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Some Examples of Bol Loops

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In this paper we give eight constructions of Bol loops of order $4n$, with two generators, consequently not Moufang. It is proved that these loops are non-isomorphic. All the loops are found to satisfy both Lagrange's theorem and Sylow's main theorem. We have therefore been able to provide more non-isomorphic Bol loops of small orders.

V tomto článku uvádíme osm konstrukcí Bolových lup řádu $4n$, které mají dva generátory a nejsou tedy Moufangovými lupami. Je dokázáno, že tyto lupy jsou neizomorfní. Je zjištěno, že všechny tyto lupy splňují jak Lagrangeovu, tak hlavní Sylowovu větu. Byli jsme tudíž schopni získat více neizomorfních Bolových lup malých řádů.

В этой статье мы показываем восемь построений луп Бола порядка $4n$, которые порождены двумя элементами и следовательно на являются лупами Муфанг. Показано, что эти лупы не изоморфны и что они удовлетворяат теоремам Лагранжа и Силова. Итак, мы получили более неизоморфных луп Бола малых порядков.

I. Introduction

A loop $G(\circ)$ is a Bol loop if and only if

$$(1) \qquad (xy \circ z)\, y = x(yz \circ y)$$

for all $x, y, z \in G$. Bol loops first appeared in a work of Bol [1] on geometry. Further studies have been carried out on this class of loops including algebraic properties, construction and classification; Robinson [8], Burn [4], Solarin and Sharma [9, 10, 11], Sharma [13].

In this study, we give eight constructions of Bol loops of order $4n$, where n is a positive integer, each gives a Bol loop of order 16, with at least one element of order 8. One of these constructions gives the only Bol loop or order 12 with two generators and also a Bol loop of order 24 (See T_1). Also, we show that the loops from these construction are non-isomorphic.

In section II, we shall state all the constructions, and for conservation of space, we shall only prove one to satisfy the Bol identity. In section III we present the dis-

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cussion on isomorphism and establish that the eight constructions are non-isomorphic. We shall refer to the six Bol loops of order 8 given by Burn [4] as $\Pi_i(8)$ $i = 1, 2, 3, 4, 5, 6$ according to Burn's numbering, without stating them. Similarly, we shall refer to the Bol loops of order 16 in our earlier paper [12] as Π_1 to Π_{21}. Bruck [3] defined the left nucleus N_λ, the middle nucleus N_μ, and the right nucleus N_ϱ of a loop $(L, \circ)$ as follows:

$$N_\lambda = \{x \in L \mid x \circ yz = xy \circ z \ \forall y, z \in L\}$$
$$N_\mu = \{y \in L \mid x \circ yz = xy \circ z \ \forall x, z \in L\}$$
$$N_\varrho = \{z \in L \mid x \circ yz = xy \circ z \ \forall x, y \in L\}$$

II.

Construction 1. Let $G(\circ) = C_{2n} \times C_2$ and the binary operation is defined as follows:

$$\begin{aligned}(x^{\beta_1}, a^{\alpha_1})(x^{\beta_2}, e) &= (x^{\beta_1+\beta_2}, a^{\alpha_1})\\ (x^{\beta_1}, a^{\alpha_1})(x^{\beta_2}, a) &= (x^{\beta_1+\beta_2}, a^{\alpha_1+1}) \quad \text{if} \quad \beta_2 \equiv 0 \pmod 2\\ &= (x^{5\beta_1+\beta_2}, a^{\alpha_1+1}) \quad \text{if} \quad \beta_2 \equiv 1 \pmod 2\end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$, where $n = 3, 4, 6, 12$.

Proof. We first show that $G(\circ)$ satisfies (1).

(a) $A = (x^{\beta_1}, e)$; $B = (x^{\beta_2}, e)$; $C = (x^{\beta_3}, e)$ then

$$\begin{aligned}(AB \circ C)\, B &= (x^{\beta_1+2\beta_2+\beta_3}, e)\\ A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, e)\,.\end{aligned}$$

(b) $A = (x^{\beta_1}, e)$; $B = (x^{\beta_2}, e)$; $C = (x^{\beta_3}, a)$ then

$$\begin{aligned}(AB \circ C)\, B &= (x^{\beta_1+2\beta_2+\beta_3}, a) \quad \text{if} \quad \beta_3 = \text{even}\\ &= (x^{5\beta_1+6\beta_2+\beta_3}, a) \quad \text{if} \quad \beta_3 = \text{odd}\\ A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, a) \quad \text{if} \quad \beta_3 = \text{even}\\ &= (x^{5\beta_1+6\beta_2+\beta_3}, a) \quad \text{if} \quad \beta_3 = \text{odd}\end{aligned}$$

(c) $A = (x^{\beta_1}, e)$; $B = (x^{\beta_2}, a)$; $C = (x^{\beta_3}, e)$ then

$$\begin{aligned}(AB \circ C)\, B &= (x^{\beta_1+2\beta_2+\beta_3}, e) \quad \text{if} \quad \beta_2 = \text{even}\\ &= (x^{25\beta_1+6\beta_2+5\beta_3}, e) \quad \text{if} \quad \beta_2 = \text{odd}\\ A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, e) \quad \text{if} \quad \beta_2 = \text{even}\\ &= (x^{\beta_1+6\beta_2+5\beta_2}, e) \quad \text{if} \quad \beta_2 = \text{odd}\,.\end{aligned}$$

(d) $A = (x^{\beta_1}, e);\ B = (x^{\beta_2}, a);\ C = (x^{\beta_3}, a)$

$$\begin{aligned}
(AB \circ C)\,B &= (x^{\beta_1+2\beta_2+\beta_3}, a) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{even}\\
&= (x^{25\beta_1+6\beta_2+5\beta_3}, a) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{even}\\
&= (x^{5\beta_1+6\beta_2+\beta_3}, a) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{odd}\\
&= (x^{125\beta_1+26\beta_2+5\beta_3}, a) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{odd}.\\
A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, a) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{even}\\
&= (x^{\beta_1+6\beta_2+5\beta_3}, a) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{even}\\
&= (x^{5\beta_1+6\beta_2+\beta_3}, a) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{odd}\\
&= (x^{5\beta_1+26\beta_2+5\beta_3}, a) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{odd}.
\end{aligned}$$

(e) $A = (x^{\beta_1}, a);\ B = (x^{\beta_2}, e);\ C = (x^{\beta_3}, e)$

$$\begin{aligned}
(AB \circ C)\,B &= (x^{\beta_1+2\beta_2+\beta_3}, a)\\
A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, a).
\end{aligned}$$

(f) $A = (x^{\beta_1}, a);\ B = (x^{\beta_2}, e);\ C = (x^{\beta_3}, a)$

$$\begin{aligned}
(AB \circ C)\,B &= (x^{\beta_1+2\beta_2+\beta_3}, e) && \text{if } \beta_3 = \text{even}\\
&= (x^{5\beta_1+6\beta_2+\beta_3}, e) && \text{if } \beta_3 = \text{odd}\\
A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, e) && \text{if } \beta_3 = \text{even}\\
&= (x^{5\beta_1+6\beta_2+\beta_3}, e) && \text{if } \beta_3 = \text{odd}.
\end{aligned}$$

(g) $A = (x^{\beta_1}, a);\ B = (x^{\beta_2}, a);\ C = (x^{\beta_3}, e)$

$$\begin{aligned}
(AB \circ C)\,B &= (x^{\beta_1+2\beta_2+\beta_3}, a) && \text{if } \beta_2 = \text{even}\\
&= (x^{25\beta_1+6\beta_2+5\beta_3}, a) && \text{if } \beta_2 = \text{odd}\\
A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, a) && \text{if } \beta_2 = \text{even}\\
&= (x^{\beta_1+6\beta_2+5\beta_3}, a) && \text{if } \beta_2 = \text{odd}.
\end{aligned}$$

(h) $A = (x^{\beta_1}, a);\ B = (x^{\beta_2}, a);\ C = (x^{\beta_3}, a)$

$$\begin{aligned}
(AB \circ C)\,B &= (x^{\beta_1+2\beta_2+\beta_3}, e) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{even}\\
&= (x^{5\beta_1+6\beta_2+\beta_3}, e) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{odd}\\
&= (x^{25\beta_1+6\beta_2+5\beta_3}, e) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{even}\\
&= (x^{125\beta_1+26\beta_2+5\beta_3}, e) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{odd}.\\
A(BC \circ B) &= (x^{\beta_1+2\beta_2+\beta_3}, e) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{even}\\
&= (x^{5\beta_1+6\beta_2+\beta_3}, e) && \text{if } \beta_2 = \text{even}, && \beta_3 = \text{odd}\\
&= (x^{\beta_1+6\beta_2+5\beta_3}, e) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{even}\\
&= (x^{5\beta_1+26\beta_2+5\beta_3}, e) && \text{if } \beta_2 = \text{odd}, && \beta_3 = \text{odd}.
\end{aligned}$$

Since $(AB \circ C)\, B = A(BC \circ B)$ is all cases whenever $25 \equiv 1 \bmod 2n$, that is $n = 2, 3, 4, 6, 12$.

Also (e, e) is the two sided identity.

Moreover, if

$$\begin{aligned} A &= (x^{\beta}, e)\,, \quad \text{then} \quad A^{-1} = (x^{-\beta}, e) \\ \text{if} \quad A &= (x^{\beta}, a)\,, \quad \text{then} \\ A^{-1} &= (x^{-\beta}, a) \quad \text{if} \quad \beta = \text{even} \\ &= (x^{-5\beta}, a) \quad \text{if} \quad \beta = \text{odd} \end{aligned}$$

therefore the inverses are defined.

Also for non-associativity, let $A = (x^{\beta_1}, e)$; $B = (x^{\beta_2}, e)$; $C = (x^{\beta_3}, a)$ where β_2 and β_3 are odd integers, then

$$AB \circ C = (x^{5\beta_1 + 5\beta_2 + \beta_3}, a)$$

and

$$A \circ BC = (x^{\beta_1 + 5\beta_2 + \beta_3}, a)$$

$$AB \circ C \neq A \circ BC \quad \text{whenever} \quad 4 \not\equiv 0 \pmod{2n}$$

thus the construction is non-associative except when $n = 2$ which gives the group $C_4 \times C_2$.

Hence it is a Bol loop of order $4n$, $n = 3, 4, 6, 12$.

Construction 1, gives the group $C_4 \times C_2$ when $n = 2$; the Bol loop of order 12 with two generators, when $n = 3$; Π_{14} when $n = 4$; T_1 when $n = 6$.

Construction 2. Let $G(\circ) = C_{2n} \times C_2$ and the binary operation is defined as follows:

$$\begin{aligned} (x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, e) &= (x^{\beta_1 + \beta_2}, a^{\alpha_1}) \\ (x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, a^{\alpha_2}) &= (x^{\beta_1 + \beta_2}, a^{\alpha_1 + \alpha_2}) \quad \text{if } \beta_2 \text{ is even} \\ &= (x^{\beta_2 - \beta_1}, a^{\alpha_1 + \alpha_2}) \quad \text{if } \beta_2 \text{ is odd} \end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$, where n is any positive integer greater than or equal to two. G is Π_5 (8), when $n = 2$; the Bol loop of order 12 with two generators, when $n = 3$; Π_5, when $n = 4$.

Construction 3. Let $G(\circ) = C_{2n} \times C_2$ and the binary operation is defined as follows:

$$\begin{aligned} (x^{\beta_1}, e) \circ (x^{\beta_2}, e) &= (x^{\beta_1 + \beta_2}, e) \\ (x^{\beta_1}, a) \circ (x^{\beta_2}, e) &= (x^{\beta_1 + \beta_2}, a) \\ (x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, a) &= (x^{\beta_2 - \beta_1 + n\alpha_1}, a^{\alpha_1 + 1}) \quad \text{if} \quad \beta_2 = \text{odd} \\ &= (x^{3\beta_1 + \beta_2 + n\alpha_1}, a^{\alpha_1 + 1}) \quad \text{if} \quad \beta_2 = \text{even} \end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$, where n is any positive even integer greater than two. G is D_4 when $n = 2$; Π_1 when $n = 4$.

Construction 4. Let $G(\circ) = C_{2n} \times C_2$ and the binary operation is defined as follows:

$$\begin{aligned}
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, e) &= (x^{\beta_1+\beta_2}, a^{\alpha_1}) \\
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, a^{\alpha_2}) &= (x^{\beta_1+\beta_2}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{\beta_2-\beta_1+n\alpha_1}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 1 \pmod 4 \\
&= (x^{3\beta_1+\beta_2}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 3 \pmod 4
\end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$ where n is any positive even integer. G is Π_5 (8) when $n = 2$; Π_3 when $n = 4$.

Construction 5. Let $G(\circ) = C_{2n} \times C_2$ and the binary operation is defined as follows:

$$\begin{aligned}
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, e) &= (x^{\beta_1+\beta_2}, a^{\alpha_1}) \\
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, a^{\alpha_2}) &= (x^{\beta_1+\beta_2}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_1 \equiv 0 \pmod 2, \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{\beta_1+\beta_2+n\alpha_2}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_1 \equiv 1 \pmod 2, \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{\beta_2-\beta_1}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 1 \pmod 2
\end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$, where n is any positive even integer. G is Π_5 (8) when $n = 2$; Π_7, when $n = 4$.

Construction 6. Let $G(\circ) = C_{2n} \times C_2$ and the binary operation is defined as follows:

$$\begin{aligned}
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, e) &= (x^{\beta_1+\beta_2}, a^{\alpha_1}) \\
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, a^{\alpha_2}) &= (x^{3\beta_1+\beta_2+n\alpha_2}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{5\beta_1+\beta_2}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 1 \pmod 2
\end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$ where n is any positive even integer. G is Π_2 (8) when $n = 2$; Π_{12} when $n = 4$.

Construction 7. Let $G(\circ) = C_{2n} \times C_2$ nad the binary operation is defined as follows:

$$\begin{aligned}
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, e) &= (x^{\beta_1+\beta_2}, a^{\alpha_1}) \\
(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, a^{\alpha_2}) &= (x^{\beta_1+\beta_2+n\alpha_1}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_2 \equiv 1 \pmod 2 \\
&= (x^{3\beta_1+\beta_2+n\alpha_1}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_1 \equiv 0 \pmod 2, \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{3\beta_1-\beta_2+n\alpha_1}, a^{\alpha_1+\alpha_2}) && \text{if} \quad \beta_1 \equiv 1 \pmod 2, \quad \beta_2 \equiv 0 \pmod 2
\end{aligned}$$

then $G(\circ)$ is a Bol loop of order $4n$, where n is any positive even integer. G is Π_2 (8) when $n = 2$; Π_{13} when $n = 4$.

Construction 8. Let $G(\circ) = C_{2n} \times C_2$, and the binary operation is defined as follows:

$$(x^{\beta_1}, a^{\alpha_1}) \circ (x^{\beta_2}, e) = (x^{\beta_1+\beta_2}, a^{\alpha_1})$$

$$
\begin{aligned}
(x^{\beta_1}, e) \circ (x^{\beta_2}, a) &= (x^{3\beta_1+\beta_2}, a) \quad \text{if} \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{5\beta_1+\beta_2}, a) \quad \text{if} \quad \beta_2 \equiv 0 \pmod 2 \\
(x^{\beta_1}, a) \circ (x^{\beta_2}, a) &= (x^{3\beta_1-\beta_2}, e) \quad \text{if} \quad \beta_2 \equiv 0 \pmod 2 \\
&= (x^{\beta_1+5\beta_2}, e) \quad \text{if} \quad \beta_1 \equiv 0 \pmod 2, \quad \beta_2 \equiv 1 \pmod 2 \\
&= (x^{\beta_1+\beta_2}, e) \quad \text{if} \quad \beta_1 \equiv 1 \pmod 2, \quad \beta_2 \equiv 1 \pmod 2
\end{aligned}
$$

then $G(\circ)$ is a Bol loop of order $4n$, where n is any positive even integer. G is Π_2 (8) when $n = 2$; Π_{16} when $n = 4$.

III. Discussion on Isomorphism

Our approach in this section shall be to consider the order of the elements in the constructions to distinguish non-isomorphic loops. Two loops shall be consider non-isomorphic if they contain different number of elements of the same order. Whenever, two loops contain the same number of elements we shall go further to consider the order of elements in their nuclei. If these conoide both cases, we shall consider commutative patterns of both loops.

Theorem 1. All the eight constructions stated above are non-isomorphic.

Proof:

(a) let us consider the number of elements of order 2 in each construction.

(i) construction 1

elements of order 2 are given by

(1) (x^n, e)

(2) $$\left.\begin{aligned}(x^\beta, a) . (y^\beta, a) &= (x^{2\beta}, e) = (e, e) \quad \text{if} \quad \beta \equiv 0 \pmod 2 \\ &= (x^{6\beta}, e) = (e, e) \quad \text{if} \quad \beta \equiv 1 \pmod 2\end{aligned}\right\} \text{(A)}$$

the only possible solutions (A) are (x^n, a), (e, a), therefore there are three elements of order 2 in construction 1.

(ii) Construction 2

elements of order 2 are given by

(1) (x^n, e)

(2) $$\left.\begin{aligned}(x^\beta, a)(x^\beta, a) = (x^{2\beta}, e) &= (e, e) \quad \text{if} \quad \beta \equiv 0 \pmod 2 \\ &= (e, e) \quad \text{if} \quad \beta \equiv 1 \pmod 2\end{aligned}\right\} \text{(B)}$$

therefore the elements of order 2 are (x^n, e), (e, a), (x^n, a) and $(x^\beta, a)\ \forall \beta \equiv 1 \pmod 2$, hence there are $n + 3$ elements of order 2 in contruction 2.

Following the above procedure we obtain

(iii) construction 3, one element of order 2
(iv) construction 4, 3 elements of order 2
(v) construction 5, $n + 3$ elements of order 2
(vi) construction 6, 1 element of order 2
(vii) construction 7, 1 element of order 2
(viii) construction 8, 3 elements of order 2

from above isomorphism is only possible in the following sets $\{(i), (iv), (viii)\}$; $\{(iii), (vi), (vii)\}$, $\{(ii), (v)\}$ (b) consider elements of order 4.

(ix) construction 1*), 4 and 8

For constructions 1 and 8

(1) $$(x^{n/2}, e)\,; \quad (x^{3n/2}, e)$$

(2) $$(x^{\beta}, a) \circ (x^{\beta}, a) = (x^{2\beta}, e) = (x^{n}, e)\,; \quad \beta = n/2\,, \quad \tfrac{3}{2}n$$

hence constructions 1 and 8 contain 4 elements of order 4.

For construction 4

(1) $$(x^{n/2}, e)\,; \quad x(x^{3n/2}, e)$$

(2) if $\beta \equiv 1 \pmod 4$ then $(x^{\beta}, a) \circ (x^{\beta}. a) = (x^{n}, e)$ implies $\beta = 4k + 1$, $k = 0, 1, \ldots$ $\ldots, (n/4) - 1$, given $n/4$ elements of order 4.

If $\beta \equiv 3 \pmod 4$, $(x^{\beta}, a) \circ (x^{\beta}, a) = (x^{4\beta}, e) = (x^{n}, e)$ implies $\beta = 4k + 3$, $k = 0, 1, \ldots, (n/4) - 1$; given another $n/4$ elements of order 4. There exist $(n/2) + 4$ elements of order 4 in construction 4.

Hence construction 4 is not isomorphic to either of constructions 1 and 8.

To show that constructions 1 and 8 are non-isomorphic, we consider the commutative pattern of the elements.

Construction 1:

$$\begin{aligned}
(x^{n/2}, a) \circ (x^{\beta}, e) &= (x^{n/2+\beta}, a) \\
(x^{\beta}, e) \circ (x^{n/2}, a) &= (x^{n/2+\beta}, a) \\
(x^{n/2}, a) \circ (x^{\beta}, a) &= (x^{n/2+\beta}, e) \quad \text{if} \quad \beta \equiv 0 \pmod 2 \\
(x^{\beta}, a) \circ (x^{n/2}, a) &= (x^{n/2+\beta}, e) \quad \text{for all } \beta \\
(x^{n/2}, a) \circ (x^{\beta}, a) &= (x^{5n/2+\beta}, e) \\
&= (x^{n/2+\beta}, e) \quad \text{if} \quad \beta \equiv 1 \pmod 2
\end{aligned}$$

hence $(x^{n/2}, a)$ commutes with all elements.

*) Construction 8 is valid only when n is an even integer, therefore n is considered as even integer in construction 1, for isomorphism.

Construction 8:

$$(x^{n/2}, a) \circ (x^{\beta}, e) = (x^{n/2+\beta}, a) \quad \text{if} \quad \beta \equiv 1 \pmod 2$$
$$(x^{\beta}, e) \circ (x^{n/2}, a) = (x^{3\beta+n/2}, a)$$

hence $(x^{n/2}, a)$ does not commute with (x^{β}, e).

Consequently construction 1 is not isomorphic to construction 8.

(x) Constructions 3, 6 and 7

consider element (x^{β}, a); $(x^{\beta}, a) \circ (x^{\beta}, a) = (x^{4\beta+n}, e)$

in all the 3 constructions whenever β is $\beta \equiv 0 \pmod 2$ Moreover,

In construction 3, whenever β is odd $(x^{\beta}, a)(x^{\beta}. a) = (x^{n}, e)$ implies (x^{β}, a) is of order 4 for $\beta \equiv 1 \pmod 2$.

But, in constructions 6 and 7 (x^{β}, a) is of order $2n$ whenever β is odd, therefore constructions 3 is not isomorphic to either of constructions 6 or 7. Consideration of commutative pattern as above shows that constructions 6 and 7 are non-isomorphic.

(xi) Constructions 2 and 5

In the case of constructions 2 and following early approaches show that the order of elements are the same. Also three elements of order 2 commute with all elements in construction 2, while only one element of order 2 commutes with all elements in construction 5, therefore the two constructions are non-isomorphic. This completes the proof of the theorem.

3.

In this section we give the representations of T_1 and state some of its properties, $N_{\lambda}, N_{\mu}, N_{\varrho}$.

T_1

$R(1) = (1)$

$R(2) = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12)\ (13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 23\ 24)$

$R(3) = (1\ 3\ 5\ 7\ 9\ 11)\ (2\ 4\ 6\ 8\ 10\ 12)\ (13\ 15\ 17\ 19\ 21\ 23)\ (14\ 16\ 18\ 20\ 22\ 24)$

$R(4) = (1\ 4\ 7\ 10)\ (2\ 5\ 8\ 11)\ (3\ 6\ 9\ 12)\ (13\ 16\ 19\ 22)\ (14\ 17\ 20\ 23)\ (15\ 18\ 21\ 24)$

$R(5) = (1\ 5\ 9)\ (2\ 6\ 10)\ (3\ 7\ 11)\ (4\ 8\ 12)\ (13\ 17\ 21)\ (14\ 18\ 22)\ (15\ 19\ 23)$
$(16\ 20\ 24)$

$R(6) = (1\ 6\ 11\ 4\ 9\ 2\ 7\ 12\ 10\ 3\ 8)\ (13\ 18\ 23\ 16\ 21\ 14\ 19\ 24\ 17\ 22\ 15\ 20)$

$R(7) = (1\ 7)\ (2\ 8)\ (3\ 9)\ (4\ 10)\ (5\ 11)\ (6\ 12)\ (13\ 19)\ (14\ 20)\ (15\ 21)\ (16\ 22)$
$(17\ 23)\ (18\ 24)$

$R(8) = (1\ 8\ 3\ 10\ 5\ 12\ 7\ 2\ 9\ 4\ 11\ 6)\ (13\ 20\ 15\ 22\ 17\ 24\ 19\ 14\ 16\ 23\ 18)$

$R(9) = (1\ 9\ 5)\ (2\ 10\ 6)\ (3\ 11\ 7)\ (4\ 12\ 8)\ (13\ 21\ 17)\ (14\ 22\ 18)\ (15\ 23\ 19)$
$(16\ 24\ 20)$

$R(10) = (1\ 10\ 7)\ (2\ 11\ 8\ 5)\ (3\ 12\ 9\ 6)\ (13\ 22\ 19\ 16)\ (14\ 23\ 20\ 17)\ (15\ 24\ 21\ 18)$

$R(11) = (1\ 11\ 9\ 7\ 5\ 3)\ (2\ 12\ 10\ 8\ 6\ 4)\ (13\ 23\ 21\ 19\ 17\ 15)\ (14\ 24\ 22\ 20\ 18\ 16)$

$R(12) = (1\ 12\ 10\ 11\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2)\ (13\ 24\ 23\ 22\ 21\ 20\ 19\ 18\ 17\ 16\ 15\ 14)$

$R(13) = (1\ 13)\ (2\ 14)\ (3\ 15)\ (4\ 16)\ (5\ 17)\ (6\ 18)\ (7\ 19)\ (8\ 20)\ (9\ 21)\ (10\ 22)\ (11\ 23)$
$(12\ 24)$

$R(14) = (1\ 14\ 7\ 20)\ (2\ 19\ 8\ 13)\ (3\ 24\ 9\ 18)\ (4\ 17\ 10\ 23)\ (5\ 22\ 11\ 16)\ (6\ 15\ 12\ 21)$

$R(15) = (1\ 15\ 5\ 19\ 9\ 23)\ (2\ 16\ 6\ 20\ 10\ 24)\ (3\ 17\ 7\ 21\ 11\ 13)\ (4\ 18\ 8\ 22\ 12\ 14)$

$R(16 = (1\ 16\ 7\ 22)\ (2\ 21\ 8\ 15)\ (3\ 14\ 9\ 20)\ (4\ 19\ 10\ 13)\ (5\ 24\ 11\ 18)\ (6\ 17\ 12\ 23)$

$R(17) = (1\ 17\ 9\ 13\ 5\ 21)\ (2\ 18\ 10\ 14\ 6\ 22)\ (3\ 19\ 11\ 15\ 7\ 23)\ (4\ 20\ 12\ 16\ 8\ 24)$

$R(18) = (1\ 18\ 7\ 24)\ (2\ 23\ 8\ 17)\ (3\ 16\ 9\ 22)\ (4\ 21\ 10\ 15)\ (5\ 14\ 11\ 20)\ (6\ 19\ 12\ 13)$

$R(19) = (1\ 19)\ (2\ 20)\ (3\ 21)\ (4\ 22)\ (5\ 23)\ (6\ 24)\ (7\ 13)\ (8\ 14)\ (9\ 15)\ (10\ 16)$
$(11\ 17)\ (12\ 18)$

$R(20) = (1\ 20\ 7\ 14)\ (2\ 13\ 8\ 19)\ (3\ 18\ 9\ 24)\ (4\ 23\ 10\ 17)\ (5\ 16\ 11\ 22)\ (6\ 21\ 12\ 15)$

$R(21) = (1\ 21\ 5\ 13\ 9\ 17)\ (2\ 22\ 6\ 14\ 10\ 18)\ (3\ 23\ 7\ 15\ 11\ 19)\ (4\ 24\ 8\ 16\ 12\ 20)$

$R(22) = (1\ 22\ 7\ 16)\ (2\ 15\ 8\ 21)\ (3\ 20\ 9\ 14)\ (4\ 13\ 10\ 19)\ (5\ 18\ 11\ 24)\ (6\ 23\ 12\ 17)$

$R(23) = (1\ 23\ 9\ 19\ 5\ 15)\ (2\ 24\ 10\ 20\ 6\ 16)\ (3\ 11\ 11\ 21\ 7\ 17)\ (4\ 14\ 12\ 8\ 18)$

$R(24) = (1\ 24\ 7\ 18)\ (2\ 17\ 8\ 23)\ (3\ 22\ 9\ 16)\ (4\ 15\ 10\ 21)\ (5\ 20\ 11\ 14)\ (6\ 13\ 12\ 19).$

$N_\lambda = \{1, 4, 7, 10, 13, 16, 19, 22\}$

$N_\mu = N_\varrho = \{1, 3, 5, 7, 9, 11\}$.

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