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Geometric Hyperquasigroups and Line Spaces

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Geometric hyperquasigroups are defined and investigated from the point of view of their relations to line spaces. Some general results on line spaces are proved showing the rôle the sizes of lines play.

Geometrické hyperkvazigrupy jsou definovány a vyšetřovány z hlediska jejich vztahů k přímkovým prostorům. Jsou dokázány některé obecné výsledky, ukazující jakou roli hrají velikosti přímek.

Определяются геометрические гиперквазигруппы и они изучаются с точки зрения их соотношения к пространствам прямых. Доказываются некоторые общие результаты, показывающие роль количества прямых.

Let $(G, \circ)$ be a *hypergroupoid*, i.e. a non-empty set G together with a binary composition law — hyperproduct — which is a mapping $\circ : G \times G \to P'(G)$, where $P'(G) = P(G) \setminus \emptyset$, $P(G)$ the power set of G. A hypergroupid $(G, \circ)$ is called a *hyperquasigroup* if

$$\forall a, b \in G\,, \quad \exists x \in G : a \circ x \ni b\,, \tag{1}$$

$$\forall a, b \in G\,, \quad \exists y \in G : y \circ a \ni b\,. \tag{2}$$

A *hypergroup* is a hyperquasigroup for which the associative law holds:

$$\forall a, b, c \in G\,, \quad (a \circ b) \circ c = a \circ (b \circ c)\,, \tag{3}$$

where

$$\begin{aligned}(a \circ b) \circ c &= \bigcup(x \circ c : x \in a \circ b)\,,\\ a \circ (b \circ c) &= \bigcup(a \circ x : x \in b \circ c)\,.\end{aligned} \tag{4}$$

Giving a hypergroupoid is equivalent to giving an incidence structure $\{x \circ y : : (x, y) \in G \times G\}$ in G, i.e. a family of non-empty subsets of G indexed by $G \times G$. Thus, the theory of hypergroupoids, in particular of hypergroups, is contained in that of incidence structures and can be viewed as an algebraization of the latter ones.

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Assume for the hypergroupoid $(G, \circ)$ the following hold:

(5) $\quad \forall a \in G, \quad a \circ a = a$ (idempotent law),

(6) $\quad \forall a, b \in G, \quad a, b \in a \circ b,$

(7) $\quad \forall a, b \in G, \quad \forall x, y \in a \circ b, \quad x \neq y \Rightarrow a \circ b \subseteq x \circ y.$

From (7) and (6) it follows that

(8) $\quad \forall a, b \in G, \quad \forall x, y \in a \circ b, \quad x \neq y \Rightarrow a \circ b = x \circ y.$

By (8), (5), (6), the commutative law, i.e.

(9) $$\forall a, b \in G, \quad a \circ b = b \circ a,$$

and the following weak associative law

(10) $\quad \forall a, b \in G, \quad \forall c \in a \circ b \Rightarrow a \circ (b \circ c) = (a \circ b) \circ c = a \circ b$

hold. Furthermore, (1) and (2) hold; therefore, $(G, \circ)$ is a hyperquasigroup. Next, consider the family $\{x \circ y : (x, y) \in G \times G \setminus \Delta_G\}$ of subsets of G and let $\mathscr{L}$ be the proper family associated with it. Thus, $(G, \mathscr{L})$ is a line space, since there is a unique element in $\mathscr{L}$ through any two distinct elements in G. Conversely, let $(G, \mathscr{L})$ be a line space; if a hyperproduct is defined on G by

$$x \circ y = \begin{cases} x \text{ when } x = y, \\ \text{line through } x \text{ and } y \text{ when } x \neq y, \end{cases}$$

then $(G, \circ)$ is a hypergroupoid for which (5), (6), and (7) hold. Therefore, the hypergroupoids for which (5), (6), and (7) hold will be called *geometric hyperquasigroups.* The subhyperquasigroups of such a hyperquasigroup $(G, \circ)$ form a closure system $\mathscr{C}$ coincident with the closure system of the subspaces in the line space associated with $(G, \circ)$. If $\mathscr{C}$ is combinatorial, i.e.

(10) $\quad \forall C \in \mathscr{C}, \quad \forall x \in G \setminus C \Rightarrow \overline{C \cup \{x\}}$ covers C,

then $(G, \circ)$ will be said to be combinatorial and the well known properties of matroids hold.

The associative law (3) for a geometric hyperquasigroup $(G, \circ)$ can be proved to be equivalent to Veblen axiom for the associated line space $(G, \mathscr{L})$; i.e. (3) is the algebraic translation of Veblen axiom. Thus,

(11) *The geometric hyperquasigroup $(G, \circ)$ is a hypergroup $\Leftrightarrow$ $(G, \mathscr{L})$ is a projective space.*

In what follows $(G, \circ)$ will always be a finite geometric hyperquasigroup and

$(G, \mathscr{L})$ the line space associated with it. Set $R = |\mathscr{L}|$, $k = |G|$,

$$m = \min_{(x,y) \in G \times G \setminus \Delta_G} |x \circ y| , \qquad n = \max_{(x,y) \in G \times G \setminus \Delta_G} |x \circ y| .$$

Obviously, both $m \geqq 2$ and $n < k$ will be assumed, otherwise $(G, \circ)$ is the trivial hypergroup.

The number t_s of lines containing s points each will be called the *index s character* of $(G, \mathscr{L})$ $(m \leqq s \leqq n)$. The characters $t_m, \ldots, t_n$ of $(G, \mathscr{L})$ satisfy thc following equalities:

$$\sum_{s=m}^{n} t_s = R , \quad \sum_{s=m}^{n} s(s-1)\, t_s = k(k-1) ; \tag{12}$$

the first equality (12) is obvious; the second one is obtained computing in two different ways the pairs consisting of a pair of distinct points in G and the line through them.

If $m = n$, i.e. any line in $(G, \mathscr{L})$ has size m, then $(G, \mathscr{L})$ is a Steiner system $S(2, m, k)$ and $(G, \circ)$ is a Steiner hyperquasigroup (a Steiner quasigroup when $m = 3$); in this case from (12)

$$R = k(k-1)/m(m-1)$$

follows. Furthermore, it is straightforward to prove that the number of lines through any point in G is $r = (k-1)/(m-1)$.

In the sequel the following will be assumed:

(i) $m < n$;

(ii) through any point in G there are exactly r lines of $\mathscr{L}$.

With such a $(G, \mathscr{L})$ the parameters (k, R, r, m, n) are associated.

Next, consider the number of pairs (x, l), where $x \in G$ and l is a line in $\mathscr{L}$ through x. This number can be computed in two ways; when lines are looked at, $N = \sum_{s=m}^{n} st_s$ is obtained; when points in G are considered, then $N = kr$. Thus, another equality for characters holds; namely,

$$\sum_{s=m}^{n} st_s = kr . \tag{13}$$

Consequently, by (12) and (13),

$$\begin{aligned} &\sum_{s=m}^{n} t_s = R , \\ &\sum_{s=m}^{n} st_s = kr , \\ &\sum_{s=m}^{n} s^2 t_s = k(k+r-1) . \end{aligned} \tag{14}$$

Let N and M be any two integers; from equalities (14) the following is obtained:

$$\sum_{s=m}^{n}(N-s)(s-M)\,t_s = -\sum_{s=m}^{n} s^2 t_s + (N+M)\sum_{s=m}^{n} st_s - NM\sum_{s=m}^{n} t_s =$$
$$= -k(k+r-1) + (N+M)\,kr - NMR =$$
$$= -(k^2 - k((N+M)\,r - r + 1) + NMR)\,.$$

Set

$$f(k, R, r; N, M) = k^2 - k((N+M)\,r - r + 1) + NMR\,; \tag{15}$$

then

$$\sum_{s=m}^{n}(N-s)(s-M)\,t_s = -f(k, R, r; N, M)\,. \tag{16}$$

$(G, \mathscr{L})$ will be said to be an h *character space* if precisely h of its characters are different from zero. If the indices of the non-zero characters are $s_1 = m, s_2, \ldots, s_{h-1}, s_h = n$, $s_1 < s_2 < \ldots < s_h$, $(G, \mathscr{L})$ will be said to be of type $(s_1, \ldots, s_h)$. Line spaces $(G, \mathscr{L})$ may be investigated looking at the number of their non-zero characters and the difficulties increase together with this number. The one character line spaces are Steiner systems as already pointed out.

Next, two character line spaces of type (m, n) will be considered. Setting $N = n$, $M = m$ in (16), since $t_{m+1} = \ldots = t_{n-1} = 0$,

$$f(k, R, r; n, m) = k^2 - k((n+m)\,r - r + 1) + mnR = 0\,, \tag{17}$$

and the following result is proved.

I. *The parameters* (k, R, r, m, n) *of a two character line space* $(G, \mathscr{L})$ *satisfy equation* (17); *therefore,*

$$\Delta = ((n+m)\,r - r + 1)^2 - 4nmR$$

must be a square integer.

Denote by u_j the number of lines through a point x in G having j points, $j = m, n$; then $u_n + u_m = r$, $(n-1)\,u_n + (m-1)\,u_m = k - 1$, so that

$$u_m = \frac{r(n-1) - k + 1}{n - m}\,, \quad u_n = \frac{k - 1 - r(m-1)}{n - m}\,; \tag{18}$$

consequently,

$$r(m-1) + 1 < k < r(n-1) + 1\,. \tag{19}$$

II. *If* $(G, \mathscr{L})$ *is of type* (m, n), *then through any point in* G *exactly* u_j *lines with* j *points pass,* $j = m, n$; u_m *and* u_n *are given by* (18), *thus the right hand sides must be integers, i.e.* $n - m$ *divides both* $r(n-1) - k + 1$ *and* $k - 1 - r(m-1)$. *Furthermore,* (19) *holds.*

In the general case, i.e. $(G, \mathscr{L})$ is an h character line space with parameters (k, R, r, m, n), substituting n and m for N and M in (16),

$$\sum_{s=m+1}^{n-1} (n - s)(s - m)\, t_s = -f(k, R, r; n, m) \tag{20}$$

is obtained. Since the left hand side of (20) is $\geqq 0$, and equal to zero iff $(G, \mathscr{L})$ has just two characters,

$$k^2 - k((k + m)r - r + 1) + mnR \leqq 0\,, \quad \text{and} \tag{21}$$

$$k^2 - k((n + m)r - r + 1) + nmR = 0 \Leftrightarrow (G, \mathscr{L}) \quad \text{is a two character space.}$$

Therefore, equation (17) must have (positive) real roots, i.e. its discriminant must be non-negative:

$$\Delta = ((n + m)r - r + 1)^2 - 4nmR \geqq 0\,. \tag{22}$$

Furthermore, denoting by k_1, k_2 these roots $(k_1 \leqq k_2)$,

$$k_1 \leqq k \leqq k_2\,, \tag{23}$$

the equalities holding iff $(G, \mathscr{L})$ is a two character space.

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