B. L. Sharma
Classification of Bol loops of order 18

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 25 (1984), No. 1, 37--44

Persistent URL: http://dml.cz/dmlcz/142527

Classification of Bol Loops of Order 18

B. L. SHARMA
Department of Mathematics, University of Ife.*)

Received 30 March 1983

In this paper we classify Bol loops of order 18 and prove that (up to isomorphism) there are exactly two (non-associative) Bol loops of order 18.

V tomto článku je dána klasifikace Bolových lup řádu 18. Dokazujeme, že (až na izomorfizmus) existují právě dvě (neasociativní) Bolovy lupy řádu 18.

В этой статье мы даем классификацию луп Бола порядка 18 и показываем, что (с точностью до изоморфизма) существуют точно две (неассоциативные) лупы Бола порядка 18.

1. Introduction

Recently Burn [5] has proved that there exist exactly six non-associative Bol loops of order 8. Solarin and Sharma [11] have shown that (up to isomorphism) there exist exactly two Bol loops of order twelve which are not groups and Moufang loops. The object of this paper is to prove that there are exactly two (non-associative) Bol loops of order 18 up to isomorphism.

For the definition of a loop, the reader should consult Bruck [4]. Albert [1] defines a right multiplication $R(a)$ as a permutation of a loop $(L, \cdot)$,

$$R(a) : X \to X \,.\, a \,.\, \forall X \in L.$$

we call the set $\Pi = \{R(a) \mid a \in L\}$ the right regular representation of $(L, \cdot)$ or briefly, the representation of L.

A Bol loop L is a loop in which we have

(1) $$(Xy \,.\, z)\, y = X(yz \,.\, y) \quad \text{for all} \quad X, y, z \in L\,.$$

Strictly speaking, (1) defines a right Bol loop. For Bol loop, see [2], [7] and [10]. The following theorems due to Burn [5] will be used in the investigations.

Theorem 1. If Π is the representation of a loop L, then L is a Bol loop if and only if $\alpha, \beta \in \Pi$ implies $\alpha\beta\alpha \in \Pi$.

Theorem 2. If Π is the representation of a finite Bol loop and $\alpha \in \Pi$, then α is a product of disjoint cycles of equal length.

*) Department of Mathematics, University of Ife, Ile-Ife. Nigeria.

Theorem 3. The order of an element of a finite Bol loop divides the order of the loop.

Bruck [3] defines the left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ϱ of a loop $(L, \cdot)$ as follows:

$$N_\lambda = \{\text{all } X \in L \mid X \,.\, yz = Xy \,.\, z, \text{ all } y, z \in L\},$$
$$N_\mu = \{\text{all } y \in L \mid X \,.\, yz = Xy \,.\, z, \text{ all } X, z \in L\},$$
$$N_\varrho = \{\text{all } z \in L \mid X \,.\, yz = Xy \,.\, z, \text{ all } X, y \in L\},$$

In this paper we presume that Π is the representation of a Bol loop $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ containing no element of order 18. It is well-known that a Bol loop must be generated by at least two elements. In case a Bol loop is generated by one element then it must be a cyclic group.

2. Main results

Theorem 4. If Π is the representation of a (non-associative) Bol loop of order 18, then Π can not contain an element of order 9.

Proof. We assume that $\alpha \in \Pi$, such that $\alpha^9 = 1$. Without loss of generality, we take

$\alpha = R(2) = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\ (10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18)$

$\alpha^2 = R(3) = (1\ 3\ 5\ 7\ 9\ 2\ 4\ 6\ 8)\ (10\ 12\ 14\ 16\ 18\ 11\ 13\ 15\ 17)$

$\alpha^3 = R(4) = (1\ 4\ 7)\ (2\ 5\ 8)\ (3\ 6\ 9)\ (10\ 13\ 16)\ (11\ 14\ 17)\ (12\ 15\ 18)$

$\alpha^4 = R(5) = (1\ 5\ 9\ 4\ 8\ 3\ 7\ 2\ 6)\ (10\ 14\ 18\ 13\ 17\ 12\ 16\ 11\ 15)$

$\alpha^5 = R(6) = (1\ 6\ 2\ 7\ 3\ 8\ 4\ 9\ 5)\ (10\ 15\ 11\ 16\ 12\ 17\ 13\ 18\ 14)$

$\alpha^6 = R(7) = (1\ 7\ 4)\ (2\ 8\ 5)\ (3\ 9\ 6)\ (10\ 16\ 13)\ (11\ 17\ 14)\ (12\ 18\ 15)$

$\alpha^7 = R(8) = (1\ 8\ 6\ 4\ 2\ 9\ 7\ 5\ 3)\ (10\ 17\ 15\ 13\ 11\ 18\ 16\ 14\ 12)$

$\alpha^8 = R(9) = (1\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2)\ (10\ 18\ 17\ 16\ 15\ 14\ 13\ 12\ 11)$

Without loss of generality we assume, that $R(10), \ldots, R(18)$ are of order 2. We assume

(2) $\qquad R(10) = (1\ 10)\ (2\ X)\ (3\ y)\ (4\ z)\ (5\ a)\ (6\ b)\ (7\ c)\ (8\ s)\ (a\ t),$

where $X \in \{11, 12, 13, 14, 15, 16, 17, 18\}$. If $X = 18$, then $R(10)\,R(9)\,R(10) = R(2)$. Thus $2\,R(10)\,R(9)\,R(10) = 2\,R(2) \Rightarrow y = 17$. Similarly we obtain $z = 16$, $a = 15$, $b = 14$, $c = 13$, $s = 12$ and $t = 11$. Hence

(3) $\qquad R(10) = (1\ 10)\ (2\ 18)\ (3\ 17)\ (4\ 16)\ (5\ 15)\ (6\ 14)\ (7\ 13)\ (8\ 12)\ (9\ 11),$

In case $X = 11$ in (2), then $R(10)\,R(2)\,R(10) = R(2)$. Thus $2\,R(10)\,R(2)\,R(10) = 2\,R(2) \Rightarrow y = 2$. Similarly we obtain $z = 13$, $a = 14$, $b = 15$, $c = 16$, $s = 17$ and $t = 18$. Thus $R(10) = (1\ 10)\,(2\ 11)\,(3\ 12)\,(4\ 13)\,(5\ 14)\,(6\ 15)\,(7\ 16)\,(8\ 17)\,(9\ 18)$. But $R(2)\,R(10)\,R(2) = R(12) = (1\ 12\ 5\ 16\ 9\ 2\ 13\ 6\ 17)\,(\ ) \notin \Pi$. Hence X can not be equal to 11. Proceeding in the same way we can show that $X \notin \{11, 12, 13, 14, 15, 16, 17\}$. Thus we get only one $R(10)$ given by (3).

Further we assume that

$$R(11) = (1\ 11)(2\ X_1)(3\ y_1)(4\ z_1)(5\ a_1)(6\ b_1)(7\ c_1)(8\ s_1)(9\ t_1),$$
$$X_1 \in \{10, 12, 13, 14, 15, 16, 17, 18\}.$$

In case $X_1 = 10$, then $R(11)\,R(9)\,R(11) = R(2)$. Also $2\,R(11)\,R(9)\,R(11) = 2\,R(2)$ $y_1 = 18$. Similarly $z_1 = 17$, $a_1 = 16$, $b_1 = 15$, $c_1 = 14$, $s_1 = 13$ and $t_1 = 12$. Thus

$$R(11) = (1\ 11)(2\ 10)(3\ 18)(4\ 17)(5\ 16)(6\ 15)(7\ 14)(8\ 13)(9\ 12) \tag{4}$$

In case $X_1 = 12$, then $R(11)\,R(2)\,R(11) = R(2)$. Also $2\,R(11)\,R(2)\,R(11) = 2\,R(2) \Rightarrow$ $\Rightarrow y_1 = 13$. Similarly $z_1 = 14$, $a_1 = 15$, $b_1 = 16$, $c_1 = 17$, $s_1 = 18$ and $t_1 = 10$. Thus

$$R(11) = (1\ 11)(2\ 12)(3\ 13)(4\ 14)(5\ 15)(6\ 16)(7\ 17)(8\ 18)(9\ 10).$$

But $R(2)\,R(11)\,R(2) = R(13) = (1\ 13\ 5\ \ldots)(\ ) \notin \Pi$. Thus X_1 can not be equal to 12. Following the same argument we can show easily $X \notin \{12, 13, 14, 15, 16, 17, 18\}$. Thus we obtain only one $R(11)$ given by (4).

Following the above arguments, we obtain

$$R(12) = (1\ 12)\ (2\ 11)\ (3\ 10)\ (4\ 18)\ (5\ 17)\ (6\ 16)\ (7\ 15)\ (8\ 14)\ (9\ 13)$$
$$R(13) = (1\ 13)\ (2\ 12)\ (3\ 11)\ (4\ 10)\ (5\ 18)\ (6\ 17)\ (7\ 16)\ (8\ 15)\ (9\ 14)$$
$$R(14) = (1\ 14)\ (2\ 13)\ (3\ 12)\ (4\ 11)\ (5\ 10)\ (6\ 18)\ (7\ 17)\ (8\ 16)\ (9\ 15)$$
$$R(15) = (1\ 15)\ (2\ 14)\ (3\ 13)\ (4\ 12)\ (5\ 11)\ (6\ 10)\ (7\ 18)\ (8\ 17)\ (9\ 16)$$
$$R(16) = (1\ 16)\ (2\ 15)\ (3\ 14)\ (4\ 13)\ (5\ 12)\ (6\ 11)\ (7\ 10)\ (8\ 18)\ (9\ 17)$$
$$R(17) = (1\ 17)\ (2\ 16)\ (3\ 15)\ (4\ 14)\ (5\ 13)\ (6\ 12)\ (7\ 11)\ (8\ 10)\ (9\ 18)$$
$$R(18) = (1\ 18)\ (2\ 17)\ (3\ 16)\ (4\ 15)\ (5\ 14)\ (6\ 13)\ (7\ 12)\ (8\ 11)\ (9\ 10)$$

Thus we get the only representation Π which is the representation of the dihedral group D_9. This completes the proof of the Theorem 4.

In order to cut down the length of the paper we state without proofs the theorems because we have not been able to get the representation of a (non-associative) Bol loop of order 18.

Theorem 5. If Π is the representation of a (non-associative) Bol loop of order 18, such that Π contains nine elements of order 2 and eight elements of order 3, then Π is isomorphic to the representation of the group $(c_3 \times c_3) \times c_2$.

Theorem 6. If Π is the representation of a (non-associative) Bol loop of order 18 and $\alpha, \beta, \lambda, \delta \in \Pi$, such that $\alpha^6 = \beta^6 = \lambda^6 = \delta^6 = 1$, $\alpha^3 = \beta^3 = \lambda^3 = \delta^3$ and $\alpha^2 \neq$ $\neq \beta^2 \neq \lambda^2 \neq \delta^2$, then Π is isomorphic to the representation of the abelian group $c_6 \times c_3$.

Theorem 7. If Π is the representation of a (non-associative) Bol loop of order 18 and $\alpha, \beta, \lambda \in \Pi$, such that $\alpha^6 = \beta^6 = \lambda^6 = 1$, $\alpha^2 = \beta^2 = \lambda^2$, $\alpha^3 \neq \beta^3 \neq \lambda^3$ and the remaining elements are of order 3, then Π is given by one of the following representations:

Π_1

$R(2) = (1\ 2\ 3\ 4\ 5\ 6)\ (7\ 8\ 9\ 10\ 11\ 12)\ (13\ 14\ 15\ 16\ 17\ 18)$

$R(3) = (1\ 3\ 5)\ (2\ 4\ 6)\ (7\ 9\ 11)\ (8\ 10\ 12)\ (13\ 15\ 17)\ (14\ 16\ 18)$

$R(4) = (1\ 4)\ (2\ 5)\ (3\ 6)\ (7\ 10)\ (8\ 11)\ (9\ 12)\ (13\ 16)\ (14\ 17)\ (15\ 18)$

$R(5) = (1\ 5\ 3)\ (2\ 6\ 4)\ (7\ 11\ 9)\ (8\ 12\ 10)\ (13\ 17\ 15)\ (14\ 18\ 16)$

$R(6) = (1\ 6\ 5\ 4\ 3\ 2)\ (7\ 12\ 11\ 10\ 9\ 8)\ (13\ 18\ 17\ 16\ 15\ 14)$

$R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ 15\ 4\ 17\ 6\ 13)\ (8\ 16\ 10\ 18\ 12\ 14)$

$R(8) = (1\ 8\ 13)\ (2\ 16\ 7)\ (3\ 10\ 15)\ (4\ 18\ 9)\ (5\ 12\ 17)\ (6\ 14\ 11)$

$R(9) = (1\ 9)\ (2\ 17)\ (3\ 11)\ (4\ 13)\ (5\ 7)\ (6\ 15)\ (8\ 18)\ (10\ 14)\ (12\ 16)$

$R(10) = (1\ 10\ 17)\ (2\ 18\ 11)\ (3\ 12\ 13)\ (4\ 14\ 7)\ (5\ 8\ 15)\ (6\ 16\ 9)$

$R(11) = (1\ 11\ 5\ 9\ 3\ 7)\ (2\ 13\ 6\ 17\ 4\ 15)\ (8\ 14\ 12\ 18\ 10\ 16)$

$R(12) = (1\ 12\ 15)\ (2\ 14\ 9)\ (3\ 8\ 17)\ (4\ 16\ 11)\ (5\ 10\ 13)\ (6\ 18\ 7)$

$R(13) = (1\ 13\ 8)\ (2\ 7\ 16)\ (3\ 15\ 10)\ (4\ 9\ 18)\ (5\ 17\ 12)\ (6\ 11\ 14)$

$R(14) = (1\ 14\ 3\ 16\ 5\ 18)\ (2\ 8\ 4\ 10\ 6\ 12)\ (7\ 15\ 9\ 17\ 11\ 13)$

$R(15) = (1\ 15\ 12)\ (2\ 9\ 14)\ (3\ 17\ 8)\ (4\ 11\ 16)\ (5\ 13\ 10)\ (6\ 7\ 18)$

$R(16) = (1\ 16)\ (2\ 10)\ (3\ 18)\ (4\ 12)\ (5\ 14)\ (6\ 8)\ (7\ 17)\ (9\ 13)\ (11\ 15)$

$R(17) = (1\ 17\ 10)\ (2\ 11\ 18)\ (3\ 13\ 12)\ (4\ 7\ 14)\ (5\ 15\ 8)\ (6\ 9\ 16)$

$R(18) = (1\ 18\ 5\ 16\ 3\ 14)\ (2\ 12\ 6\ 10\ 4\ 8)\ (7\ 13\ 11\ 17\ 9\ 15)$

$N_\lambda = \{1, 2, 3, 4, 5, 6\}$, $N_\mu = N_\varrho = \{1, 3, 5\}$.

Π_2

$R(2)$ to $R(6)$ are same as in Π_1,

$R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ 17\ 4\ 13\ 6\ 15)\ (8\ 18\ 10\ 14\ 12\ 16)$

$R(8) = (1\ 8\ 13)\ (2\ 16\ 7)\ (3\ 10\ 15)\ (4\ 18\ 9)\ (5\ 12\ 17)\ (6\ 14\ 11)$

$R(9) = (1\ 9)\ (2\ 13)\ (3\ 11)\ (4\ 15)\ (5\ 7)\ (6\ 17)\ (8\ 14)\ (10\ 16)\ (12\ 18)$

$R(10) = (1\ 10\ 17)\ (2\ 18\ 11)\ (3\ 12\ 13)\ (4\ 14\ 7)\ (5\ 8\ 15)\ (6\ 16\ 9)$

$R(11) = (1\ 11\ 5\ 9\ 3\ 7)\ (2\ 15\ 6\ 13\ 4\ 17)\ (8\ 16\ 12\ 14\ 10\ 18)$

$R(12) = (1\ 12\ 15)\ (2\ 14\ 9)\ (3\ 8\ 17)\ (4\ 16\ 11)\ (5\ 10\ 13)\ (6\ 18\ 7)$

$R(13) = (1\ 13\ 8)\ (2\ 7\ 16)\ (3\ 15\ 10)\ (4\ 9\ 18)\ (5\ 17\ 12)\ (6\ 11\ 14)$

$R(14) = (1\ 14\ 5\ 18\ 3\ 16)\ (2\ 12\ 6\ 10\ 4\ 8)\ (7\ 15\ 11\ 13\ 9\ 17)$

$R(15) = (1\ 15\ 12)\ (2\ 9\ 14)\ (3\ 17\ 8)\ (4\ 11\ 16)\ (5\ 13\ 10)\ (6\ 7\ 18)$

$R(16) = (1\ 16\ 3\ 18\ 5\ 14)\ (2\ 8\ 4\ 10\ 6\ 12)\ (7\ 17\ 9\ 13\ 11\ 15)$

$R(17) = (1\ 17\ 10)\ (2\ 11\ 18)\ (3\ 13\ 12)\ (4\ 7\ 14)\ (5\ 15\ 8)\ (6\ 9\ 16)$

$R(18) = (1\ 18)\ (2\ 10)\ (3\ 14)\ (4\ 12)\ (5\ 16)\ (6\ 8)\ (7\ 13)\ (9\ 15)\ (11\ 17)$

$N_\lambda = \{1, 3, 5, 8, 10, 12, 13, 15, 17\}\,, \quad N_\mu = N_\varrho = \{1, 3, 5\}\,.$

Proof. Without loss of generality take $\alpha = R(2)$, then $\alpha^2 = R(3)$, $\alpha^3 = R(4)$, $\alpha^4 = R(5)$, $\alpha^5 = R(6)$. $R(2), R(3), \ldots, R(6)$ are the same as given in Π_1. Let us assume $\beta = R(7)$. By our assumption $\alpha^2 = \beta^2 = R(3) = R^2(7)$. Thus

$$R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ X\ 4\ y\ 6\ z)\ (8\ p\ 10\ q\ 12\ r)\,,$$
$$X, p \in \{13, 14, 15, 16, 17, 18\} \quad \text{and} \quad X \neq p\,.$$

(i) Take $X = 13$ and $p = 14$ then $R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ 13\ 4\ 15\ 6\ 17)\ (8\ 14\ 10\ 16\ 12\ 18)$ $R(2)\,R(7)\,R(2) = R(14) = (1\ 14)\ (\ 210)\ (3\ 16)\ (4\ 12)\ (5\ 18)\ (6\ 8)\ (7\ 15)\ (9\ 17)\ (11\ 13)$. $R(4)\,R(7)\,R(4) = R(18) = (1\ 18\ 3\ 14\ 5\ 16)\,(2\ 8\ 4\ 10\ 6\ 12)\,(7\ 13\ 9\ 15\ 11\ 17)$. Thus by assumption $R(8)$, $R(10)$, $R(12)$, $R(13)$, $R(15)$ and $R(17)$ are of order 3. Without loss of generality, we assume $R(8) = (1\ 8\ 13)\ (2\ a\ b)\ (3\ c\ d)\ (4\ l\ m)\ (5\ s\ t)\ (6\ u\ v)$, where $a \in \{14, 16, 18\}$ and $b \in \{7, 9, 11\}$. In case $a = 14$ and $b = 7$, then $R(8)\,R(6)\,R(8) = R(2)$. Thus $2\,R(8)\,R(6)\,R(8) = 2\,R(2) \Rightarrow 13\,R(8) = 3$. It is a contradiction. Hence b can not be equal to 7. Following the same argument we can show that $b \notin \{7, 11\}$.

In case $a = 14$ and $b = 9$, then $R(8)\,R(2)\,R(8) = R(2)$. Also $2\,R(8)\,R(2)\,R(8) = 2\,R(2) \Rightarrow d = 15$. Similarly we obtain $c = 10$, $l = 16$, $m = 11$, $s = 12$, $t = 17$, $u = 18$ and $v = 7$. Hence, $R(8) = (1\ 8\ 13)\ (2\ 14\ 9)\ (3\ 10\ 15)\ (4\ 16\ 11)\ (5\ 12\ 17)$

In case $a = 14$ and $b = 9$, then $R(8)\,R(2)\,R(8) = R(2)$. Also $2\,R(8)\,R(2)\,R(8) = 2\,R(2) \Rightarrow d = 15$. Similarly we obtain $c = 10$, $l = 16$, $m = 11$, $s = 12$, $t = 17$, $u = 18$ and $v = 7$. Hence,

$$(5) \qquad R(8) = (1\ 8\ 13)\ (2\ 14\ 9)\ (3\ 10\ 15)\ (4\ 16\ 11)\ (5\ 12\ 17)\ (6\ 18\ 7)$$

$R(7) = R(7)\,R(8)\,R(7)$ and $R(2)\,R(8)\,R(2) = R(15)$. Thus we get the representation Π which is isomorphic to Π_1 under the isomorphism $(2\ 6)\ (3\ 5)\ (7\ 16\ 9\ 14\ 11\ 18)\ (8\ 15\ 10\ 13\ 12\ 17)$.

In case $a = 16$ and $b = 7$, then $R(8)\,R(6)\,R(8) = R(2)$. Thus $2\,R(8)\,R(6)\,R(8) = 2R(2)\ d = 15$. Similarly we obtain $c = 10$, $l = 18$, $m = 9$, $s = 12$, $t = 17$, $u = 14$ and $v = 11$. Thus

$$(6) \qquad R(8) = (1\ 8\ 13)\ (2\ 16\ 7)\ (3\ 10\ 15)\ (4\ 18\ 9)\ (5\ 12\ 17)\ (6\ 14\ 11)$$

$R(17) = R(2)\,R(8)\,R(2)$ and $R(12) = R(3)\,R(7)\,R(3)$. Thus we get the representation Π which is isomorphic to Π_1 under the isomorphism (3 5) (6 18 7) (2 16 9 4 14 11) (8 10) (13 17). Following the above argument we can prove easily that $b \notin \{9, 11\}$.

In case $a = 18$ and $b = 11$, then $R(8)\,R(4)\,R(8) = R(2)$. Thus $2\,R(2) = 2\,R(8)\,R(4)\,R(8) \Rightarrow d = 15$. Similarly we get $c = 10$, $l = 14$, $m = 7$, $s = 12$, $t = 17$, $u = 16$ and $v = 9$. Thus

$$(7) \qquad R(8) = (1\ 8\ 13)\ (2\ 18\ 11)\ (3\ 10\ 15)\ (4\ 14\ 7)\ (5\ 12\ 17)\ (6\ 16\ 9)$$

$R(12) = R(3)\,R(8)\,R(3)$ and $R(17) = R(4)\,R(8)\,R(4)$. Thus we get the representation Π which is isomorphic to Π_1 under the isomorphism (2 11 18) (3 5) (6 7 16 4 9 14) (13 17) (8 10).

(ii) Take $X = 13$ and $p = 16$, then

$R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ 13\ 4\ 15\ 6\ 17)\ (8\ 16\ 10\ 18\ 12\ 14)$. But $R(7)\,R(2)\,R(7) = R(16) = (1\ 16)\ (\ )\ldots(\ ) \in \Pi$. Also $R(6)\,R(7)\,R(6) = R(16) = (1\ 16\ 5\ 14\ 3\ 18)\ (\ )(\ ) \in \Pi$. It is impossible. Hence $\varrho \neq 16$. Similarly we can show that $\varrho \notin \{16, 18\}$.

(iii) Take $X = 15$ and $p = 16$, then

$R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ 15\ 4\ 17\ 6\ 13)\ (8\ 16\ 10\ 18\ 12\ 14)$. Now we take $R(8)$ given by (5). Thus $R(2)\,R(7)\,R(2) = R(16)$, $R(4)\,R(7)\,R(4) = R(14)$. Hence we get the representation Π which is isomorphic to Π_1 under the isomorphism (2 14) (4 16) (6 18) (8 15) (10 17) (12 13).

In case we consider $R(8)$ given by (6), then $R(15) = R(7)\,R(8)\,R(7)$, $R(16) = R(2)\,R(7)\,R(2)$ and $R(14) = R(4)\,(R7)\,R(4)$, we get the representation Π_1.

In case we consider $R(8)$ given by $R(7)$, then $R(17) = R(7)\,R(8)\,R(7)$ and $R(2)\,R(7)\,R(2) = R(16)$. We get the representation Π which is isomorphic to Π_1 under the isomorphism (2 7) (4 9) (6 11) (10 17) (12 13) (8 15).

Following the above argument we can easily show that $p \notin \{14, 18\}$.

(iv) Take $X = 17$ and $p = 18$, the

$R(7) = (1\ 7\ 3\ 9\ 5\ 11)\ (2\ 17\ 4\ 13\ 6\ 15)\ (8\ 18\ 10\ 14\ 12\ 16)$. Now we consider $R(8)$ given by (5). Thus $R(15) = R(7)\,R(8)\,R(7)$ and $R(17) = R(6)\,R(8)\,R(6)$. Thus we get the representation Π which is isomorphic to Π_2 under the isomorphism (2 6) (3 5) (7 14) (8 13) (9 18) (10 17) (11 16) (12 15).

In case we take $R(8)$ given by (6). Thus $R(17) = R(7)\,R(8)\,R(7)$ and $R(12) = R(8)\,R(17)\,R(8)$. Thus we get the representation Π_2.

Finally we see $R(8)$ given by (3). Thus $R(12) = R(3)\,R(8)\,R(3)$ and $R(17) = R(4)\,R(8)\,R(4)$. Thus we get the representation Π which is the representation of the group $s_3 \times c_3$.

Following the above argument we can prove that $p \notin \{14, 16\}$.

Finally all the representation when $X \in \{14, 16, 18\}$ are isomorphic to the representation obtained above under conjugation by (1 2 3 4 5 6) (13 8 14 9 15 10 16 11 17 12 18 7). This complete the proof the theorem 7.

Remark: Since the left nucleus N_λ of Π_1 can not be isomorphic to the left nucleus N_λ of Π_2, hence Π_1 is not isotopic to Π_2. It suggests the open problem: "Are the Bol loops of order $2p^2$ (p being prime) isomorphic to their loop isotopes"?

Theorem 8. If Π is the representation of a (non-associative) Bol loop of order 18, then Π is given by one of the representations Π_1 and Π_2.

Proof. The proof follows from Theorem 4 to theorem 7.

3. The representations Π_1 and Π_2 suggest the following constructions for Bol loops of order $2p^2$ (p being prime number $\geqq 3$). The proofs are straightforward

Theorem 9. Let G be an abelian group generated by the elements X and y such that $X^n = y^n = r$, and c_2 be generated by a. Let l, m, p, q be arbitrary integers $0 \leqq l, m, p, q < n$, and the identities in each group be denoted by e. Let $H = G \times c_2$ and the multiplication is defined as follows:

$$
\begin{aligned}
(X^l y^m, e) \circ (X^p y^q, e) &= (X^{l+p} y^{m+q}, e) \\
(X^l y^m, e) \circ (X^p y^q, a) &= (X^{l+p} y^{m+q}, a) \\
(X^l y^m, a) \circ (X^p y^q, e) &= (X^{l+p} y^{m-q}, a) \\
(X^l y^m, a) \circ (X^p y^q, a) &= (X^{l+p-q} y^{m-q}, e) .
\end{aligned}
$$

Then $H(\circ)$ is a Bol loop of order $2n^2$ ($n \geqq 2$). The representation Π_1 and the Theorem 9 (in a different way) have been given by Solarin and Sharma [9].

Theorem 9. Let X and a be the generators of the cyclic groups c_{2p} nad c_p respectively, such that $X^{2p} = a^p = e$ (the identities in each group is represented by e). Let $H = c_{2p} \times c_p$ and the multiplication be defined as follows:

(i) $(X^l, e) \circ (X^m, e) = (X^{l+m}, e)$

$\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots$

$\circ (X^m, a^{p-1}) = (X^{l+m}, a^{p-1})$

(ii) $(X^l, a^\varrho) \circ (X^m, e) = (X^{l+m}, a^\varrho)$ if m is even .

$\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots$

$\circ (X^m, a^{p-1}) = (X^{l+m}, a^{\varrho+p-1})$.

(iii) $(X^l, a^\varrho) \circ (X^m, e) = (X^{l+m+2\varrho}, a^{-\varrho})$ if m is odd .

$\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots$

$\circ (X^m, a^{(p-1)/2}) = (X^{l+m+2(\varrho+(p-1)/2)}, a^{-\varrho+(p-1)/2})$.

(iv) $(X^l, a^\varrho) \circ (X^m, a^{(p+1)/2}) = (X^{l+m}, a^{-\varrho+(p+1)/2})$ if m is odd .

(v) $(X^l, a^\varrho) \circ (X^m, a^{(p+3)/2}) = (X^{l+m+2(\varrho+(p+3)/2)}, a^{-\varrho+(p+3)/2})$ if m is odd.

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\circ (X^m, a^{p-1}) = (X^{l+m+2(\varrho+p-1)}, a^{-\varrho+p-1}).$$

Then $H(\circ)$ is a Bol loop of order $2p^2$.

4. We give below the table to show how for the problem of classification of (non-associative) Bol loops have been solved

Bol loops

Order	Number	Remark
8	6	
12	2	
15	2	
16	25	If a Bol loop constains an element of order 8
18	2	

The Bol loops of order 8, 12, 15 and 16 have been classified by Burn [5], Solarin and Sharma [11], Niederreiter and Robinson [6]. The classification of the Bol loops of order 16, and 24 are under investigation.

The author is indebted to his colleague Mr. A. R. T. Solarin, for checking, with a computer the representations satisfying the conditions of Theorem 1.

The author is also grateful to the Ife University Research Committee for financial support for this project.

References

[1] ALBERT, A. A.: "Quasigroup I". Trans. Amer. Math. Soc. 54 (1943), pp. 507—519.

[2] BOL, G.: "Gowebe and Gruppen". Math. Ann. 114 (1937), pp. 414—431.

[3] BRUCK, R. H.: "Contribution to the theory of loops". Trans. Amer. Math. Soc. 60 (1946), pp. 245—354.

[4] BRUCK, R. H.: "A Survey of Binary Systems". Berlin, Springer-Verlag. 1958.

[5] BURN, R. P.: "Finite Bol loops". Math. Proc. Camb. Phil. Soc. 84 (1978), pp. 377—385.

[6] NIEDERREITER, H. & ROBINSON, H. K.: "Bol loops of order pq". Proc. Camb. Phil. Soc. (1981), 89, pp. 241—256.

[7] ROBINSON, D. A.: "Bol loops". Trans. Amer. Math. Soc. 123 (1966), pp. 341—354.

[8] SHARMA, B. L. and Solarin, A. R. T.: "On Bol loops of order 16 (I)". Kyungpook Math. Jour. (accepted for publication).

[9] SOLARIN, A. R. T. and SHARMA, B. L.: "On the construction of Bol loops". Ann. Univ. "AL. CUZA". Iasi. Sec. I. a Vol. 27, (1980), pp. 13—17.

[10] SOLARIN, A. R. T. and SHARMA, B. L.: "On the construction of Bol loops II". Ann. Univ. "AL. CUZA". Iasi. (accepted for publication).

[11] SOLARIN, A. R. T. and SHARMA, B. L.: "Bol loops of order 12". Ann. Univ. "AL CUZA". Iasi. (accepted for publication).