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## Sheaves of Metric Lattices

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The sheaf of sections of the bundle associated with a given presheaf of complete metric lattices of suitable sort over a hereditarily paracompact base is isomorphic to the latter.

Пучки метрических решеток. Пучок резов накрывающего пространства данного предпучка полных метрических решеток удобного сорта над наследственно паракомпактным базисом изоморфный данному предпучку.

Svazek řezů bandlu příslušného danému předsvazku vhodných metrických svazů, úplných v dané metrice, nad dědičně parakompaktní bází, je izomorfní původnímu předsvazku.

### Introduction

In [1], K. H. Hofmann proved that the sheaf of sections of the bundle associated with a given sheaf of Banach  $C(X)$ -modules of a suitable sort over a hereditarily paracompact base  $X$  is isomorphic to the latter. It is only natural to try to find out how much this result rests on the fact that it is a sheaf of Banach  $C(X)$ -modules, and whether the developed machinery would work also in a more general case. A possibility seems to be offered in the form of sheaves of metric spaces, but since we cannot do without the fact that an isometry of a complete metric space onto a dense subset of another one is an isometry onto the latter, we must keep the requirement of completeness. Also we need there to multiply elements by some suitable functions to make an infinite family of elements be locally finite. Thus complete metric linear spaces over  $C(X)$  should seemingly be what we want. But when trying to generalise the results of [1] to this case, we meet some heavy hardships. While the machinery works for Banach spaces as the norm behaves well towards the multiplication by the functions from  $C(X)$ , it fails to work for the metric linear spaces over  $C(X)$  because the metric behaves poorly towards the multiplication. We need multiply the elements by locally finite partitions of unity, and if the elements have small norm, the outcoming sum keeps it, while in the metric case the distance of it from zero may become big. This poor behavior of metric towards multiplication causes the theory to fail to work in the metric case.

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But as the additive partitions of unity in [1] was in agreement with some sorts of metrics – namely with norms – which behaved well towards the multiplication, one may be led to taking the lattice – partition of unity and to trying to sort out a class of metrics that are in agreement with multiplying the elements by this partition. For this we need take the metric linear spaces with a new structure, namely with that of upper semilattice, and then to sort out the metrics which are well behaved towards it. We are thus lead to the notion of  $C(Y, P)$  – area, which is a “module” over the set of all continuous functions on a topological space  $Y$  with the values in  $P = \langle -1, 1 \rangle$ . This may seem somewhat artificial, to assume that we have the multiplication only by the functions from  $C(Y, P)$  instead of by all continuous functions on  $Y$ , but on the other hand, what we need is to multiply by partitions of unity, and the functions which they consist of have values only in  $\langle 0, 1 \rangle$ , wherefore it perhaps isn’t so great a sin to show what is necessary indeed and what is needlessly strong.

The purpose of this paper is to bring over the K. H. Hofmann’s results to a class of sheaves of complete metric spaces, namely to the sheaves of  $C(Y, P)$  – areas. In the first section we adopt the means, developed by K. H. Hofmann in [1] for sheaves of Banach spaces, to the case of those of metric spaces. In the section two the notion of  $C(Y, P)$  – area is introduced and its properties which are needed later found. Then, following K. H. Hofmann’s line of [1], we develop some means that enable us to prove in the spirit of [1] that the sheaf of sections of the bundle associated with a given sheaf of complete  $C(Y, P)$  – areas of suitable sort over a hereditarily paracompact base is isomorphic to the latter.

## 1. Presheaves of metric spaces with contractions

*1.1. Notation.* A map of a metric space  $(X_1, d_1)$  into another one  $(X_2, d_2)$  is called contraction if  $d_2(f(x), f(y)) \leq d_1(x, y)$  for all  $x, y \in X_1$ .

The category of all metric (complete metric) spaces with contractions as morphisms is denoted by  $\mathfrak{M}\mathfrak{C}$ .

A category  $\mathfrak{R}$  is called inductive if for every presheaf  $\mathcal{S} = \{X_\alpha |_{\varrho_{\alpha\beta}} \langle A \leq \rangle\}$  from it there is  $\underline{\lim} \mathcal{S} = \langle I |_{\{\xi_\alpha | \alpha \in A\}} \rangle$  in  $\mathfrak{R}$ . (here  $\xi_\alpha : X_\alpha \rightarrow I$  are the natural  $\mathfrak{R}$ -morphisms). We shall often write  $\underline{\lim} \mathcal{S} = I$ , for short.

The following lemma was proven by K. H. Hofmann [1, Lemma 1.6–1.9] for the category of Banach spaces with contractions. Our proof follows the line of that of Hofmann’s.

*1.2. Lemma.* Both  $\mathfrak{M}$  and  $\mathfrak{M}\mathfrak{C}$  are inductive. Let  $\mathcal{S} = \{(X_\alpha, d_\alpha) |_{\varrho_{\alpha\beta}} \langle A \leq \rangle\}$  be a presheaf from  $\mathfrak{M}\mathfrak{C}$ , let  $\langle (I^0, D) |_{\{\xi_\alpha | \alpha \in A\}} \rangle$  be its inductive limit in  $\mathfrak{M}$ , and let  $(I, D)$  be the completion of  $(I^0, D)$ . Then  $\langle (I, D) |_{\{\xi_\alpha | \alpha \in A\}} \rangle$  is inductive limit of  $\mathcal{S}$  in  $\mathfrak{M}\mathfrak{C}$ . Moreover, the following holds:

A. If  $\alpha, \beta \in A$ ,  $a \in X_\alpha$ ,  $b \in X_\beta$ , then  $a, b$  represent the same element of  $I^0$  (meaning

that  $\xi_\alpha(a) = \xi_\beta(b)$  iff there is  $\gamma \geq \alpha, \beta$  such that for  $a' = \varrho_{\alpha\gamma}(a), b' = \varrho_{\beta\gamma}(b)$  we have, setting  $A(\gamma) = \{\delta \in A \mid \delta \geq \gamma\}$ :

$$\lim \{d_\delta(\varrho_{\gamma\delta}(a'), \varrho_{\gamma\delta}(b')) \mid \delta \in A(\gamma)\} = 0.$$

B. If  $p, q \in I$  such that there are representatives  $a, b$  of  $p, q$  in an  $X_\alpha$  (in which case  $p, q \in I^0$ ) then

$$(*) \quad D(p, q) = \lim \{d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b)) \mid \beta \in A(\alpha)\} = \inf \{\text{the same set}\}.$$

*Proof.* Given a presheaf  $\mathcal{S} = \{(X_\alpha, d_\alpha) \mid \varrho_{\alpha\beta} \mid \langle A \leq \rangle\}$  from  $\mathfrak{M}(\mathfrak{MC})$ , we can make the inductive limit  $\langle L \mid \{\eta_\alpha \mid \alpha \in A\} \rangle$  of  $\mathcal{S}' = \{X_\alpha \mid \varrho_{\alpha\beta} \mid \langle A \leq \rangle\}$  in the category of sets. If  $p, q \in L$  then there are their representatives  $a, b \in X_\alpha$  for an  $\alpha \in A$ , meaning that  $\eta_\alpha(a) = p, \eta_\alpha(b) = q$ . As the  $\varrho_{\alpha\beta}$ 's are contractions, the function  $f_{\alpha\beta}(\beta) = d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b))$  is nonincreasing on  $A(\alpha) = \{\beta \in A \mid \beta \geq \alpha\}$ ; there thus is

$$(**) \quad D'_{\alpha\beta}(p, q) = \lim \{d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b)) \mid \beta \in A(\alpha)\} = \inf \{\text{the same set}\}.$$

It is easy to check that, as  $f_{\alpha\beta}$  is nonincreasing on  $A(\alpha)$ , the number  $D'_{\alpha\beta}(p, q)$  does not depend on the choice of  $\alpha$  and of the representatives  $a, b \in X_\alpha$  of  $p, q$ . We may thus write  $D'$  instead of  $D'_{\alpha\beta}$ . So we get a function  $D'$  on  $L \times L$  which is easy to be seen to fulfil the triangle inequality. It may happen that  $D'(p, q) = 0$  notwithstanding that  $p \neq q$ ; from this reason we denote by  $I^0$  the set of all equivalence classes of  $L$  by “ $p, q \in L, p \sim q$  iff  $D'(p, q) = 0$ ”, and by  $\varphi: L \rightarrow I^0$  the map sending  $p \in L$  onto its equivalence class  $\varphi(p)$ . Now, clearly the function  $D$  defined on  $I^0 \times I^0$  as  $D(\varphi(p), \varphi(q)) = D'(p, q)$  is a metric on  $I^0$ , and (\*\*) readily yields that the  $\xi_\alpha = \varphi\eta_\alpha: (X_\alpha, d_\alpha) \rightarrow (I^0, D)$  are contractions. We shall show that  $\langle (I^0, D) \mid \{\xi_\alpha \mid \alpha \in A\} \rangle$  is the inductive limit of  $\mathcal{S}$  in  $\mathfrak{M}$ . For this end we take a fan of contractions  $\{f_\alpha: (X_\alpha, d_\alpha) \rightarrow (X, d) \mid \alpha \in A\}$  between  $\mathcal{S}$  and a metric space  $(X, d)$ , meaning that  $f_\beta\varrho_{\alpha\beta} = f_\alpha$  for all  $\alpha, \beta \in A, \alpha \leq \beta$ . As  $\langle L \mid \{\eta_\alpha \mid \alpha \in A\} \rangle = \varinjlim \mathcal{S}'$  in the category of sets, there is a unique  $f': L \rightarrow X$  with  $f'\eta_\alpha = f_\alpha$  for all  $\alpha \in A$ . If  $p, q \in L$  and if  $a, b \in X_\alpha$  are their representatives then from  $d(f'(p), f'(q)) = d(f_\beta(\varrho_{\alpha\beta}(a)), f_\beta(\varrho_{\alpha\beta}(b))) \leq d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b))$  for all  $\beta \geq \alpha$ , and from (\*\*) we get  $d(f'(p), f'(q)) \leq \lim \{d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(b)) \mid \beta \geq \alpha\} = D'(p, q)$ , thus  $f(p) = f(q)$  whenever  $D'(p, q) = 0$ , hence  $f'$  yields a map  $f: I^0 \rightarrow X$  with  $f\varphi(s) = f'(s)$  for all  $s \in L$ . Further,  $d(f\varphi(p), f\varphi(q)) = d(f'(p), f'(q)) \leq D'(p, q) = D(\varphi(p), \varphi(q))$  so  $f: (I^0, D) \rightarrow (X, d)$  is a contraction and  $f\xi_\alpha = f\varphi\eta_\alpha = f'\eta_\alpha = f_\alpha$  for all  $\alpha \in A$ . We have shown that there is a contraction  $f: (I^0, D) \rightarrow (X, d)$  with  $f\xi_\alpha = f_\alpha$  for all  $\alpha \in A$ . Let  $g: (I^0, D) \rightarrow (X, d)$  be another contraction with  $g\xi_\alpha = g\varphi\eta_\alpha = f_\alpha$  for all  $\alpha \in A$ . Then  $g\varphi = f'$  as  $f': L \rightarrow X$  is the unique map with  $f'\eta_\alpha = f_\alpha$  for all  $\alpha \in A$ . As  $\varphi(L) = I^0$ , we get from  $g\varphi = f\varphi = f'$  that  $g = f$  which proves our lemma for  $\mathfrak{M}$ .

If we are in  $\mathfrak{MC}$ , we — having already made  $(I^0, D)$  — make the completion  $(\hat{I}^0, \hat{D})$  of  $(I^0, D)$  and denote it by  $(I, D)$  as  $\hat{D}$  is just an extension of  $D$ . If now  $(X, d)$  is complete and if  $\{f_\alpha: (X_\alpha, d_\alpha) \rightarrow (X, d) \mid \alpha \in A\}$  is a fan of contractions then,

by what we have just proven, there is a unique contraction  $f : (I^0, D) \rightarrow (X, d)$  with  $f\xi_\alpha = f_\alpha$  for all  $\alpha \in A$ . There is a unique extension  $\hat{f} : (I, D) \rightarrow (X, d)$  of  $f$  to the whole of  $I$  which is a contraction thanks the density of  $I^0$  in  $I$ . Any other contraction  $g : (I, D) \rightarrow (X, d)$  such that  $g\xi_\alpha = f$  must be equal to  $f$  on  $I^0$  and hence equal to  $\hat{f}$  on  $I$  as  $I^0$  is dense in  $I$ . We are done.

It should be noticed that, by 1.2A,  $a \in X_\alpha$ ,  $b \in X_\beta$  represent the same element in  $I$  not only when  $\varrho_{\alpha\gamma}(a) = \varrho_{\beta\gamma}(b)$  for a  $\gamma \geq \alpha, \beta$ , as it is in the usual categories.

**1.3. Notation.** Let  $\mathcal{S} = \{(X_U, d_U) | \varrho_{UV} | X\}$  be a presheaf from  $\mathfrak{M}$  ( $\mathfrak{MC}$ ) over a topological space  $X$ .

A. For  $x \in X$  let  $\mathcal{B}(x) = \{U \subset X \mid U \text{ open, } x \in U\}$ , let  $\leq$  be the partial order in  $\mathcal{B}(x)$  defined as “ $U \leq V$  iff  $V \subset U$ ”, and let  $\mathcal{S}_x = \{(X_U, d_U) | \varrho_{UV} | \langle \mathcal{B}(x) \leq \rangle\}$ . By 1.2, there is  $\varinjlim \mathcal{S}_x = \langle (E_x^0, D_x) | \{\xi_{Ux} \mid U \in \mathcal{B}(x)\} \rangle$  in  $\mathfrak{M}(\varinjlim \mathcal{S}_x = \langle (E_x, D_x) | \{\xi_{Ux} \mid U \in \mathcal{B}(x)\} \rangle$  in  $\mathfrak{MC}$ ). The metric space  $(E_x^0, D_x)$  ( $(E_x, D_x)$ ) is called the stalk of  $\mathcal{S}$  over  $x$ ; it is thus a metric (complete metric) space with a metric  $D_x$ . If  $\mathcal{S}$  is from  $\mathfrak{MC}$  then  $(E_x, D_x)$  is just a completion of  $(E_x^0, D_x)$ . If  $r, s \in E_x$  such that there is  $U \in \mathcal{B}(x)$  and some representatives  $a, b \in X_U$  of  $r, s$  (in which case  $r, s \in E_x^0$ ), then

$$(*) \quad D_x(r, s) = \lim \{d_V(\varrho_{UV}(a), \varrho_{UV}(b)) \mid V \in \mathcal{B}(x), V \subset U\} = \inf \{\text{the same set}\} .$$

B. The set  $E^0 = \bigcup \{E_x^0 \mid x \in X\}$  ( $E = \bigcup \{E_x \mid x \in X\}$ ) with the projection  $p : E^0 \rightarrow X$  ( $E \rightarrow X$ ) defined as  $p(r) = x$  for all  $r \in E_x^0$  ( $r \in E_x$ ) is called bundle of  $\mathcal{S}$ .

C. If  $U \subset X$  is open,  $a \in X_U$ , we denote by  $\hat{a}$  the map  $\hat{a} : U \rightarrow E$  defined by  $\hat{a}(x) = \xi_{Ux}(a)$  for  $x \in U$ , and set  $A_U = \{\hat{a} \mid a \in X_U\}$ .

D. Let  $U \subset X$  be open. Any map  $s : U \rightarrow E$  such that  $ps = \text{identity}$  is called section over  $U$ . We say that  $s$  is bounded if there is  $a \in X_U$  such that  $\sup \{D_x(\hat{a}(x), s(x)) \mid x \in U\}$  is finite. The set of all bounded sections on  $U$  is denoted by  $\tilde{\Gamma}(U)$ . If  $s, t \in \tilde{\Gamma}(U)$ , we set  $\tilde{d}_U(s, t) = \sup \{D_x(s(x), t(x)) \mid x \in U\}$ .

**1.4. Lemma.** Under the conditions of 1.3 we have

- (a):  $\hat{a} \in \tilde{\Gamma}(U)$  for each  $a \in X_U$ , and if  $a, b \in X_U$  then  $\tilde{d}_U(\hat{a}, \hat{b}) \leq d_U(a, b)$ ;
- (b): The function  $\tilde{d}_U$  defined on  $\tilde{\Gamma}(U) \times \tilde{\Gamma}(U)$  is a metric;

thus by (a), the map  $p_U : (X_U, d_U) \rightarrow (\tilde{\Gamma}(U), \tilde{d}_U)$  which sends any  $a \in X_U$  onto  $\hat{a} \in \tilde{\Gamma}(U)$  is a contraction.

*Proof.* (a): The boundedness of  $\hat{a}$  readily follows from the definition of  $\tilde{\Gamma}(U)$ ; since for each  $x \in U$  we have  $D_x(\hat{a}(x), \hat{b}(x)) \leq d_U(a, b)$  – see the definition of  $\hat{a}(x)$  in 1.3.C and (\*) in 1.3.A, (a) follows. (b): Clearly  $\tilde{d}_U$  is a metric if it is finite; so we prove the finiteness. If  $s, t \in \tilde{\Gamma}(U)$  then  $\tilde{d}_U(s, \hat{a}) < \infty$ ,  $\tilde{d}_U(t, \hat{b}) < \infty$  for some  $a, b \in X_U$  hence  $\tilde{d}_U(s, t) \leq \tilde{d}_U(s, \hat{a}) + \tilde{d}_U(\hat{a}, \hat{b}) + \tilde{d}_U(t, \hat{b}) < \tilde{d}_U(s, \hat{a}) + d_U(a, b) + \tilde{d}_U(t, \hat{b}) < \infty$ . We are done.

Also the next lemma and its proof follow and extend those of K. H. Hofmann's [1, Prop. 3.13] from the category of Banach spaces with contractions to  $\mathfrak{MC}$ .

**1.5. Lemma.** Let  $\mathcal{S} = \{(X_U, d_U) |_{\varrho_{UV}} | X\}$  be a presheaf from  $\mathfrak{MC}$ ,  $E$  its bundle. If  $U \subset X$  is open,  $a \in X_U, \varepsilon > 0$ , let  $O(U, a, \varepsilon) = \{r \in E \mid x = p(r) \in U, D_x(r, \hat{a}(x)) < \varepsilon\}$ . Then

- (a):  $\varphi(x) = D_x(\hat{a}(x), \hat{b}(x))$  is upper semicontinuous on  $U$  for any  $a, b \in X_U$ ;
- (b):  $\mathcal{B} = \{O(U, a, \varepsilon) \mid U \subset X \text{ is open, } a \in X_U, \varepsilon > 0\}$  is a base of a topology  $t$  in  $E$  which yields in the stalks  $E_x$  the same topology  $t_x$  as  $D_x$ .

*Proof.* (a): Let  $a, b \in X_U, x \in U, \varepsilon > 0$ . As  $D_x(\hat{a}(x), \hat{b}(x)) = \lim \{d_V(\varrho_{UV}(a), \varrho_{UV}(b)) \mid V \text{ open, } x \in V \subset U\}$ , there is an open  $V \subset U$  with  $x \in V$  such that  $d_V(\varrho_{UV}(a), \varrho_{UV}(b)) < D_x(\hat{a}(x), \hat{b}(x)) + \varepsilon$ . Since for  $c = \varrho_{UV}(a), d = \varrho_{UV}(b)$  we have  $\hat{c}(z) = \hat{a}(z), \hat{d}(z) = \hat{b}(z)$  for all  $z \in V$ , we may assume  $U = V$ . Then  $D_y(\hat{a}(y), \hat{b}(y)) \leq d_U(a, b) < D_x(\hat{a}(x), \hat{b}(x)) + \varepsilon$  for all  $y \in U$  which means the upper semicontinuity of  $\varphi(y)$  at  $x$ .

(b): Given  $r \in E$  — we set  $p(r) = x$  — and its metric nbd  $N_\varepsilon = \{q \in E_x \mid D_x(r, q) < \varepsilon\}$  in  $E_x$ , then the  $D_x$  — density of  $E_x^0 = \bigcup \{\xi_{Ux}(X_U) \mid U \subset X \text{ open, } x \in U\}$  in the stalk  $p^{-1}(x) = E_x$  yields that there is an open nbd  $V \subset U$  of  $x$  and an  $a \in X_V$  with  $\hat{a}(x) = \xi_{Vx}(a) \in N_{\varepsilon/2}$ . Then  $r \in O(V, a, \varepsilon/2) \cap E_x \subset N_\varepsilon$  so  $t_x$  is finer than  $D_x$ . On the other hand, if  $r \in O(U, a, \varepsilon) \cap E_x$  then  $D_x(r, \hat{a}(x)) < \varepsilon, \varepsilon' = \varepsilon - D_x(r, \hat{a}(x)) > 0$  and  $N_{\varepsilon'} \subset O(U, a, \varepsilon) \cap E_x$  so  $t_x$  is metrisable by  $D_x$ . We have also proven that any  $r \in E$  is in a set from  $\mathcal{B}$ . All what remains to prove is that if  $O(U, a, \varepsilon) \cap O(V, b, \delta) \neq \emptyset$  then it contains a  $O(W, c, \eta)$ . So let  $r \in O(U, a, \varepsilon) \cap O(V, b, \delta)$ , let  $p(r) = x$ . Set  $\eta = \frac{1}{2} \min(\varepsilon - D_x(\hat{a}(x), r), \delta - D_x(\hat{b}(x), r))$ . The density of  $E_x^0$  in  $E_x$  yields that there is an open  $W'$  with  $x \in W' \subset U \cap V$  and a  $c \in X_{W'}$  such that  $D_x(\hat{c}(x), r) < \eta$ . As  $D_x(\hat{c}(x), \hat{a}(x)) \leq D_x(\hat{a}(x), r) + D_x(r, \hat{c}(x)) < \eta + D_x(\hat{a}(x), r) \leq \frac{1}{2}(\varepsilon - D_x(\hat{a}(x), r)) + D_x(\hat{a}(x), r) = \frac{1}{2}(\varepsilon + D_x(\hat{a}(x), r))$ , and as  $\varphi(y) = D_y(\hat{c}(y), \hat{a}(y))$  is upper semicontinuous, there is an open nbd  $W_1$  of  $x, W_1 \subset W'$  with  $\varphi(y) < \frac{1}{2}(\varepsilon + D_x(\hat{a}(x), r))$  on  $W_1$ . Likewise on an open  $W_2 \subset W'$  with  $x \in W_2$  we have for  $\psi(y) = D_y(\hat{c}(y), \hat{b}(y))$ :  $\psi(y) < \frac{1}{2}(\delta + D_x(\hat{b}(x), r))$ . For  $W = W_1 \cap W_2$  we have  $O(W, c, \eta) \subset O(U, a, \varepsilon) \cap O(V, b, \delta)$ ; indeed, for  $q \in O(W, c, \eta)$  we have — putting  $y = p(q) \in W$ :  $D_y(\hat{a}(y), q) \leq D_y(\hat{a}(y), \hat{c}(y)) + D_y(\hat{c}(y), q) < \frac{1}{2}(\varepsilon + D_x(\hat{a}(x), r)) + \eta \leq \frac{1}{2}(\varepsilon + D_x(\hat{a}(x), r)) + \frac{1}{2}(\varepsilon - D_x(\hat{a}(x), r)) = \varepsilon$  so  $q \in O(U, a, \varepsilon)$ . Likewise  $q \in O(V, b, \delta)$  which proves (b).

**1.6. Notation.** Let  $\mathcal{S} = \{(X_U, d_U) |_{\varrho_{UV}} | X\}$  be a presheaf from  $\mathfrak{MC}$ , let  $U \subset X$  be open, let  $E$  be the bundle of  $\mathcal{S}$ . If  $t$  is the topology defined in  $E$  by the set  $\mathcal{B}$  from the foregoing lemma, we denote by  $\Gamma(U)$  the set of all continuous bounded sections on  $U$ .

1.7. *Lemma.* Under the conditions of 1.6 we have

(a):  $\hat{a} \in \Gamma(U)$  for each  $a \in X_U$ ; thus the map  $p_U$  from 1.4b sends  $X_U$  into  $\Gamma(U)$  wherefore  $A_U \subset \Gamma(U)$ .

(b): If  $r, s \in \Gamma(U)$  then  $\varphi(x) = D_x(r(x), s(x))$  is upper semicontinuous on  $U$ .

*Proof.* (a): The continuity of  $\hat{a}$  follows from the definition of the topology in  $E$ ; By 1.4a,  $\hat{a}$  is bounded.

(b): If  $x \in U$ ,  $\varepsilon > 0$  then there are  $a, b \in X_U$  with  $A = D_x(\hat{a}(x), r(x)) < \varepsilon/4$ ,  $B = D_x(\hat{b}(x), s(x)) < \varepsilon/4$ . By 1.5a,  $\psi(y) = D_y(\hat{a}(y), \hat{b}(y))$  is upper semicontinuous on  $U$  so from  $\psi(x) \leq \varphi(x) + A + B < \varphi(x) + \varepsilon/2$  it follows  $\psi(y) \leq \varphi(x) + \varepsilon/2$  on an open nbd  $V \subset U$  of  $x$ . As  $M = \{z \in E \mid p(z) \in V, D_z(\hat{a}(p(z)), z) < \varepsilon/4\}$  resp.  $N = \{z \in E \mid p(z) \in V, D_z(\hat{b}(p(z)), z) < \varepsilon/4\}$  are nbds of  $r(x)$  resp.  $s(x)$ , there is an open  $W \subset V$  with  $x \in W$  such that  $r(z) \in M$ ,  $s(z) \in N$  for  $z \in W$  as  $r, s$  are continuous. Then for  $y \in W$  we have  $\varphi(y) \leq \psi(y) + D_y(\hat{a}(y), r(y)) + D_y(\hat{b}(y), s(y)) < \varphi(x) + \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varphi(x) + \varepsilon$ .

Extending K. H. Hofmann's proof from the category of Banach spaces with contractions [1, Prop. 3.22] to  $\mathfrak{MC}$  we get

1.8. *Lemma.* Let  $\mathcal{S} = \{(X_U, d_U) \mid \varrho_{UV} \mid X\}$  be from  $\mathfrak{MC}$ . TFAE:

(1): If  $U \subset X$  is open  $a, b \in X_U$  and if  $\mathcal{V}$  is an open cover of  $U$  then  $d_U(a, b) = \sup \{d_V(\varrho_{UV}(a), \varrho_{UV}(b)) \mid V \in \mathcal{V}\}$ ;

(2): Given an open  $U \subset X$ ,  $a, b \in X_U$ , an open cover  $\mathcal{V}$  of  $U$  and  $\varepsilon > 0$ , then there is  $V \in \mathcal{V}$  such that  $d_V(\varrho_{UV}(a), \varrho_{UV}(b)) > d_U(a, b) - \varepsilon$ ;

(3): The natural map  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  is an isometry into  $\Gamma(U)$  for any open  $U \subset X$  (see 1.7a).

*Proof.* (1)  $\Rightarrow$  (2) is clear. Let  $a, b \in X_U$  and let (2) hold. By 1.4a,  $\tilde{d}_U(\hat{a}, \hat{b}) \leq d_U(a, b)$ . If  $<$  held, then there would be  $\tilde{d}_U(\hat{a}, \hat{b}) < c < d_U(a, b)$ ; it means  $D_x(\hat{a}(x), \hat{b}(x)) < c$  for any  $x \in U$ . By (\*) in 1.3, for every  $x \in U$  there is an open nbd  $V_x \subset U$  of  $x$  such that  $d_{V_x}(\varrho_{UV_x}(a), \varrho_{UV_x}(b)) < c$ . Then  $\mathcal{V} = \{V_x \mid x \in U\}$  is an open cover of  $U$  and  $\sup \{d_V(\varrho_{UV}(a), \varrho_{UV}(b)) \mid V \in \mathcal{V}\} \leq c$  which contradicts to (2) whereby (2)  $\Rightarrow$  (3) is proven. Let (3) hold; given  $a, b \in X_U$  and an open cover  $\mathcal{V}$  of  $U$ , we have – setting  $a_V = \varrho_{UV}(a)$ ,  $b_V = \varrho_{UV}(b)$  :  $d_U(a, b) = \tilde{d}_U(\hat{a}, \hat{b}) = \sup \{D_x(\hat{a}(x), \hat{b}(x)) \mid x \in U\} = \sup \{\sup \{D_x(\hat{a}_V(x), \hat{b}_V(x)) \mid x \in V\} \mid V \in \mathcal{V}\} = \sup \{\tilde{d}_V(\hat{a}_V, \hat{b}_V) \mid V \in \mathcal{V}\} \leq \sup \{d_V(a_V, b_V) \mid V \in \mathcal{V}\} \leq d_U(a, b)$  so  $d_U(a, b) = \sup \{d_V(a_V, b_V) \mid V \in \mathcal{V}\}$ , hence (1) holds.

Following K. H. Hofmann we define

1.9. *Definition.*  $\mathcal{S}$  is called monopresheaf if it fulfils any of the conditions 1–3 of 1.8. Thus we have

1.10. *Theorem.* Let  $\mathcal{S} = \{(X_U, d_U) |_{\varrho_{UV}} X\}$  be a monopresheaf from  $\mathfrak{MC}$ . Then for any open  $U \subset X$  the natural map  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  is an isometry into  $\Gamma(U)$ .

1.11. *Definition.* A presheaf  $\mathcal{S} = \{(X_U, d_U) |_{\varrho_{UV}} X\}$  from  $\mathfrak{M}$  is called sheaf if it fulfils the following for any open  $U \subset X$ :

COND 1: If  $a, b \in X_U$  and if for an open cover  $\mathcal{V}$  of  $U$  we have  $\varrho_{UV}(a) = \varrho_{UV}(b)$  for all  $V \in \mathcal{V}$  then  $a = b$ .

COND 2: a) Given an open cover  $\mathcal{V}$  of  $U$  and a family  $\mathcal{F}_{\mathcal{V}} = \{a_V \in X_V \mid V \in \mathcal{V}\}$  such that  $\varrho_{V \cap W}(a_V) = \varrho_{W \cap V}(a_W)$  whenever  $V \cap W \neq \emptyset$  – we call such a family smooth – then there is an  $a \in X_U$  with  $\varrho_{UV}(a) = a_V$  for all  $V \in \mathcal{V}$ ;

b) If  $\mathcal{G}_{\mathcal{V}} = \{b_V \in X_V \mid V \in \mathcal{V}\}$  is another smooth family and  $b \in X_U$  such that  $\varrho_{UV}(b) = b_V$  for all  $V \in \mathcal{V}$  then  $d_U(a, b) = \sup \{d_V(a_V, b_V) \mid V \in \mathcal{V}\}$ .

1.12. *Remark.* It readily follows from COND 2,b that every sheaf is a monopresheaf. Further, it is easy to see that COND 1 is equivalent to the 1-1 ness of the map  $p_U : X_U \rightarrow \Gamma(U)$ . Also the element  $a \in X_U$  determined by  $\mathcal{F}_{\mathcal{V}}$  in COND 2a is unique because of COND 1.

## 2. $C(Y, P) - K - \text{areas}$

2.1. *Definition.* Let us recall that a semigroup is a pair  $(S, \vee)$  where  $S$  is a set and  $\vee : S \times S \rightarrow S$  is a map such that for all  $a, b, c \in S$  we have  $(a \vee b) \vee c = a \vee (b \vee c)$ ; it is called commutative if  $a \vee b = b \vee a$ ; it is called an upper semilattice if it is commutative and  $a \vee a = a$ .

Let  $\odot$  be a commutative semigroup operation in the set  $R$  of real numbers such that

- (a)  $\odot : R \times R \rightarrow R$  is continuous,
- (b)  $x \odot y \geq x' \odot y'$  if  $x \leq x', y \leq y'$ ,
- (c)  $0 \odot 0 = 0$ .

A metric  $\odot$  – faithful semigroup (upper semilattice) is defined to be a triple  $(S, d, \vee)$ , where  $S$  is a set,  $d$  is a metric on  $S$  and  $\vee$  is a commutative semigroup (upper semilattice) operation on  $S$ , such that for any  $a, b, x, y \in S$

$$(d) \quad d(a \vee b, x \vee y) \leq d(a, x) \odot d(b, y).$$

The usual addition in the reals shall be denoted by  $+$ , the usual upper semilattice operation by  $\vee_R$  meaning  $x \vee_R y = x$  iff  $y \leq x$ .

2.2. *Example. A.* Let  $(X, d)$  be a metric linear space with a translation – invariant metric  $d$ , let  $\ddagger$  be the addition in  $X$ . Then  $(X, d, \ddagger)$  is a metric  $+$  – faithful semigroup.



B. Let  $(X, d)$  be a metric space ordered by  $\leq$ . Then  $X$  becomes an upper semilattice with the operation  $\vee$  defined as “ $x, y \in X$  then  $x \vee y$  iff  $y \leq x$ .” It is easy to check that if  $\leq$  fulfils the condition C: “If  $x, y, z \in X$ ,  $x \leq y \leq z$  then  $d(x, y) \vee_R d(y, z) \leq d(x, z)$ ”, then  $(X, d, \vee)$  is an  $\vee_R$  – faithful upper semilattice.

2.3. *Definition.* A group upper semilattice is a commutative group  $(G, \dot{+})$  which is an upper semilattice with the operation  $\vee$  such that for any  $x, y, z \in G$

$$(*) \quad (x \vee y) \dot{+} z = (x \dot{+} z) \vee (y \dot{+} z).$$

It shall be denoted by  $(G, +, \vee)$ . We shall often write  $+$  instead of  $\dot{+}$  where it cannot lead to a misunderstanding.

2.4. *Lemma.* Given a group upper semilattice  $(G, +, \vee)$  then we have for any  $x, y \in G$  (denoting by  $0$  the neutral element of  $G$  and by  $-a$  the inverse element of  $a \in G$ ):  $x + y = x \vee y - ((-x) \vee (-y))$ , hence setting  $x^+ = x \vee 0$ ,  $x^- = (-x) \vee 0$ , we have  $x = x^+ - x^-$ .

*Proof.* From  $(*)$  we get  $x \vee y - ((-x) \vee (-y)) = x + 0 \vee (y - x) - ((-y) + (y - x) \vee 0) = x + y$  as  $\vee$  is commutative.

2.5. *Definition.* The set of all continuous functions on a topological space  $Y$  with values in the interval  $P = \langle -1, 1 \rangle$  ( $Q = \langle 0, 1 \rangle$ ) is denoted by  $C(Y, P)$  ( $C(Y, Q)$ ). The set of all constant functions from  $C(Y, P)$  ( $C(Y, Q)$ ) is denoted by  $P(Q)$ .

A  $C(Y, P)$  – area is a structure  $(X, d, +, \vee, \circ)$  where  $X$  is a set,  $d$  is a metric on  $X$ ,  $+$  is a commutative group operation on  $X$ ,  $\vee$  is an upper semilattice operation on  $X$ , and  $\circ : C(Y, P) \times X \rightarrow X$  is a map such that

A:  $(X, d, \vee)$  is a  $\vee_R$  – faithful upper semilattice, i.e.

$$(1) \quad d(x \vee y, u \vee v) \leq d(x, u) \vee_R d(y, v) \text{ for any } x, y, u, v \in X.$$

B:  $(X, d, +)$  is a metric group meaning

$$(2) \quad d(x + y, u + v) \leq d(x, u) + d(y, v) \text{ for any } x, y, u, v \in X.$$

C:  $(X, +, \vee)$  is a group upper semilattice, i.e.

$$(3) \quad (x \vee y) + z = (x + z) \vee (y + z) \text{ for any } x, y, z.$$

D: The map  $\circ : C(Y, P) \times X \rightarrow X$  sending  $(f, x) \in C(Y, P) \times X$  onto  $f \circ x$  fulfils the conditions below for every  $x, y \in X$ ,  $f, g \in C(Y, P)$ , and any constant functions  $c_1, c_2 \in C(Y, Q)$ :

$$(4) \quad 1 \circ x = x,$$

$$(5) \quad (c_1 \vee_R c_2) \circ x = (c_1 \circ x) \vee (c_2 \circ x) \text{ for any } x \in X^+, \text{ where } X^+ = \{x^+ = x \vee 0 \mid x \in X\} - \text{see 2.4,}$$

$$(6) \quad (f + g) \circ x = f \circ x + g \circ x \text{ whenever } f + g \in C(Y, P),$$

$$(7) \quad d(f \circ x, f \circ y) \leq d(x, y).$$

We shall often write  $fx$  instead of  $f \circ x$ , for short.

A map  $F : (X_1, d_1, +_1, \vee_1, \circ_1) \rightarrow (X_2, d_2, +_2, \vee_2, \circ_2)$  between two  $C(Y, P)$  – areas is called an  $A$  – homomorphism if for all  $x, y \in X_1, f \in C(Y, P)$

- (1')  $F(x +_1 y) = F(x) +_2 F(y)$ ,
- (2')  $F(x \vee_1 y) = F(x) \vee_2 F(y)$ ,
- (3')  $F(f \circ_1 x) = f \circ_2 F(x)$ .

The category of all  $C(Y, P)$  – areas (metric – complete ones) with the contractive  $A$  – homomorphisms shall be denoted by  $\mathfrak{A}_Y\mathfrak{M}(\mathfrak{A}_Y\mathfrak{M}\mathfrak{C})$ . Where possible, the index  $Y$  shall be left out.

If instead of (7) we assume only

(7') There is a constant  $K$  such that  $d(fx, fy) \leq Kd(x, y)$  for all  $x, y \in X, f \in C(Y, P)$ , then  $(X, d, +, \vee, \circ)$  is called a  $C(Y, P) – K$  – area. The category of all those – with a fixed  $K$  – is denoted by  $\mathfrak{A}_Y\mathfrak{M}(K) (\mathfrak{A}_Y\mathfrak{M}\mathfrak{C}(K))$ .

2.6. Remark. A. Let  $(X, +)$  be a commutative group such that there is a map of  $L = C(Y, P) \times X \rightarrow X$  sending  $(f, x) \in L$  onto  $fx$ . Denoting by  $0$  the neutral element of  $X$  and by  $-a$  the inverse element of  $a \in X$ , then (1)  $\Leftrightarrow$  (3)  $\Rightarrow$  (2)  $\Rightarrow$  (4) below, for any  $f, g \in C(Y, P), x \in X$ .

- (1)  $(f - g)x = fx - gx$  whenever  $f - g \in C(Y, P)$ ,
- (2)  $(-g)x = -(gx)$ ,
- (3)  $(f + g)x = fx + gx$  whenever  $f + g \in C(Y, P)$ ,
- (4)  $0x = 0$ .

B. Let a metric group  $(X, d, +)$  with the addition  $+$  be partially ordered by  $\leq$  meaning that for all  $x, y, z, u, v \in X$

- A' (1')  $x + z \leq y + z$  if  $x \leq y$ ,
- B': (2')  $d(x + y, u + v) \leq d(x, u) + d(y, v)$ ,

such that for any  $x, y \in X$  there is their least upper bound  $x \vee y$  and such that this upper semilattice operation  $\vee$  makes  $(X, d, \vee)$  into an  $\vee_R$  – faithful upper semilattice (by 2.2B, it is enough that  $\leq$  be an order, and that  $d, \leq$  fulfil the condition C of 2.2B). Let  $Y$  be a topological space, let  $C(Y)$  be the set of all continuous bounded functions on  $Y$  and let  $(X, d, +)$  be a  $C(Y)$  module meaning that

D': There is a map  $\circ : C(Y) \times X \rightarrow X$  sending  $(f, x) \in C(Y) \times X$  onto  $fx$  such that for every  $x, y \in X, f \in C(Y)$  and any constants  $c_1, c_2 \in \mathcal{Q}, c_1 \leq c_2$  we have

- (4')  $1x = x$ ,
- (5'')  $c_1x \leq c_2x$  if  $x \in X^+$ ,
- (6')  $(f + g)x = fx + gx$ ,
- (7')  $d(fx, fy) \leq d(x, y)$  if  $f \in C(Y, P)$ ,

then (3'), (5') below hold for any  $x, y, z \in X$  and any  $c_1, c_2 \in Q$ ,  $c_1 \leq c_2$ , wherefore  $(X, d, +, \vee, \circ)$  is a  $C(Y, P)$  - area.

$$C'. (3') (x \vee y) + z = (x + z) \vee (y + z),$$

$$(5') (c_1 \vee_R c_2) x = (c_1 x) \vee (c_2 x) \text{ if } x \in X^+.$$

If instead of (7') we have (7''): There is a constant  $K$  with  $d(fx, fy) \leq Kd(x, y)$  for all  $x, y \in X, f \in C(Y, P)$ , then  $(X, d, +, \vee, \circ)$  is a  $C(Y, P) - K$  area.

*Addition.* We have  $(5''') \Rightarrow (5'') \Leftrightarrow (5')$  for  $(5''')$ : For all  $x, y \in X$  with  $x \leq y$  and any  $c, d \in Q$  we have (a):  $cx \leq x$  if  $x \in X^+$ ; (b):  $cx \leq cy$ ; (c):  $c(dx) = (cd)x$ .

*Proof.* A. Let (3) hold. Then  $0x = (0 + 0)x = 0x + 0x$  hence  $0x = 0$  and (4) holds. Further, setting  $g = -f$  in (3) we get by (4)  $0 = 0x = fx + (-f)x$  hence (2) holds. Furthermore,  $(f - g)x = (f + (-g))x = fx + (-g)x = fx - gx$  by (2), hence (1) holds. If (1) holds, we put  $f = g$  and get  $0x = 0$  so (4) holds. Putting  $f = 0$  in (1), we get  $(-g)x = 0x - gx = -gx$ , hence (2) holds. Further,  $(f + g)x = (f - (-g))x = fx - (-g)x = fx + gx$  hence (3) holds. If (2) holds, we set  $g = 0$  and get  $0x = -0x$  hence  $0x = 0$ .

B. It is easy to check C'. We check (5'): If  $c_1, c_2 \in Q, x \in X^+$ , then (5'') yields  $(c_1 \vee_R c_2)x \geq c_j x, j = 1, 2$ , so  $(c_1 \vee_R c_2)x \geq (c_1 x) \vee (c_2 x)$ . If  $u \in X, u \geq c_j x, j = 1, 2$ , then  $u \geq (c_1 \vee_R c_2)x$  for  $c_1 \vee_R c_2$  is one of  $c_1, c_2$ , whence (5') holds. It remains to check the Addition. If (5'') holds,  $x \in X^+, 0 \leq c_1 \leq c_2 \leq 1$ , then  $(c_1 x) \vee (c_2 x) = (c_1 \vee_R c_2)x = c_2 x$  hence  $c_1 x \leq c_2 x$  and (5') holds. If (5''') holds,  $x \in X^+, 0 \leq c_1 < c_2 \leq 1$  then  $0 \leq c_1/c_2 \leq 1$ ; by (a),  $(c_1/c_2)x \leq x$  and by (b, c)  $c_1 x = (c_2(c_1/c_2))x = c_2((c_1/c_2)x) \leq c_2 x$  and (5'') holds.

**2.7. Proposition.** If  $(X, d, +, \vee, 0)$  is a  $C(Y, P) - K -$  area, then there is an extension  $+\hat{\phantom{+}}, \vee\hat{\phantom{\vee}}, \text{ and } \circ\hat{\phantom{\circ}}$  of  $+, \vee,$  and  $\circ$  to the completion  $(\hat{X}, \hat{d})$  of  $(X, d)$  such that  $(\hat{X}, \hat{d}, +\hat{\phantom{+}}, \vee\hat{\phantom{\vee}}, \circ\hat{\phantom{\circ}})$  is a  $C(Y, P) - K -$  area. Further, if  $+\tilde{\phantom{+}}, \vee\tilde{\phantom{\vee}}, \text{ and } \circ\tilde{\phantom{\circ}}$  is another extension of  $+, \vee, \circ$  to  $(\hat{X}, \hat{d})$  such that  $(\hat{X}, \hat{d}, +\tilde{\phantom{+}}, \vee\tilde{\phantom{\vee}}, \circ\tilde{\phantom{\circ}})$  is a  $C(Y, P) - K -$  area, then  $+\tilde{\phantom{+}} = +\hat{\phantom{+}}, \vee\tilde{\phantom{\vee}} = \vee\hat{\phantom{\vee}}, \circ\tilde{\phantom{\circ}} = \circ\hat{\phantom{\circ}}$ . If  $f: (X, d, +, \vee, \circ) \rightarrow (X_1, d_1, +_1, \vee_1, \circ_1)$  is a contractive  $A -$  homomorphism where  $(X_1, d_1)$  is complete, then there is a unique extension  $\hat{f}: (\hat{X}, \hat{d}, +\hat{\phantom{+}}, \vee\hat{\phantom{\vee}}, \circ\hat{\phantom{\circ}}) \rightarrow (X_1, d_1, +_1, \vee_1, \circ_1)$  of  $f$  which is an  $A -$  homomorphism.

*Proof.* Let  $x, y \in \hat{X}, \{x_n\}, \{y_n\} \subset X, x_n \rightarrow x, y_n \rightarrow y$ . By 2.5(1),  $d(x_n \vee y_n, x_m \vee y_m) \leq d(x_n, x_m) \vee_R d(y_n, y_m)$  so  $\{x_n \vee y_n\}$  is Cauchy. We denote its limit in  $\hat{X}$  by  $x \vee\hat{\phantom{\vee}} y$ . If  $\{x'_n\}, \{y'_n\} \subset X$  are some other sequences tending to  $x, y$  then  $d(x_n \vee y_n, x'_n \vee y'_n) \leq d(x_n, x'_n) \vee_R d(y_n, y'_n) \rightarrow 0$  so  $x \vee\hat{\phantom{\vee}} y$  does not depend on the sequences. If  $x, y, z \in \hat{X}, \{x_n\}, \{y_n\}, \{z_n\} \subset X$  tending to them then  $x \vee\hat{\phantom{\vee}} \vee\hat{\phantom{\vee}} (y \vee\hat{\phantom{\vee}} z) = \lim x_n \vee (y_n \vee z_n) = \lim (x_n \vee y_n) \vee z_n = (x \vee\hat{\phantom{\vee}} y) \vee\hat{\phantom{\vee}} z$  and

likewise commutativity can be proven. Further,  $x \vee \wedge x = \lim x_n \vee x_n = \lim x_n = x$  wherefore  $\vee \wedge$  is an upper semilattice operation.

We show that  $(\hat{X}, \hat{d}, \vee \wedge)$  is  $\vee_R$ -faithful. Indeed, if  $x, y, u, v \in \hat{X}$ ,  $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\} \subset X$  tending to them, then by 2.5(1)  $\hat{d}(x \vee \wedge y, u \vee \wedge v) = \lim d(x_n \vee y_n, u_n \vee v_n) \leq \lim d(x_n, u_n) \vee_R d(y_n, v_n) = \lim d(x_n, u_n) \vee_R \lim d(y_n, v_n) = \hat{d}(x, y) \vee_R \hat{d}(u, v)$  as desired. Thus  $(\hat{X}, \hat{d}, \vee \wedge)$  is an  $\vee_R$ -faithful upper semilattice. Let  $\vee \sim$  be another extension of  $\vee$  to  $\hat{X}$  such that  $(\hat{X}, \hat{d}, \vee \sim)$  is a  $\vee_R$ -faithful upper semilattice. Taking the metric  $d_1$  on  $X \times X$  such that  $d_1((a, b), (c, d)) = d(a, c) + d(b, d)$ , we can easily check that  $\hat{X} \times \hat{X}$  with the metric  $d_2$  defined as  $d_2((x, y), (u, v)) = \hat{d}(x, u) + \hat{d}(y, v)$  is a completion of  $(X \times X, d_1)$ . By 2.5(1)  $\vee : (X \times X, d_1) \rightarrow (\hat{X}, \hat{d})$  is uniformly continuous, thus there is a unique uniformly continuous extension  $\vee^- : (\hat{X} \times \hat{X}, d_2) \rightarrow (\hat{X}, \hat{d})$  of  $\vee$ . As by 2.5(1), both  $\vee \wedge, \vee \sim : (\hat{X} \times \hat{X}, d_2) \rightarrow (\hat{X}, \hat{d})$  are uniformly continuous, they are equal one to another, being both equal to  $\vee$  on  $X \times X$ . We have thus uniquely extended  $\vee$  to  $(\hat{X}, \hat{d})$ , having used 2.5(1). Likewise we can uniquely extend  $+$  to  $(\hat{X}, \hat{d})$  using 2.5(2) and show that  $\vee \wedge, + \wedge$  fulfil 2.5(3).

By 2.5(7), if  $f \in C(Y, P)$  then the map  $n_f : (X, d) \rightarrow (X, d)$  sending  $x \in X$  onto  $fx$  is uniformly continuous so there is a unique extension  $\hat{n}_f$  of  $n_f$  to the whole of  $\hat{X}$ . If  $f = 1$  then  $n_f$  is identical hence so is  $\hat{n}_f$  so  $1x = x$  for all  $x \in \hat{X}$ . If  $c_1, c_2 \in \mathcal{Q}, x \in \hat{X}^+$  then there is  $\{x_n\} \subset X^+$  tending to  $x$  (we have  $x = y \vee 0$  for a  $y \in \hat{X}$ ; if  $\{y_n\} \subset X, y_n \rightarrow y$ , we set  $x_n = y_n \vee 0$ ); then for each  $n$  we have by 2.5(5)  $(c_1 \vee_R c_2) x_n = (c_1 x_n) \vee \wedge \vee \wedge (c_2 x_n)$ , and passing to limits we get  $(c_1 \vee_R c_2) x = (c_1 x) \vee \wedge \vee \wedge (c_2 x)$  as  $\vee \wedge$  is continuous. If  $f, g, f + g \in C(Y, P)$ , then  $n_{f+g} = n_f + n_g$  on  $X$  so  $(f + g)x = \hat{n}_{f+g}(x) = \hat{n}_f(x) + \hat{n}_g(x) = fx + gx$  for  $x \in \hat{X}$ . Finally if  $x, y \in \hat{X}, f \in C(Y, P)$ , then  $\hat{d}(fx, fy) = \lim d(fx_n, fy_n) \leq K \lim d(x_n, y_n) = K\hat{d}(x, y)$  whenever  $\{x_n\}, \{y_n\} \subset X$  tend to  $x, y$ . Let a map  $m : C(Y, P) \times (\hat{X}, \hat{d}) \rightarrow (\hat{X}, \hat{d})$  fulfil 2.5(7). Then for any fixed  $f$  the map  $m(f, \cdot) : (\hat{X}, \hat{d}) \rightarrow (\hat{X}, \hat{d})$  is uniformly continuous and hence if  $m(f, x) = n_f(x) = fx$  for all  $x \in X$ , then  $m(f, x) = \hat{n}_f(x) = fx$  on  $\hat{X}$ , hence  $\circ \wedge$  is unique. If  $f : (X, d, +, \vee, \circ) \rightarrow (X_1, d_1, +_1, \vee_1, \circ_1)$  is a contractive map then it is uniformly continuous so there is its extension  $\hat{f}$  to  $\hat{X}$  which is unique and easy to be shown to be an  $A$ -homomorphism if so is  $f$ .

**2.8. Proposition.** For a fixed  $K$ , the category  $\mathfrak{U}_Y \mathfrak{M}(K)$  ( $\mathfrak{U}_Y \mathfrak{MC}(K)$ ) of all  $C(Y, P) - K$ -areas (metric complete ones), with the contractive  $A$ -homomorphisms as morphisms is inductive.

*Proof.* Fix  $Y, K$ . Let  $\mathcal{S} = \{(X_\alpha, d_\alpha, +_\alpha, \vee_\alpha, \circ_\alpha) | \mathcal{Q}_{\alpha\beta} | \langle A \leq \rangle\}$  be a presheaf from  $\mathfrak{U}_Y \mathfrak{M}(K)$ . Then  $\mathcal{S}_1 = \{(X_\alpha, d_\alpha) | \mathcal{Q}_{\alpha\beta} | \langle A \leq \rangle\}$  is from  $\mathfrak{M}$ ; let  $\langle (I^0, D) | \{\xi_\alpha | \alpha \in A\} \rangle = \varinjlim \mathcal{S}_1$  in  $\mathfrak{M}$  (see 1.2), let  $p, q \in I^0$ , let  $a, b \in X_\alpha$  be some representatives of  $p, q$  in an  $X_\alpha$ , and let  $p \vee q$  be the element of  $I^0$  represented by  $a \vee_\alpha b$ . If  $c, d \in X_\beta$  represent  $p, q$ , too, then for  $r$  represented by  $c \vee_\beta d$  we have by (\*) in 1.2B and by 2.5(1), for  $\gamma \geq \alpha, \beta$  and for  $a_\gamma = \mathcal{Q}_{\alpha\gamma}(a), b_\gamma = \mathcal{Q}_{\alpha\gamma}(b), c_\gamma = \mathcal{Q}_{\beta\gamma}(c), d_\gamma = \mathcal{Q}_{\beta\gamma}(d)$ :

$: D(p \vee q, r) = \lim \{d_\gamma(a_\gamma \vee_\gamma b_\gamma, c_\gamma \vee_\gamma d_\gamma) \mid \gamma \geq \alpha, \beta\} \leq \lim \{d_\gamma(a_\gamma, c_\gamma) \mid \gamma \geq \alpha, \beta\} \vee_R$   
 $\vee_R \{d_\gamma(b_\gamma, d_\gamma) \mid \gamma \geq \alpha, \beta\} = \lim \{d_\gamma(a_\gamma, c_\gamma) \mid \gamma \geq \alpha\} \vee_R \lim \{d_\gamma(b_\gamma, d_\gamma) \mid \gamma \geq \alpha\} =$   
 $= D(p, p) \vee_R D(q, q) = 0$  hence  $p \vee q$  does not depend on the choice of  $X_\alpha$  and of  
the representatives (recall that  $F(\gamma) = d_\gamma(a_\gamma, c_\gamma)$ ,  $G(\gamma) = d_\gamma(b_\gamma, d_\gamma)$  are nonincreasing).  
If  $p, q, r \in I^0$ ,  $a, b, c \in X_\alpha$  their representatives, then  $(p \vee q) \vee r = \xi_\alpha((a \vee_\alpha b) \vee_\alpha$   
 $\vee_\alpha c) = \xi_\alpha(a \vee_\alpha (b \vee_\alpha c)) = p \vee (q \vee r)$ . Likewise we can show that  $p \vee p = p$   
and commutativity, whence  $\vee$  is an upper semilattice operation. We show that  
 $(I^0, D, \vee)$  is  $\vee_R$ -faithful. If  $p, q, r, s \in I^0$ ,  $a, b, c, d \in X_\alpha$  their representatives,  
then by (\*) in 1.2 and by 2.5(1),  $D(p \vee q, r \vee s) = \lim \{d_\beta(\varrho_{\alpha\beta}(a \vee_\alpha b),$   
 $\varrho_{\alpha\beta}(c \vee_\alpha d)) \mid \beta \geq \alpha\} = \lim \{d_\beta(\varrho_{\alpha\beta}(a) \vee_\beta \varrho_{\alpha\beta}(b), \varrho_{\alpha\beta}(c) \vee_\beta \varrho_{\alpha\beta}(d)) \mid \beta \geq \alpha\} \leq$   
 $\leq \lim \{d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(c)) \vee_R d_\beta(\varrho_{\alpha\beta}(b), \varrho_{\alpha\beta}(d)) \mid \beta \geq \alpha\} = \lim \{d_\beta(\varrho_{\alpha\beta}(a), \varrho_{\alpha\beta}(c))$   
 $\cdot \mid \beta \geq \alpha\} \vee_R \lim \{d_\beta(\varrho_{\alpha\beta}(b), \varrho_{\alpha\beta}(d)) \mid \beta \geq \alpha\} = d(p, r) \vee_R d(q, s)$  as desired. Thus  
 $(I^0, D, \vee)$  is a  $\vee_R$ -faithful upper semilattice. In likewise natural way addition can  
be brought over to  $I^0$  with the help of 2.5(2) such that, having denoted it by  $+$ ,  
 $(I^0, D, +)$  is a metric group (i.e. 2.5(2) holds), and that 2.5(3) is fulfilled. Given  
 $p \in I^0$ ,  $a \in X_\alpha$  its representative,  $f \in C(Y, P)$ , we set  $f \circ_\alpha p$  to be the element represented  
by  $f \circ_\alpha a$ . Likewise as we have proven that  $p \vee q$  does not depend on the represen-  
tatives, we can prove that  $f \circ p$  does not either. We show that 2.5(5) holds for  $\circ$ .  
If  $c_1, c_2 \in \mathcal{Q}$ ,  $p \in (I^0)^+$  then  $p = q \vee 0$  for a  $q \in I^0$  (see 2.5(5), 2.4). If  $a \in X_\alpha$  repre-  
sents  $q$  then  $a \vee 0 = a^+ \in X_\alpha^+$  represents  $p$  and by 2.5(5) we have  $(c_1 \vee_R c_2) \circ_\alpha a =$   
 $= (c_1 \circ_\alpha a) \vee_\alpha (c_2 \circ_\alpha a)$  hence  $(c_1 \vee_R c_2) p = (c_1 p) \vee (c_2 p)$  as desired. The veri-  
fication of the other conditions of 2.5D is easier still. As the  $\varrho_{\alpha\beta}$ 's are  $A$ -homo-  
morphisms, we can easily check that  $(I^0, D, +, \vee, 0)$  has the required properties  
of inductive limits.

If  $\mathcal{S}$  is from  $\mathfrak{U}_Y\mathfrak{MC}(K)$ , our statement easily follows from 2.7 as the inductive  
limit of  $\mathcal{S}$  in  $\mathfrak{U}_Y\mathfrak{MC}(K)$  is just the completion of that in  $\mathfrak{U}_Y\mathfrak{M}(K)$  with the operations  
extended by 2.7. The proof is thereby finished.

**2.9. Corollary.** Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) \varrho_{UV} \mid X\}$  be a presheaf from  
 $\mathfrak{U}_Y\mathfrak{MC}(K)$  over a topological space  $X$ , let  $E$  be its bundle. Then

(a): For every  $x \in X$  the stalk  $E_x$  over  $x$  is an  $C(Y, P) - K$ -area with the opera-  
tions  $+_x, \vee_x, \circ_x$  defined as the natural bringover of these from  $\mathcal{S}_x$  (see 2.8, 2.7).

(b): If  $U \subset X$  is open, then the set  $\tilde{\Gamma}(U)$  of all bounded sections over  $U$  in  $E$   
with its natural metric  $\tilde{d}_U$  (see 1.3D) and with the operations  $+_{\tilde{U}}, \vee_{\tilde{U}}, \circ_{\tilde{U}}$  pointwise  
defined by  $(r \vee_{\tilde{U}} s)(x) = r(x) \vee_x s(x)$  for  $x \in U$  - and likewise for  $+_{\tilde{U}}, \circ_{\tilde{U}}$  - is  
a  $C(Y, P) - K$ -area.

*Proof.* (a) readily follows from 2.7, 2.8 (b): Let  $r, s, u, v \in \tilde{\Gamma}(U)$ . Then by 2.1b,  
 $\tilde{d}_U(r \vee_{\tilde{U}} s, u \vee_{\tilde{U}} v) = \sup \{D_x(r(x) \vee_x s(x), u(x) \vee_x v(x)) \mid x \in U\} \leq \sup \{D_x(r(x),$   
 $u(x)) \vee_R D_x(v(x), s(x)) \mid x \in U\} \leq \sup \{D_x(r(x), u(x)) \mid x \in U\} \vee_R \sup \{D_x(v(x),$   
 $s(x)) \mid x \in U\} = \tilde{d}_U(r, u) \vee_R \tilde{d}_U(v, s)$ . Likewise we verify the other requirements of  
2.5, and (b) is done.

2.10. *Proposition.* Under the conditions of 2.9, (a): We can define the operations  $\vee, +$  in  $E$  stalkwise.

More precisely, if  $p : E \rightarrow X$  is the natural projection (see 1.3B), we denote by  $E \times_X E = \{(r, s) \in E \times E \mid p(r) = p(s)\}$  the pullback of  $E \times E$  over  $X$ . If  $(r, s) \in E \times_X E$ ,  $x = p(r) = p(s)$ , we set  $r \vee s = r \vee_x s$ ,  $r + s = r +_x s$ , and get so two maps  $\vee, + : E \times_X E \rightarrow E$ . Let  $t$  be the natural topology defined in  $E$  by 1.5b. Then, under this topology,  $\vee, +$  are continuous.

(b): The set  $\Gamma(U)$  of all continuous bounded sections over  $U$  is closed under the operations  $\vee \sim, + \sim$  meaning that  $r \vee \sim s, r + \sim s \in \Gamma(U)$  if  $r, s \in \Gamma(U)$ .

(c): The natural map  $p_U : (X_U, d_U, +_U, \vee_U, \circ_U) \rightarrow (\Gamma(U), \tilde{d}_U, +_{\tilde{U}}, \vee_{\tilde{U}}, \circ_{\tilde{U}})$  (see 1.4b, 1.7a) is an  $A$  – homomorphism.

*Proof.* Given  $(\alpha_0, \beta_0) \in E \times_X E$  and its  $t$ -nbd  $O = O(\hat{a}, U, \varepsilon)$  (see 1.3B). Setting  $p(\alpha_0) = x$ , we have  $\varepsilon - D_x(\hat{a}(x), \alpha_0 \vee \beta_0) > 0$  so  $0 < \varepsilon_1 = \frac{1}{4}(\varepsilon - D_x(\hat{a}(x), \alpha_0 \vee \beta_0)) + D_x(\hat{a}(x), \alpha_0 \vee \beta_0) = \varepsilon/4 + \frac{3}{4}D_x(\hat{a}(x), \alpha_0 \vee \beta_0) < \varepsilon$  and  $D_x(\hat{a}(x), \alpha_0 \vee \beta_0) < \varepsilon_1 < \varepsilon$ . By 1.5b,  $t$  yields the same topology in  $p^{-1}(x) = E_x$  as  $D_x$ , and  $\vee : (E_x, D_x) \times (E_x, D_x) \rightarrow (E_x, D_x)$  is continuous as it coincides with  $\vee_x$ , which is by 2.5(1) continuous. There thus is  $O_1 = O(\hat{b}, V, \delta)$  with  $\alpha_0 \in O_1, 0 < \delta \leq \frac{1}{2}(\varepsilon - \varepsilon_1)$  and  $O_2 = O(\hat{c}, W, \eta)$  with  $\beta_0 \in O_2, 0 < \eta \leq \frac{1}{2}(\varepsilon - \varepsilon_1)$ , such that for any  $p \in E_x \cap O_1, q \in E_x \cap O_2$  we have  $p \vee q \in O(\hat{a}, U, \varepsilon_1)$  namely  $D_x(\hat{b}(x) \vee \hat{c}(x), \hat{a}(x)) < \varepsilon_1$ . The upper semicontinuity of  $\varphi(y) = D_y(\hat{e}(y), f(y))$  on  $U$  for any open  $U \subset X$  and any  $e, f \in X_U$  (see 1.5a) yields  $D_y(\hat{a}(y), (b \vee c)^\wedge(y)) = D_y(\hat{a}(y), \hat{b}(y) \vee \hat{c}(y)) < \varepsilon_1$  on an open nbd  $W_1 \subset U \cap V \cap W$  of  $x_0$ . Now if  $y \in W_1, r, s \in E_y, r \in O(b, W_1, \delta), s \in O(c, W, \eta)$ , we have  $D_y(r \vee s, \hat{a}(y)) \leq D_y(r \vee s, \hat{b}(y) \vee \hat{c}(y)) + D_y(\hat{b}(y) \vee \hat{c}(y), \hat{a}(y)) \leq D_y(r, \hat{b}(y)) + D_y(s, \hat{c}(y)) + D_y(\hat{b}(y) \vee \hat{c}(y), \hat{a}(y)) < \delta + \eta + \varepsilon_1 \leq \frac{1}{2}(\varepsilon - \varepsilon_1) + \frac{1}{2}(\varepsilon - \varepsilon_1) + \varepsilon_1 = \varepsilon$  which proves (a) for  $\vee$ . The same proof works for  $+$  (only we use 2.5(2) instead of 2.5(1)) whereby (a) is settled.

(b): If  $r, s \in \Gamma(U)$ , then  $F = (r, s) : U \rightarrow (E, t) \times (E, t)$  defined as  $F(x) = (r(x), s(x))$  is continuous; moreover,  $F$  maps  $U$  into  $E \times_X E$ . Thus the map  $\varphi = r \vee_{\tilde{U}} s : U \rightarrow (E, t)$  is the composition of  $F$  followed by  $\vee : E \times_X E \rightarrow E$  – which is continuous by (a), so the whole map  $\varphi$  is. The same proof works for the addition. We show that  $r \vee_{\tilde{U}} s$  is bounded. As  $r, s$  are bounded, there is  $a, b \in X_U$  such that  $\tilde{d}_U(\hat{a}, r), \tilde{d}_U(\hat{b}, s)$  are finite. Then  $\tilde{d}_U((a \vee_U b)^\wedge, r \vee_{\tilde{U}} s) = \tilde{d}_U(\hat{a} \vee_{\tilde{U}} \hat{b}, r \vee_{\tilde{U}} s) \leq \tilde{d}_U(\hat{a}, r) \vee_R \tilde{d}_U(\hat{b}, s)$  which is finite, hence  $r \vee_{\tilde{U}} s$  is bounded (see 1.3D). Likewise we can show that  $r +_{\tilde{U}} s$  is bounded. We are done.

(c): Let  $U \subset X$  be open,  $a, b \in X_U$ . By 2.8, for every  $x \in U$  the natural map  $\xi_{U,x} : (X_U, d_U, +_U, \vee_U, \circ_U) \rightarrow (E_x, d_x, +_x, \vee_x, \circ_x)$  is an  $A$  – homomorphism so  $(a \vee_U b)^\wedge(x) = \hat{a}(x) \vee_x \hat{b}(x)$  at any  $x \in U$ , which says that  $p_U(a \vee_U b) = p_U(a) \vee_{\tilde{U}} p_U(b)$ ; likewise  $(a +_U b)^\wedge(x) = \hat{a}(x) +_x \hat{b}(x)$ ,  $(fa)^\wedge(x) = f \hat{a}(x)$ , hence  $p_U(a +_U b) = p_U(a) +_x p_U(b)$ ,  $p_U(fa) = f p_U(a)$  which with 1.4b proves (c).

Now we generalize the notion of locally finite family, which is due to K. H. Hofmann, to  $\mathfrak{MC}$  as follows:

2.11. *Definition.* Let  $\mathcal{S} = \{(X_U, d_U) |_{\varrho_{UV}} X\}$  be a presheaf from  $\mathfrak{MC}$ ,  $U \subset X$  open. A subset  $M \subset \Gamma(U)$  is called locally finite if for every  $x \in U$  there is an open nbd  $V \subset U$  of  $x$  and a finite set  $F \subset M$  such that for each  $r \in M$  there is  $s \in F$  with  $r(y) = s(y)$  for all  $y \in V$ .

B. Let  $\mathcal{S}$  be from  $\mathfrak{Y}, \mathfrak{MC}(K)$ . A set  $M \subset \Gamma(U)$  is called  $\vee_{\tilde{U}}$ -closed if for every locally finite  $N \subset M$  such that  $r = \vee_{\tilde{U}} N = \vee_{\tilde{U}} \{s \mid s \in N\} \subset \tilde{\Gamma}(U)$  (i.e.  $r$  is bounded;  $r$  is defined as  $r(x) = \vee_x \{s(x) \mid s \in N\}$  for  $x \in U$ ) we have  $r \in M$ .

Following K. H. Hofmann we get in our case

2.12. *Lemma.* Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) |_{\varrho_{UV}} X\}$  be a sheaf (see 1.11) from  $\mathfrak{Y}, \mathfrak{MC}(K)$ ,  $(E, t)$  its bundle (see 1.3A, 1.5b), let  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  (see 1.4b, 1.7a) be the natural map sending  $X_U$  onto  $\{\hat{a} \mid a \in X_U\} = A_U \subset \Gamma(U)$ . Then for any locally finite family  $N \subset A_U$  we have  $\vee_{\tilde{U}} N = \vee_{\tilde{U}} \{n/n \in N\} \in A_U$  wherefore  $A_U$  is  $\vee_{\tilde{U}}$ -closed.

*Proof.* Let  $N \subset A_U$  be locally finite; for every  $x \in U$  there is an open nbd  $\vee_x \subset U$  of  $x$  and a finite set  $F_x \subset N$  such that for every  $r \in N$  there is  $s \in F_x$  with  $r|_{\vee_x} = s|_{\vee_x}$ . We set  $\tilde{F}_x = \{a \in X_U \mid \hat{a} \in F_x\}$ ,  $a_{\vee_x} = \vee_U \tilde{F}_x$ ,  $b_{\vee_x} = \varrho_{U\vee_x}(a_{\vee_x})$ ; we get a family  $\mathcal{F} = \{b_{\vee_x} \mid x \in U\}$ . We shall show that  $\mathcal{F}$  is smooth (see 1.11). Let  $V_x \cap V_y \neq \emptyset$ ; setting  $u = \vee_{\tilde{U}} N$ ,  $c = \varrho_{\vee_x \vee_x \cap \vee_y}(b_{\vee_x})$ ,  $d = \varrho_{\vee_y \vee_x \cap \vee_y}(b_{\vee_y})$  we have for each  $z \in V_x \cap \vee_y$ :  $\hat{c}(z) = u(z) = \hat{d}(z)$ ; by 1.12,  $c = d$  which is the smoothness of  $\mathcal{F}$ .

By COND 2 of 1.11, there is  $b \in X_U$  with  $\varrho_{U\vee_x}(b) = b_{\vee_x}$  for all  $x \in U$ . As  $\hat{b}(x) = b_{\vee_x}(x) = u(x)$  for all  $x \in U$  we get  $u \in A_U$ . The following lemma is in the spirit of [1, Lemma 4.8, p. 35].

2.13. *Lemma.* Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) |_{\varrho_{UV}} X\}$  be a presheaf from  $\mathfrak{X}, \mathfrak{MC}(K)$ ,  $X$  regular, let  $U \subset X$  be open and paracompact, let  $M \subset \Gamma(U)$  such that

- (1)  $M$  is  $\vee_{\tilde{U}}$ -closed;
- (2)  $M$  is a subgroup of  $\Gamma(U)$  with respect to  $+\tilde{U}$ , and  $fm \in M$  for any  $f \in C(X, Q)$ ,  $m \in M$ ;
- (3)  $M(x) = \{m(x) \mid m \in M\}$  is dense in  $\Gamma(U)(x) = \{\sigma(x) \mid \sigma \in \Gamma(U)\}$  for all  $x \in U$ ;
- (4) The multiplication of the sections  $\sigma \in \Gamma(U)$  by the functions from  $C(X, Q)$  is pointwise meaning that  $(f \circ_{\tilde{U}} \sigma)(x) = f \circ_x \sigma(x) = f(x) \circ_x \sigma(x)$  for any  $x \in U$ ,  $\sigma \in \Gamma(U)$ ,  $f \in C(X, Q)$ .

Then  $M$  is dense in  $(\Gamma(U), \tilde{d}_U)$ .

*Proof.* If  $\sigma \in \Gamma(U)^+$ ,  $\varepsilon > 0$  then by (3), for every  $x \in U$  there is  $m_x \in M$  with  $D_x(m_x(x), \sigma(x)) < \varepsilon$ . As  $\varphi(y) = D_y(m_x(y), \sigma(y))$  is by Lemma 1.7b upper semi-continuous, there is an open nbd  $V_x \subset U$  of  $x$  such that  $\varphi(y) < \varepsilon$  on  $V_x$ . Since  $X$  is regular, we may assume  $\overline{V_x} \subset U$ . As  $\mathcal{V} = \{V_x \mid x \in U\}$  is an open cover of  $U$  and  $U$

is paracompact, there is an open locally finite refinement  $\mathcal{W} = \{W_x \mid x \in U\}$  of  $\mathcal{V}$  and a family  $\mathcal{G} = \{g_x \mid x \in U\}$  with  $g_x \in C(U, Q)$ , such that – setting  $S_x = \{y \in U \mid g_x(y) > 0\}$  – we have  $\bar{S}_x \subset W_x \subset \bar{W}_x \subset V_x$  for all  $x \in U$ , and that  $\bigvee_R \{g_x(y) \mid x \in U\} = 1$  for every  $y \in U$  ( $\mathcal{G}$  is a locally finite lattice – partition of unity subordinated to  $\mathcal{V}$ ). For every  $x \in U$  let  $f_x$  be the extension of  $g_x$  to  $X$  by zero on  $X - U$ ; then  $f_x \in C(X, Q)$  for  $\bar{V}_x \subset U$ . Setting  $\mathcal{F} = \{f_x m_x \mid x \in U\}$ , we have  $\mathcal{F} \subset M$  by (2). Further,  $\mathcal{F}$  is locally finite. Indeed, for every  $z \in U$  there is an open nbd  $N_z$  of  $z$  such that only finitely many  $f_x$ 's are nonzero on  $N_z$ . By (4), if  $f \in C(X, Q)$ ,  $f = 0$  on  $N_z$ ,  $\sigma \in \Gamma(U)$ , then  $(f\sigma)(t) = f\sigma(t) = f(t)\sigma(t)$  on  $N_z$  which shows that only finitely many  $f_x m_x$ 's may be nonzero on  $N_z$  which is the local finiteness of  $\mathcal{F}$ . Let us set  $m = \bigvee_{\tilde{U}} \mathcal{F} = \bigvee_{\tilde{U}} \{f_x m_x \mid x \in U\}$ , and let  $F_z$  be the finite set of all the  $x$ 's for which  $f_x m_x$  is nonzero on  $N_z$ . As  $m = 0 \vee_{\tilde{U}} (\bigvee_{\tilde{U}} \{f_x m_x \mid x \in F_z\})$ , we get from (2) and 2.10b that  $m$  is a continuous section over  $U$ , and for any  $z \in U$  we have  $D_z(\sigma(z), m(z)) = D_z((\bigvee_{x \in U} f_x(z)) \sigma(z), (\bigvee_{x \in U} f_x m_x)(z)) = D_z(\bigvee_{x \in U} (f_x(z) \sigma(z)), \bigvee_{x \in U} (f_x(z) m_x(z))) = D_z(0 \vee (\bigvee_{x \in U, z \in W_x} (f_x(z) \cdot \sigma(z))), 0 \vee (\bigvee_{x \in U, z \in W_x} (f_x(z) m_x(z)))) \leq \bigvee_{x \in U, z \in W_x} D_z(f_x(z) \sigma(z), f_x(z) m_x(z)) \leq \bigvee_{x \in U, z \in W_x} K D_z(\sigma(z), m_x(z)) \leq K \bigvee_{x \in U, z \in V_x} D_z(\sigma(z), m_x(z)) \leq K\varepsilon$ , which shows that  $m$  is bounded and hence  $m \in M$  by (1) on the one hand, and that  $m$  is  $K\varepsilon$  – close to  $\sigma$  on the other. In order that all the above inequalities be clear, we must bear in mind that  $(E_z, D_z, +_z, \vee_z, c_z)$  is, by 2.9a, a  $C(X, P) - K -$  area. The first equality thus holds as  $\bigvee_R \{f_x(z) \mid x \in U\} = 1$  and  $1 \sigma(z) = \sigma(z)$ , the next as, by the definition of maximum of a locally finite family (see 2.11B), we have  $(\bigvee_{x \in U} f_x m_x)(z) = \bigvee_{x \in U} (f_x m_x)(z)$ , and because  $(\bigvee_{x \in U} f_x(z)) \sigma(z) = \bigvee_{x \in U} f_x(z) \sigma(z)$  by 2.5(5), the third as  $(f_x m_x)(z) = f_x(z) \cdot m_x(z)$  by (4), the next as  $f_x(z) = 0$  for  $z \notin W_x$ , the fifth as  $D_z(0 \vee b, 0 \vee d) \leq \leq D_z(0, 0) \vee_R D_z(b, d)$ , the sixth as  $D_z(fa, fb) \leq K D_z(a, b)$  for  $f \in C(X, Q)$ , the seventh as  $W_x \subset V_x$ , the eight as  $D_z(\sigma(z), m_x(z)) < \varepsilon$  on  $V_x$ .

We have thus shown that  $\sigma \in \Gamma(U)^+$  can be  $\varepsilon$  – approximated by an  $m \in M$ . If  $\sigma \in \Gamma(U)$  is any, we have  $\sigma = \sigma^+ - \sigma^-$  with  $\sigma^+, \sigma^- \in \Gamma(U)^+$ ; We  $\varepsilon$  – approximate  $\sigma^+, \sigma^-$  by  $m, n \in M$ , and then  $m - n \in M$ ,  $D_z(\sigma^+(z) - \sigma^-(z), m(z) - n(z)) \leq \leq D_z(\sigma^+(z), m(z)) + D_z(-\sigma^-(z), -n(z)) < (1 + K) \varepsilon$  as  $D_z(-a, -b) \leq K D_z(a, b)$  by 2.5(7). We are done.

The following notion was introduced by K. H. Hofmann for presheaf of Banach spaces [1, 2.14, p. 12]:

**2.14. Definition.** Given a presheaf  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) \mid \varrho_{UV} \mid X\}$  from  $\mathfrak{A}_X \mathfrak{MC}(K)$ ,  $M \subset X$ , we set  $I_M = \{f \in C(X, Q) \mid f = 0 \text{ on } M\}$ .  $\mathcal{S}$  is called “well supported” if for any open  $U \subset X$ ,  $f \in I_U$ ,  $a \in X_U$  we have  $f \circ_U a = 0$ .

**2.15. Lemma.** Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) \mid \varrho_{UV} \mid X\}$  be a well supported



sheaf from  $\mathfrak{A}_X \mathfrak{M}\mathfrak{C}(K)$ ,  $X$  normal,  $U, V \subset X$  open,  $\bar{V} \subset U$ ,  $a \in X_U$ . Then there is  $b \in X_X$  with  $\varrho_{XV}(b) = \varrho_{UV}(a)$ .

*Proof.* There is an open  $W \subset X$  with  $\bar{V} \subset W \subset \bar{W} \subset U$ , and  $f \in C(X, Q)$  with  $f = 1$  on  $\bar{V}$ ,  $f = 0$  on  $X - \bar{W}$ . Further,  $\{U, X - \bar{W}\}$  is an open cover of  $X$ . We set  $a_{X-W} = 0$ ,  $a_U = fa$ . As  $f \in I_{X-W}$ , we have  $f \in I_{(X-W) \cap U}$  so  $\varrho_{UU \cap (X-W)}(a_U) = \varrho_{UU \cap (X-W)}(fa) = f \varrho_{UU \cap (X-W)}(a) = 0$  for  $\mathcal{S}$  is well supported. Also  $\varrho_{X-W, U \cap (X-W)}(a_{X-W}) = 0$  hence  $\{a_{X-W}, a_U\}$  is a smooth family (see 1.11). As  $\mathcal{S}$  is a sheaf, there is  $b \in X_X$  with  $\varrho_{XU}(b) = a_U = fa$ . Then  $\varrho_{XV}(b) = \varrho_{UV}(a_U) = \varrho_{UV}(fa) = f \varrho_{UV}(a) = \varrho_{UV}(a)$  for  $f = 1$  on  $V$  and  $\mathcal{S}$  is well supported.

2.16. Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) | \varrho_{UV} | X\}$  be a well supported sheaf from  $\mathfrak{A}_X \mathfrak{M}\mathfrak{C}(K)$  over a normal  $X$ ,  $x \in X$ ,  $r \in E_x^0$  (see 1.3A, B). Then there is  $b \in X_X$  such that  $\hat{b}(x) = r$ .

*Proof.* There is an open nbd  $V \subset U$  of  $x$  and  $a \in X_V$  with  $\hat{a}(x) = r$ . There is an open nbd  $W$  of  $x$  such that  $\bar{W} \subset V$ . By the foregoing lemma, there is  $b \in X_X$  with  $\varrho_{XW}(b) = \varrho_{VW}(a)$  whence  $\hat{b}(x) = \hat{a}(x) = r$ .

We shall extend [1, Prop. 2.13, p. 11] to our case.

2.17. *Lemma.* If  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) | \varrho_{UV} | X\}$  is form  $\mathfrak{A}_X \mathfrak{M}\mathfrak{C}(K)$ ,  $X$  normal, then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) below:

(1) a)  $\mathcal{S}$  is a well supported sheaf;

b) For every  $a \in X_X$  the map  $M_a : C(X, Q) \rightarrow X_X$  sending  $f \in C(X, Q)$  onto  $fa$  is continuous at zero with respect to the sup - norm meaning: For every  $a \in X_X$   $\varepsilon > 0$  there is  $\delta > 0$  such that  $0 \leq f < \delta$  yields  $d_X(fa, 0) < \varepsilon$ ;

(2) For every  $x \in X$  we have  $I_x E_x = 0$ ;

(3) For every  $\sigma \in \Gamma(U)$ ,  $f \in C(X, Q)$ ,  $x \in U$  we have  $(f\sigma)(x) = f(x)\sigma(x)$  whence the condition (4) of 2.13 is fulfilled.

*Proof.* Let (2) hold, let  $\sigma \in \Gamma(U)$ ,  $f \in C(X, Q)$ ,  $x \in U$ . Then  $(f - f(x)) \in C(X, P)$ ,  $\varphi = (f - f(x))^+$ ,  $\psi = (f - f(x))^- \in C(X, Q)$  and they both are in  $I_x$  hence  $(\varphi\sigma)(x) = \varphi\sigma(x) = 0$ ,  $(\psi\sigma)(x) = \psi\sigma(x) = 0$  and thus  $(f\sigma)(x) - f(x)\sigma(x) = (f\sigma)(x) - (f(x)\sigma)(x) = ((f - f(x))\sigma)(x) = ((\varphi - \psi)\sigma)(x) = (\varphi\sigma - \psi\sigma)(x) = (\varphi\sigma)(x) - (\psi\sigma)(x) = 0$  and (3) holds.

Let (1) hold. As the map  $n_f : (E_x, D_x) \rightarrow (E_x, D_x)$ , where  $n_f(a) = fa$  for  $a \in E_x$  is continuous and  $E_x^0$  (see 1.3A) is dense in  $E_x$ , it is enough to show that  $I_x E_x^0 = 0$ . Let  $r \in E_x^0$ ,  $f \in I_x$ ,  $\varepsilon > 0$ ; by 2.16, there is  $a \in X_X$  with  $\hat{a}(x) = r$ . By (1b), there is  $\delta > 0$  with  $\delta \leq 1$  such that  $d_X(ga, 0) < \varepsilon$  if  $0 \leq g \leq \delta$ . As  $f \in I_x$ , there is an open nbd  $U$  of  $x$  such that  $0 \leq f < \delta$  on  $U$ . Setting  $h = \min(f, \delta)$  we have  $h \in C(X, Q)$ , and (i)  $0 \leq h \leq \delta$ , (ii)  $h = f$  on  $U$ . By (i),  $d_X(ha, 0) < \varepsilon$ ; by (ii),  $f \varrho_{XU}(a) = h \varrho_{XU}(a)$  for  $\mathcal{S}$  is well supported. So  $D_x(fr, 0) \leq d_U(f \varrho_{XU}(a), 0) = d_U(h \varrho_{XU}(a), 0) \leq d_X(ha, 0) < \varepsilon$ ; thus  $fr = 0$ .

2.18. *Theorem.* Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) |_{\mathcal{Q}_{UV}} | X\}$  be a well supported sheaf from  $\mathfrak{A}_X \mathfrak{M}\mathfrak{C}(K)$ ,  $X$  hereditarily paracompact, let the multiplication  $M_a : C(X, \mathcal{Q}) \rightarrow (X_X, d_X)$  sending  $a \in X_X$  onto  $fa \in X_X$  be continuous at zero meaning that for every  $a \in X_X$ ,  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $f \in C(X, \mathcal{Q})$ ,  $0 \leq f < \delta$  yields  $d_X(fa, 0) < \varepsilon$ . Let  $t$  be the topology in the bundle  $E$  of  $\mathcal{S}$  defined in 1.5b, let  $\Gamma(U)$  for open  $U \subset X$  be the set of all continuous bounded sections on  $U$  (see 1.6). Then for every open  $U \subset X$  the natural map  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  (see 1.4b) is an isometric isomorphism onto  $\Gamma(U)$ .

*Proof.*  $\mathcal{S}$  is a sheaf hence it is a monopresheaf by 1.12. Let  $U \subset X$  be open. By 1.10,  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  is an isometry into  $\Gamma(U)$ . By 2.12, the  $p_U$  – image  $A_U$  of  $X_U$  is  $\vee_{\tilde{U}}$  – closed hence  $A_U$  fulfils the condition (1) of 2.13. Clearly the condition (2) of 2.13 is fulfilled for  $A_U$ . By 2.16,  $\{\sigma(x) | \sigma \in A_U\}$  is dense in  $E_x^0$  for any  $x \in U$ , and as  $E_x^0$  is dense in  $E_x$ , the condition (3) of 2.13 is fulfilled by  $A_U$ . By 2.17,  $A_U$  fulfils also the cond. (4) of 2.13. Thus  $A_U$  is dense in  $\Gamma(U)$  by 2.13. Since  $p_U$  is an isometry and  $(X_U, d_U)$  is complete, we have  $A_U = \Gamma(U)$ . We're done.

*Added in proof:* The author has been told that the hypothesis that the base spaces be hereditarily paracompact in [1] has been weakened to the requirement that they be locally paracompact (see [2]) Likewise it can be easily shown that Th. 2.18 holds also for locally paracompact  $X$ . Indeed, what we have actually shown in the proof of Th. 2.18 is the following

2.18\*. *Theorem.* Let  $\mathcal{S} = \{(X_U, d_U, +_U, \vee_U, \circ_U) |_{\mathcal{Q}_{UV}} | X\}$  be a well supported sheaf from  $\mathfrak{A}_X \mathfrak{M}\mathfrak{C}(K)$ , let the multiplication  $M_a : C(X, \mathcal{Q}) \rightarrow (X_X, d_X)$  sending  $a \in X_X$  onto  $fa \in X_X$  be continuous at zero (see Th. 2.18). Denote by  $t$  the topology in the bundle  $E$  of  $\mathcal{S}$  defined in 1.5b, and by  $\Gamma(U)$ , for open  $U \subset X$ , the set of all continuous bounded sections on  $U$ . Let  $U \subset X$  be open and paracompact. Then the natural map  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  is an isometric isomorphism onto  $\Gamma(U)$ . (Taking  $U \subset X$  open and paracompact in the proof of Th. 2.18, we get the proof of Th. 2.18\*.)

Now, let the space  $X$  in Th. 2.18 be only locally paracompact,  $U \subset X$  open. By 1.10,  $p_U : (X_U, d_U) \rightarrow (\Gamma(U), \tilde{d}_U)$  is an isometric isomorphism into  $\Gamma(U)$ , and it is only to show that it is onto  $\Gamma(U)$ . Let  $r \in \Gamma(U)$ . As  $X$  is locally paracompact, there is an open cover  $\mathcal{V}$  of  $U$  with each  $V \in \mathcal{V}$  paracompact. By Th. 2.18\*,  $p_V : (X_V, d_V) \rightarrow (\Gamma(V), \tilde{d}_V)$  is an isometric isomorphism onto  $\Gamma(V)$  for each  $V \in \mathcal{V}$ , hence there is  $a_V \in X_V$  such that  $p_V(a_V) = r|_V$  for each  $V \in \mathcal{V}$  ( $r|_V$  is the restriction of  $r$  to  $V$ ). If  $V, W \in \mathcal{V}$ ,  $x \in V \cap W$ , then  $[p_{V \cap W} \mathcal{Q}_{V \cap W}(a_V)](x) = r(x) = [p_{V \cap W} \mathcal{Q}_{V \cap W}(a_W)](x)$  whence  $\tilde{d}_{V \cap W}(p_{V \cap W} \mathcal{Q}_{V \cap W}(a_V), p_{V \cap W} \mathcal{Q}_{V \cap W}(a_W)) = 0$ . As  $p_{V \cap W} : (X_{V \cap W}, d_{V \cap W}) \rightarrow (\Gamma(V \cap W), \tilde{d}_{V \cap W})$  is an isometry, we get  $\mathcal{Q}_{V \cap W}(a_V) = \mathcal{Q}_{V \cap W}(a_W)$ . As  $\mathcal{S}$  is a sheaf there is  $a \in X_U$  with  $\mathcal{Q}_{UV}(a) = a_V$  for all  $V \in \mathcal{V}$ . Then  $p_U(a) = r$  and here we are.

## References

- [1] HOFMANN, K. H.: Sheaves and Bundles of Banach Spaces. Reprints and Lecture Notes in Mathematics, Tulane University (1975).
- [2] HOFMANN, K. H. and KEIMEL, K.: Sheaf theoretical concepts in analysis: Bundles and Sheaves of Banach Spaces, Banach  $C(X)$ -modules. Lecture Notes In Mathematics, to appear.