A, (u)] = A$. If A is a finitely generated ideal, then by Zorn's Lemma A has maximal proper ideals (if $A \neq (0)$) and they are prime ideals.

3.4. Assume that the commutator is associative in A. If X is an ideal, then any maximal proper residual of X is c-prime.

Proof. (See [7], propriété 2.1.).

The *T-prime radical* of $I \in i(A)$, denoted $\sqrt[T]{I}$ (or $\sqrt{I}$ in case $T = T_0$) is the intersection of all T-prime ideals containing I.

3.5. For $Q \in i(A)$ the following are equivalent:

i) Q is T-semiprime;
ii) Q is T-radical, i.e. $\sqrt[T]{Q} = Q$;
iii) Q is an intersection of T-prime ideals.

Proof. Any T-prime is T-semiprime and an intersection of T-semiprime ideals is T-semiprime, therefore if $Q = \sqrt[T]{Q}$ then Q is T-semiprime.

For the converse, first observe that if N is a T, n-system and $a \in N$, then there is a T, m-system $M \subseteq N$ with $a \in M$. In fact, define $a_0 = a$; given $a_i \in N$, choose $a_{i+1} \in [a_i, a_i]_T \cap N$; then let $M = \{a_i \mid i \in \omega\}$. Now assume that Q is T-semiprime and $a \notin Q$. Let $M \subseteq A \setminus Q$ be a T, m-system such that $a \in M$. By Zorn's Lemma we get a maximal P amongst the ideals J such that $Q \subseteq J$ and $J \cap M = \emptyset$. Then P is T-prime, $P \supseteq Q$ and $a \notin P$: hence $a \notin \sqrt[T]{Q}$.

3.6. For any $I \in i(A)$,

$\sqrt[T]{(I)} = \{a \in A \mid \text{for every } T, m\text{-system } S, \text{ if } a \in S \text{ then } S \cap I \neq \emptyset\}$ = the smallest T-semiprime ideal containing I = the intersection of all the minimal T-prime ideals containing I.

The proof follows from previous results or Zorn's Lemma.

Remarks. For rings, our notion of c-semiprime is the classical notion of semiprime (see e.g. [8]); hence our notion of prime radical is the classical notion.

In the case of groups-where of course ideal $\equiv$ normal subgroup this notion of c-prime does not appear in the literature up to our knowledge.

Let $R_N(G)$ be the (normal) subgroup of a group G generated by the family of all normal solvable subgroups-see also 3.12 below. The following remark, due to Antonio Pasini, suggests that the notion of c-prime normal subgroup is of some interest. Assume that G has a uniform finite bound for the length of chains of normal subgroups. Then if $G = G'$, proper c-prime normal subgroups are exactly maximal normal subgroups. If $G \neq G'$ and if moreover $R_N(G)$ and $G^{(\infty)} \equiv \bigcap_{i \in \omega} G^{(i)}$ together generate G as a subgroup, then H is c-prime in G iff $H \supseteq R_N(G)$ and $H \cap G^{(\infty)}$ is maximal in $G^{(\infty)}$.

3.7. If $[\ ,\ ]_T$ is finitary in A then for any $I \in i(A)$, we have:

$\sqrt[T]{(I)} = \{a \in A \mid (a)^{(n),T} \subseteq I \text{ for some integer } n\}$.

Proof. The inclusion '$\supseteq$' is trivial. For the converse, assume $(a)^{(n),T} \nsubseteq I$, for all n. Then the set

$$Z = \{J \in i(A) \mid I \subseteq J \text{ and } (a)^{(n),T} \nsubseteq J \text{ for any } n\}$$

is not empty. We can apply Zorn's Lemma, to get an ideal M maximal in $\boldsymbol{Z}$. Take $b \notin M$. Then for some n,

$$(a)^{(n),T} \subseteq M \vee (b) ;$$

consequently

$$(a)^{(n+1),T} \subseteq M \vee [b, b]_T .$$

Hence $M \vee [b, b]_T \nsubseteq \boldsymbol{Z}$, therefore $[b, b]_T \nsubseteq M$. This shows that M is T-semiprime, contains I, $a \notin M$; hence $a \notin \sqrt[T]{I}$.

The following corollaries are almost evident:

3.8. If $[\ ,\ ]_T$ is associative and finitary in A, then for $I \in i(A)$,

$$\sqrt{(I)} = \{a \mid (a)^{n,T} \subseteq I \text{ for some } n\} .$$

3.9. If the commutator is finitary in A, $I \in i(A)$, then $a \in \sqrt{I}$ iff there is a finite $F \subseteq I$ such that $a \in \sqrt{F^{id}}$.

(We will not push further here the natural topological interpretation of 3.9 in terms of the "spectrum" of A.)

The *T-prime radical* of A is $\sqrt[T]{(0)}$, denoted $R_T(A)$, and A is a *T-prime algebra* (0) is a T-prime ideal. Then by standard proof we get:

3.10. $R_T(A/R_T(A)) = (0)$.

3.11. A is a subdirect product of T-prime algebras iff $R_T(A) = (0)$.

3.12. If $[\ ,\ ]_T$ is finitary in A, then $R_T(A)$ is the ideal of A generated by all solvable ideals of A.

4. ACC on ideals

We will briefly consider finiteness conditions on ideals, and we will use the full force of the commutator to get a general form of a famous theorem by I. S. Cohen, stating that if all prime ideals of a commutative unitary ring are finitely generated, then the ring is Noetherian (see [2], [9]).

Let K an ideal determined equational class. We say that $A \in K$ *satisfies the a.c.c.* (*on ideals*) if any properly ascending chain of ideals in A is finite; as usual this is equivalent to either of the following: every ideal of A is finitely generated or else: every non empty set of ideals has maximal elements.

We will make use of the following notion of "unspecified product" of n ideals: any word on n letters in the free commutative grupoid gives rise in a natural way to an n-ary operation in $i(A)$. By $((I_1, \ldots, I_n))$ we will denote the result of one of these operation; if $I_1 = \ldots = I_n = X$, we will use the notation $((X))^n$. *In this section prime means c-prime.*

4.1. Assume that in A any unspecified product of finitely many prime ideals is finitely generated. Then for any ideal X of A there is a finite number of prime ideals

$P_1, \ldots, P_n$ and a product $((P_1, \ldots, P_n))$ such that:

$$((P_1, \ldots, P_n)) \subseteq X \subseteq \bigcap_{i=1}^{n} P_i\,.$$

Proof. By contradiction, assume that the set U of ideals not satisfying the property is not empty. If $(X_k \mid k \in K)$ is a chain in U, then $Y = \bigcup_{k \in K} X_k \in U$: if not, there should exist prime ideals $Q_1, \ldots, Q_n$ such that

$$((Q_1, \ldots, Q_n)) \subseteq X_k \subseteq \bigcap_{i=1}^{n} Q_i$$

for some $k \in K$. By Zorn's Lemma, there is G maximal in U . G is not prime, hence $[a, b] \subseteq G$ for some $a, b \notin G$. Then $X_1 = G \vee (a)$ and $X_2 = G \vee (b)$ are not in U. Hence

$$((P_1^i, \ldots, P_{n_i}^i)) \subseteq X_i \subseteq P_1^i \cap \ldots \cap P_{n_i}^i$$

for $i \in \{1, 2\}$, $P_1^i, \ldots, P_{n_i}^i$ prime ideals. Then

$$[((P_1, \ldots, P_n)), ((P^2, \ldots, P_{n_2}^2))] \subseteq [X_1, X_2] \subseteq G \subseteq P_1^1 \cap \ldots \cap P_{n_2}^2$$

which contradicts the fact that $G \in U$.

4.2. If A satisfies a.c.c. for ideals then for every $I \in i(A)$ there is a finite number of minimal prime ideals containing I.

This is a corollary of 4.1.

4.2. Can be strenghtened as follows:

4.3. Assume that A satisfies a.c.c. on T-semiprime ideals. Then any T-semiprime ideal H is the intersection of finitely many T-prime ideals, minimal over H.

We omit the proof, which is easily patterned after that of the case of rings, see e.g. [6].

4.4. Let A satisfy a.c.c. on ideals. For $I \in i(A)$, $\sqrt{I}$ is the largest ideal X such that $((X))^m \subseteq I$ for some positive integer m.

Proof. Let $P_1, \ldots, P_k$ be prime such that

$$((P_1, \ldots, P_k)) \subseteq I \subseteq \bigcap_{i=1}^{k} P_i\,.$$

If $((X))^m \subseteq I$ for some m then $X \subseteq P_i$ for every i. Since $\sqrt{(I)} = \bigcap_{i=1}^{k} P_i$, and $X \subseteq \sqrt{I}$, we get

$$((\bigcap_{i=1}^{k} P_i))^k \subseteq ((P_1, \ldots, P_k))\,.$$

— where in the left hand side we are applying to the k-tuple $R, \ldots, R$ (where $R = \bigcap_{i=1}^{k} P_i$) the operation giving $((P_1, \ldots, P_k))$ —. Therefore $\sqrt{I}$ has the required property.

Let us observe that the assumption of 4.1 is satisfied if the commutator is finitary and every prime ideal is finitely generated. We want now to explain how these two conditions do not fall short from the simple assumption of a.c.c..

The following property of the commutator, called the "term condition", will be repeatedly used below (for a proof see [5]):

4.5. Let $p(\vec{x}, \vec{y})$ be an ideal term in $\vec{y}$, let $I, J \in i(A)$ and assume that $\vec{c}J\vec{c}'$, $\vec{b}[I, J]\,\vec{b}'$, and $\vec{b} \in I$. Then
$$p(\vec{c}, \vec{b})\,[I, J]\, p(\vec{c}', \vec{b}')\,.$$

We now assume that σ is a set of ideal terms such that:

i) $0, d_1, \ldots, d_m, p_0$ (see proposition 1.2 above) are in σ;

ii) contains some base for ideal terms;

ii) is closed under composition, i.e. if $p(\vec{x}, \vec{y})$ and $p_1(\vec{x}^1, \vec{y}^1), \ldots, p_k(\vec{x}^k, \vec{y}^k)$ are in σ then also
$$p(\vec{x}, p_1, \ldots, p_k) \in \sigma$$
as an ideal term in $y^1 * \ldots * y^k$.

Such σ will be held fixed, and moreover we assume that σ is well ordered and that $o(\sigma)$ is the corresponding ordinal. For $\alpha < o(\sigma)$, we have a pair n_α, m_α of numbers such that the α-th element of σ is of the form
$$p_\alpha(x_1, \ldots, x_{n_\alpha}, y_1, \ldots, y_{m_\alpha})$$
as an ideal term in $\vec{y}$. Observe that for $\alpha = 0$ we already know p_0.

An *A-σ-complex* (or simply an *A-complex*) is a triple $(M, 0, g)$, denoted also by M, where M is a non empty set, $0 \in M$, and g is a mapping which associates to each $\alpha < o(\sigma)$ a mapping g_α from $A^{n_\alpha} \times M^{m_\alpha}$ into M, such that:

1. $g_\alpha(\vec{a}, \vec{0}) = 0$, for $\vec{a} \in A$;
2. g_α respects composition, i.e. if
$$p_\alpha = p_\beta(\vec{x}, p_{\alpha_1}(\vec{x}^1, \vec{y}^1), \ldots, p_{\alpha_k}(\vec{x}^k, \vec{y}^k))$$
belongs to σ, and $p_{\alpha_j} \in \sigma$ for $j = 1, \ldots, k$, then
$$g_\alpha(\vec{a}, \vec{a}^1, \ldots, \vec{a}^k, \vec{m}^1, \ldots, \vec{m}^k) =$$
$$= g_\beta(\vec{a}, g_{\alpha_1}(\vec{a}^1, \vec{m}^1), \ldots, g_{\alpha_k}(\vec{a}^k, \vec{m}^k))\,,$$
for all $\vec{a}, \vec{a}^1, \ldots, \vec{a}^k \in A$, $\vec{m}^1, \ldots,, \vec{m}^k \in M$.
3. If $n_\alpha = n_\beta$ and $m_\alpha = m_\beta$ and in A: $p_\alpha(\vec{x}, \vec{y}) = p_\beta(\vec{x}', \vec{y}')$ holds identically, then $g_\alpha = g_\beta$.

The basic example is of course that of modules over a ring A.

In general if I is an ideal (i.e. a σ-ideal of A) it is considered in a natural way as an A-σ-complex. If $I \in i(A)$, A/I also is naturally an A-complex, by defining

$$g_\alpha(\vec{a}, x_1 + I, \ldots, x_{m_\alpha} + I) = p_\alpha(\vec{a}, \vec{x}) + I\,.$$

If $I, J \in i(A)$ and $I \supseteq J$, then the set $I/J = \{a + J \mid a \in I\}$ is an A-σ-complex by defining:

$$g_\alpha(\vec{a}, x_1 + J, \ldots, x_{m_\alpha} + J) = p_\alpha(\vec{a}, \vec{x}) + J\,.$$

If $(M, 0, g)$ and $(N, 0', g')$ are A-complexes, $N \subseteq M$, $0 = 0'$ and $g'_\alpha(\vec{a}, \vec{n}) = g_\alpha(\vec{a}, \vec{n})$ for all $\vec{a} \in A$, $\vec{n} \in N$, $\alpha < o(\sigma)$, we say that N is an *A-σ-subcomplex of M*. In a natural way we get the notion of a A-σ-subcomplex of M being the *intersection of a family of A-subcomplexes of M*, and consequently the notion of the A-σ-subcomplex of M *generated by a subset H* of M, denoted by $\tilde{H}^M$. Then we easily see that

$$\tilde{H}^M = \{g_\alpha(\vec{a}, \vec{b}) \mid \vec{a} \in A,\ \vec{b} \in H,\ \alpha < o(\sigma)\}\,,$$

whereas 0, g are defined in the natural way.

4.6. Assume F, G, I are ideals of A, and $[F, G] \subseteq I \subseteq F$. Then $F/[F, G]$ becomes an A/I-complex by defining

$$g_\alpha(a_1 + I, \ldots, a_{n_\alpha} + I,\ b_1 + [F, G], \ldots, b_{m_\alpha} + [F, G]) = \\ = p_\alpha(a_1, \ldots, a_{n_\alpha}, b_1, \ldots, b_{m_\alpha}) + [F, G]\,.$$

If moreover F is finitely generated as an ideal, then $F/[F, G]$ is finitely generated as an A/I-complex.

For the proof, the only thing which requires some care is that g_α is well defined: this is assured by 4.5.

Assume F, G, I are as in the hypothesis of 4.6: we explicitely note the $I/[F, G]$ is an A/I-subcomplex of $F/[F, G]$.

4.7. Assume $X, Y \in i(A)$ and $Y \subseteq X$. If Y and X/Y are finitely generated as A-complexes, then X is finitely generated as an A-complex (i.e. as an ideal).

Proof. Assume that $\{u + J \mid u \in U\}$ generates I/J for some finite $U \subseteq I$, and that some finite $H \subseteq J$ generates J. If $x \in I$ then

$$x + J = g_\alpha(\vec{a}, i_1 + J, \ldots, i_{m_\alpha} + J) = p_\alpha(\vec{a}, \vec{\imath}) + J\,,$$

for some $\alpha < o(\sigma)$, $\vec{a} \in A$, $i_1, \ldots, i_{m_\alpha} \in U$. Put $y = p_\alpha(\vec{a}, \vec{\imath})$ and $z_i = d_i(x, y)$, $i = 1, \ldots, m$. Then $z_i \in J$. Therefore

$$z_i = g_{\alpha_i}(\vec{b}^i, h^i_1, \ldots, h^i_{r_i})$$

for some $\vec{b}^i \in A$, $\vec{h}^i \in H$. Now we have:

$$x = p_0(x, y, y, d_1(x, y), \ldots, d_m(x, y)) =$$
$$= p_0(x, y, g_\alpha(\vec{a}, \vec{\imath}), g_{\alpha_1}(\vec{b}^1, \vec{h}^1), \ldots, g_{\alpha_m}(\vec{b}^m, \vec{h}^m)),$$

hence x belongs to the ideal generated by $U \cup H$.

We will be concerned with the following property of quotient algebras A/I, where $I \in i(A)$:

(Ω) $\begin{cases}\text{For } F, G \text{ ideals of } A \text{ such that } [F, G] \subseteq I \subseteq F, \text{ if } F/[F, G] \text{ is finitely generated} \\ \text{as an } A/I\text{-}\sigma\text{-complex, then also all its } A/I\text{-subcomplexes are finitely generated.}\end{cases}$

The basic example is of course that of a ring A such that A/I is Noetherian.

4.8. Assume that the commutator is finitary in A, and that any quotient algebra A/I of A which satisfies a.c.c. on its ideals satisfies also property (Ω). If I is an ideal of A, maximal amongst the non finitely generated ideals of A, then I is c-prime.

Proof. In our hypothesis on I, A/I will satisfy a.c.c.; by absurd, let there be $a, b \notin I$ such that $[a, b] \subseteq I$. Let

$$F = I \vee (a), \quad G = I \vee (b).$$

Both F and F and also $[F, G]$ are finitely generated ideals of A, and, by a simple calculation, $[F, G] \subseteq I$. $F/[F, G]$ is a finitely generated A/I-σ-complex. By (Ω), $I/[F, G]$ is also a finitely generated A/I-complex. We now consider $I/[F, G]$ as an A-complex by defining

$$g_\alpha(a, i_1 + [F, G], \ldots, i_{m\alpha} + [F, G]) = p_\alpha(\vec{a}, \vec{\imath}) + [F, G],$$

for $\vec{a} \in A$, $\vec{\imath} = i_1, \ldots, i_{m_\alpha} \in I$; (this is a good definition by 4.5) and as such $I/[F, G]$ is finitely generated. We then conclude that I would be finitely generated: contradiction.

The following is now a simple corollary, by Zorn's Lemma:

4.9. If A satisfies the same assumption as in 4.8, and if every prime ideal of A is finitely generated, then A satisfies a.c.c..

As a final remark, we observe that most of the results in the present section should hold good for P-ideals, where the set P of ideal terms be suitably chosen; but we have preferred to avoid here this extreme generality which would have yielded some loss of perspicuity.

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