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# Radiation Transfer in Circumstellar Dust Envelopes With Spherical Symmetry — Henyey's Method of Solution

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A numerical method solving the radiative transfer is developed on the basis of Henyey's approach to the multiple scattering problem, giving for model dust nebulae primarily the surface brightness distribution. Generally polarized radiation and anisotropic scattering are taken into account. The method proves efficient only if the model of circumstellar dust envelope has spherical symmetry.

На основе подхода Хени к проблеме многократного рассеяния разработан вычислительный метод решающий перенос излучения. Посредством этого метода определяется особенно распределение яркости на поверхности пылевой туманности с учетом произвольно поляризованного излучения и анизотропного рассеяния. Метод позволяет приложения только к моделям сферично симметрических пылевых околозвездных оболочек.

Na základě Henyeova přístupu k problému mnohonásobného rozptylu je vypracována numerická metoda řešení přenosu záření, která poskytuje pro modely prachových mlhovin zejména rozdělení jasu po povrchu. Je uvažováno obecně polarizované záření a anizotropní rozptyl. Tato metoda je vhodná pouze pro model cirkumstelární prachové obálky se sférickou symetrií.

## 1. Introduction

The dusty component of the circumstellar matter around protostellar objects or evolved stars plays an important role in the energy bilance of their envelopes and consequently also in their spectral appearance. The propagation of the radiative energy is strongly determined by the optical characteristics of the dust, whereas the gas associated with the dust leaves the radiation flow practically unaffected, at least in the visible and infrared region of the spectrum (with the exception of the spectral lines, of course). Thus, to construct the theoretical models of circumstellar dust clouds one must solve first of all the problem of absorption and scattering by dust grains.

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The first step to study the transfer of multiple scattered radiation in the astrophysical media was done by Henyey (1937). His method was designed for calculation of the surface brightness of reflection nebulae with a moderate optical thickness. In order to obtain analytical solution of the equation of radiative transfer, the following assumptions were adopted:

i) The reflection nebula is modelled as a slab illuminated by a star in front of it or submerged within it.

ii) The density of dust particles in the nebula is uniform, i.e. constant in space (and time).

iii) The scattering properties of the particles may be approximated by the scattering phase function of an "average particle". The variety of such artificial phase functions was achieved by using the simple analytical Henyey-Greenstein's function (Henyey, Greenstein 1941) with only one free parameter specifying the anisotropy of scattering.

The characteristic feature of the Henyey's method rests in regarding the light beam as a composition of rays scattered once, twice, etc. Hence, the radiation field in the nebula may be determined by calculating the intensity of once scattered light in the whole volume of the nebula at first, then the intensity of twice scattered light, etc. until the last values of intensity of $n$ − times scattered light do not exceed a given limit. The intensity at a given point within the envelope and in a given direction is thus obtained as the sum of the finite number $n$ of terms (appearing in the series (2)).

However, this approach should not be errorneously taken for what usually is called "Henyey's method" and relates to the solving of radiation transfer equation as the two-point boundary value problem. The aim of this paper is to generalize the Henyey's idea mentioned above to the propagation of the polarized light and to trace out the area of applicability of the constructed generalized computational procedure.

## 2. Formulation of the Problem

By introducing the monochromatic intensity I of the unpolarized radiation, the absorption coefficient $\varkappa = \varkappa(x)$ and the source term $\mathcal{S}$, the equation of radiation transfer along a straight line with coordinate $x$ can be expressed in a simple form

$$\frac{1}{\varkappa}\frac{\mathrm{d}I}{\mathrm{d}x} + I = \mathcal{S}\,. \tag{1}$$

The essential Henyey's invention how to treat the multiple scattering is based on the apparent fact that the field of the $n$ − times scattered radiation is the source for the radiation scattered once more, i.e. $n + 1$ − times. Thus, the intensity at any point within the nebula and in any direction may be interpretted as the expansion

$$I = \sum_{n=1}^{\infty} I_n \tag{2}$$

of intensities $I_1, I_2, \ldots$ of once, twice, ... scattered light. If individual terms from series (2) are put into the eq. (1), the latter equation splits into a set of equations

$$\begin{aligned}
&\frac{1}{\varkappa}\frac{\mathrm{d}I_1}{\mathrm{d}x} + I_1 = I_0 \\
&\frac{1}{\varkappa}\frac{\mathrm{d}I_2}{\mathrm{d}x} + I_2 = \mathscr{S}_2 = \int_{4\pi} \Phi I_1 \,\mathrm{d}\omega \\
&\frac{1}{\varkappa}\frac{\mathrm{d}I_n}{\mathrm{d}x} + I_n = \mathscr{S}_{n-1} = \int_{4\pi} \Phi I_{n-1} \,\mathrm{d}\omega \\
&\quad\vdots
\end{aligned} \tag{3}$$

where the phase function $\Phi(\boldsymbol{p}_0, \boldsymbol{p})$ is proportional to the intensity of the scattered light by a small particle in the direction given by unit vector $\boldsymbol{p}$, if the incident light beam comes from the direction $\boldsymbol{p}_0$. The quantitity $\Phi\,\mathrm{d}\omega$ is therefore proportional to the energy scattered into the elementary solid angle $\mathrm{d}\omega$ per time unit at a given wavelength, and it is normalized commonly as

$$\int_{4\pi} \Phi \,\mathrm{d}\omega = 1 \tag{4}$$

or

$$\int_{4\pi} \Phi \,\mathrm{d}\omega = \text{monochromatic albedo} \leqq 1\,. \tag{5}$$

Due to symmetry problems, I started with the most simple model of spherical circumstellar envelope with only radially varying both density and kind of dust grains. The outer radius of the sphere is $R_e$, the radius of the empty central "bubble" is $R_i$ and the star is assumed to illuminate the nebula from its centre as a point source of the intensity $I_*$. With respect to the strong symmetry, the intensity $I(R, \eta)$ depends only on the distance from centre $R$ and the angle $\eta$ between the vertical direction and the direction of the light beam (Fig. 1). The equations in (3) can be solved recurrently beginning from

$$I_0(R, 0) = I_* \exp\left(-\int_{R_i}^{R} \varkappa(R')\,\mathrm{d}R'\right). \tag{6}$$

Integrating the $n$-th equation, the formula

$$\begin{aligned}
I_n(x, \eta) &= C_n(\eta) \exp\left(-\int_0^x \varkappa(R')\,\mathrm{d}R'\right) + \\
&+ \int_0^x \left(\int_{4\pi} \Phi(x', \vartheta)\, I_{n-1}(x', \eta_1)\,\mathrm{d}\omega\right) \exp\left(-\int_{x'}^{x} \varkappa(x'')\,\mathrm{d}x''\right) \mathrm{d}x'
\end{aligned} \tag{7}$$

is obtained, where the coordinates $x$, $x'$, $x''$ are measured along the integration line, the angle $\eta$ is the deviation of integration line from the radial direction at a given

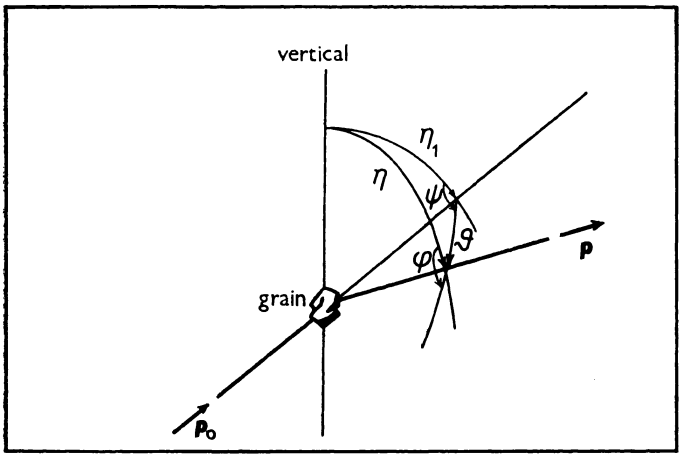


Fig. 1.

point $P$ and the meaning of the remaining angles follows from Fig. 1. For these angles $\eta_1$, $\eta$, $\vartheta$ the relation

$$\cos \eta_1 = \cos \eta \cos \vartheta + \sin \eta \sin \vartheta \cos \varphi \tag{8}$$

is valid and the source term is then transformed into

$$\mathscr{S}_n(R, \eta) = \int_0^{2\pi} \int_0^{\pi} I_{n-1}(R, \eta)\, \Phi(R, \vartheta) \sin \vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi \,. \tag{9}$$

The first term in the the sum (7) corresponds to the attenuated radiation of intensity $C_n$ entering the envelope along the integration line from without. In practically all reasonable models, there is no need to introduce this term (i.e. $C_0 = C_1 = \ldots$ $\ldots = 0$). The second term is to be evaluated either analytically or numerically at all, the points on the integration line, in which the intensity is needed.

The direct integration of the equations (3) simplifies the calculation to some extent because it does not require as many integral evaluations as in (7). As shown in Fig. 2, a set of integration straight lines parallel to the $x$-axis is drawn through the nebula in the direction to the observer. Let us call such a set the "integration warp". As the direct integrational procedure continues in steps $\Delta x$ along an integration line of a given $y$, all the values of $I_n(R, \eta)$ are generated for $R \in (y, R_e)$ and the corresponding value of $\eta$. Thanks to the spherical symmetry, the simple relation between the computed $I_n(x, y)$ and the tabellated $I_n(R = \sqrt{(x^2 + y^2)}, \eta = \operatorname{arctg}(y/x))$ sets in. Thus, the whole range of the pairs $(R, \eta)$ with $R \in (R_i, R_e)$ and $\eta \in \langle 0°, 180° \rangle$ is covered by the admittable points $(x, y)$. We may easily imagine this when consider-

ing the change of $\eta$ at the point P revolving the centre with constant $R$ and crossing the parallel lines of integration warp. Finally, the intensity $\sum_{n=1}^{\infty} I_n(y)$ of the emerging

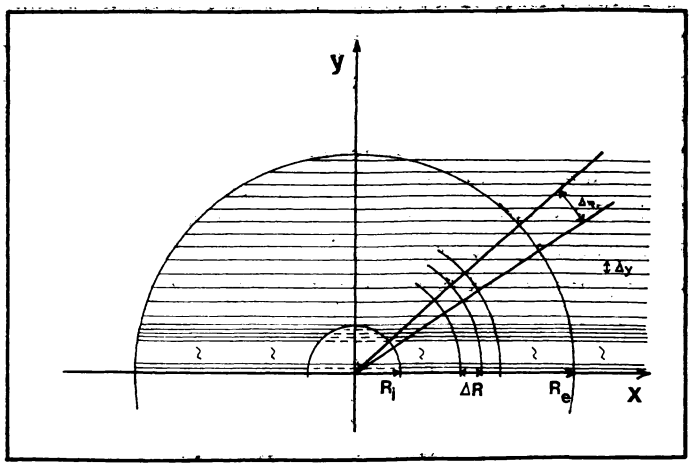


Fig. 2.

beams parallel to the $x$-axis represents the desired brightness distribution on the surface of the nebula. With respect to the symmetry, any distant observer could see the same picture as the observer from the direction of $x$.

### 3. The Adaptation of Basic Equations to Polarized Light

Similarly as in other problems dealing with noncoherent scattering, the Stokes parameters $I$, $Q$, $U$, $V$ (defined elsewhere, e.g. by Shurcliff (1962), Hulst (1957)) are used to describe the arbitrarily polarized monochromatic radiation. Since each of the Stokes parameters has a physical dimension of intensity, the generalisation from the unpolarized to the polarized radiation in the set (3) may be performed straightforwardly by replacing the symbol $I$ by the column vector

$$\boldsymbol{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}$$

and the phase function $\Phi$ by an appropriate matrix.

The source term must be expressed as the column vector of four elements, too. Thus, the number of equations in (3) rises four-times

$$\frac{1}{\varkappa}\frac{\mathrm{d}\boldsymbol{S}_n}{\mathrm{d}x} + \boldsymbol{S}_n = \mathscr{S}_{n-1}, \quad n = 1, 2, \ldots . \tag{10}$$

Apart from the numerical integration method used, the source term must be known at all the inner points, in which the integration algorithm requires its values. The most difficult problems of solution (3) arise therefore in finding the convenient and acceptable formula for the source term and in its numerical evaluation, too.

The natural frame of reference for the Stokes vector are two axes perpendicular to the light beam. One axis lies in the vertical plane determined by the radial direction and by the beam itself and the other axis is perpendicular to it. Both axes pass through the given point P. The rotational angle $\alpha$ is measured clockwise from $\boldsymbol{r}$ in the plane $(\boldsymbol{r}, \boldsymbol{l})$ when looking in the direction of the light propagation. Rotating the reference vectors $\boldsymbol{r}, \boldsymbol{l}$ in this sense, the set of vector-parameters is transformed by the rotation matrix (Shurcliff 1962)

$$\mathscr{O}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

Provided that the Mie's theory is used to describe the scattering process by dust grains (assuming their sphere-like shapes), the Stokes vector of the incident beam is transformed into that of the scattered beam by multiplication by the scattering matrix

$$\boldsymbol{M}(\vartheta) = \begin{pmatrix} 0.5(i_1 + i_2) & 0.5(i_2 - i_1) & 0 & 0 \\ 0.5(i_2 - i_1) & 0.5(i_1 + i_2) & 0 & 0 \\ 0 & 0 & \sqrt{(i_1 i_2)}\cos\sigma & -\sqrt{(i_1 i_2)}\sin\sigma \\ 0 & 0 & \sqrt{(i_1 i_2)}\sin\sigma & \sqrt{(i_1 i_2)}\cos\sigma \end{pmatrix}. \tag{12}$$

The Mie's functions $i_1(\vartheta)$, $i_2(\vartheta)$ and $\sigma(\vartheta)$ are given e.g. by Hulst (1957). The dependence on the scattering angle $\vartheta$ between the incident and scattered beams was omitted in (12) to simplify the notation. It should be emphasized that the set of Stokes parameters of the incident as well as the scattered beam is expressed with respect to the vectors $\boldsymbol{l}$ laying always in the scattering plane, whereas the vector $\boldsymbol{r}$ remains identical. Finally, according to the configuration of the scattering plane and the vertical plane shown in Fig. 1, the Stokes vector must be prepared for the scattering by rotating the frame $\boldsymbol{r}, \boldsymbol{l}$ by $\psi$, then transformed by scattering matrix and related again to the vertical plane by rotating by $-\varphi$. All the above operations are to be included in the integrand of the source term, so that the latter is put into the form

$$\mathscr{S}_n(R, \eta) = \int_0^{2\pi}\int_0^{\pi} \mathscr{O}(-\varphi)\, \boldsymbol{M}(R, \vartheta)\, \mathscr{O}(\psi)\, \boldsymbol{S}_n(R, \eta_1) \sin\vartheta \,\mathrm{d}\vartheta \,\mathrm{d}\varphi \tag{13}$$

for $n = 1, 2, \ldots$ and

$$\mathscr{S}_0(R,\eta) = \mathbf{M}(R,\eta)\begin{pmatrix} I_* \\ 0 \\ 0 \\ 0 \end{pmatrix} \exp\left(-\int_{R_i}^{R} \varkappa(R')\,\mathrm{d}R'\right) \tag{14}$$

for $n = 0$.

The computational procedure solves the modified set of equations (10). It starts with the equations for $n = 1$, calculates the $\mathbf{S}_1(x, y)$ and stores the $\mathbf{S}_1(R, \eta)$. From the stored values the source terms $\mathscr{S}_1(R, \eta)$ are calculated and then the next four equations for the individual Stokes parameters for $n = 2$ may be solved. These two steps are repeated with increasing $n$ until a given $n_{\max}$ is reached. The computation proceeds stepwise from an integration point to the next one along an integration line from the back surface to the front one, the integration lines being taken one by one untill complete exhaustion of the integration warp. If $\varkappa = 0$ is put in the empty space outside the nebula, the integration interval can be enlarged to $\langle -R_e, R_e \rangle$ for any integration line. The advantage of testing whether $\varkappa = 0$ and the immediate transfer to the next integration point in the positive case allow the modelling of the envelope consisting of one or more concentric dust shells.

When constructing a sequence of integration points on an integration line, the rapid change of $\eta$ at $x \simeq 0$ must be taken into account, what requires the more refined division in this region. The most common cosmic dust particles have the diameter approximately in the interval 100 nm and 1 μm, so that they scatter light of wavelength comparable to their diameters predominantly in the forward direction. This suggests that the scattering matrix elements strongly vary at $\vartheta \simeq 0$. Since the most intensive radiation in the circumstellar nebula moves therefore in the radial direction, the substantial contribution to the source term comes just from it despite of the number of preceding scattering events. However, the last statement is valid only for not too high a number of scatterings, may be $n \lesssim 20$. Thus, the computation of $\mathbf{S}_n(y)$ without refined division at $x \simeq 0$ would be a source of errors and it would also lead to the underestimated values. All the facts mentioned above are the reason for tabellating the vector elements $\mathbf{S}_n(x, y)$ and $\mathscr{S}_n(R, \eta)$ as the arrays $\mathbf{S}_n(IR, I\eta)$ and $\mathscr{S}_n(IR, I\eta)$ depending on the indices $IR, I\eta$ and with the fixed given table differences $\Delta R, \Delta\eta$.

It is desirable to make the division in $x$ and $y$ so fine that at least one value of $\mathbf{S}_n(x, y)$ would be at disposal for the evaluating of the table values $\mathbf{S}_n(IR, I\eta)$ at each position on integration line, which is given by the area determined by the corner points

$$\begin{aligned} &(R + \Delta R/2,\ \eta - \Delta\eta/2) \quad (R + \Delta R/2,\ \eta + \Delta\eta/2) \\ &(R - \Delta R/2,\ \eta - \Delta\eta/2) \quad (R - \Delta R/2,\ \eta + \Delta\eta/2)\,. \end{aligned} \tag{15}$$

The number of such inner points $(x, y)$ is denoted by the weight $w(IR, I\eta)$. To satisfy the demand $w(IR, I\eta) > 0$ to the largest possible extent, the steps $\Delta x, \Delta y$ must fulfil the following relations according to Fig. 3:

$$\Delta y \lesssim \Delta R \sin \eta + R \, \Delta\eta \cos \eta \,, \tag{16}$$

what turns to the $\Delta y < R_i \, \Delta\eta$ for $R \to R_i$ and $\eta \to 0$ (or $\eta \to 180°$) and $\Delta y < \Delta R$ for $\eta \to 90°$, and

$$\Delta x \lesssim R \, \Delta\eta \sin \eta \tag{17}$$

assuming the small values of all the $\Delta$.

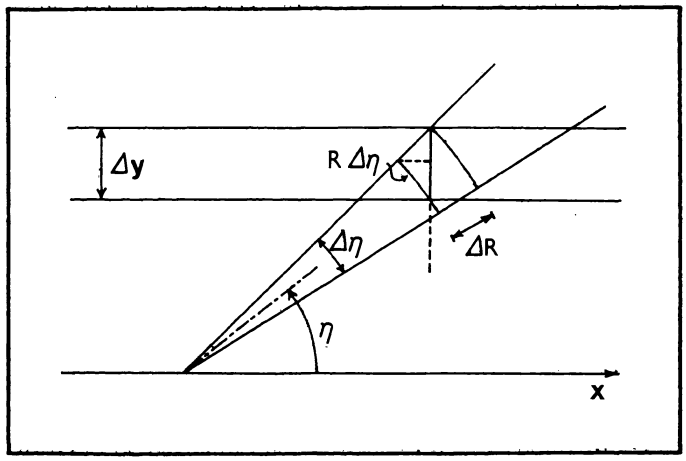


Fig. 3.

It is commonly accepted that the multiple scattering effects do not take place at small optical distances $\Delta\tau$, may be $\Delta\tau < \Delta\tau_0 \simeq 0.3$. Thus, the following definitions for the steps $\Delta$ were chosen:

fixed $\Delta$: $\Delta\eta = 5°$ (18a)

and $\Delta R = \Delta\tau_0/\varkappa(R)$, if $\varkappa \neq 0$ and if no other requirements arise (18b)

varying $\Delta$: $\Delta x = \min\left(R \, \Delta\eta \sin \eta, \, \Delta\tau_0/\varkappa(R)\right)$, (18c)

$\Delta y = \min\left(R_i \, \Delta\eta, \, \Delta R, \, \Delta\tau_0/\varkappa(R)\right)$. (18d)

According to (18d) the initial pattern of integrating lines was constructed with $\Delta y = 0.3/\varkappa(R)$ for $y \in (R_i, R_e)$ and $\Delta y = 0.03/\varkappa(R_i)$ for $y \in \langle 0, R_i)$. A special procedure has been developed to check the values $w(IR, I\eta)$ and in case of neighbouring "blind spaces" the additional integration lines and/or integration points are inserted by this procedure. The values of $\mathbf{S}_n(IR, I\eta)$ and $\mathscr{S}_n(IR, I\eta)$ are obtained by the linear interpolation at remaining isolated positions where $w(IR, I\eta) = 0$.

## 4. The Computational Procedure

The PL/1 routine has been worked out for the computer EC 1040 on the basis of the following sequence of steps:

i) The input parameters are: $R_i$, $R_e$, the tables $\varkappa(IR)$, $\boldsymbol{M}(IR, I\vartheta)$, $\Delta\eta$, $\Delta\tau_0$, $\Delta y_{max}$ (the upper limit of all $\Delta y$), $n_{max}$ (the maximum number of the terms in the series (2) which are taken into consideration), $I_*$.

ii) The construction of the integrational warp, i.e. of the table $y(Iy)$ by using (18d) and the following remarks, then the construction of the net of integration points, the evaluation of $w(IR, I\eta)$ and, finally, the refinement of division if necessary.

iii) The determination of $\mathscr{S}_1(IR, I\eta)$ according to (14). The counter of scatterings $n$ is put to equal 1 in this step.

iv) Starting from $y = 0$, an integration procedure is applied to solve (10) along each integration line of $y$ given in the table $y(Iy)$ until the $y \simeq R_e$ is reached. The integration procedure starts behind the nebula (initial $x = -R_e$) with initial values $I = Q = U = V = 0$ and proceeds by steps $\Delta x$ to the upper surface until the terminal value $x = R_e$ is reached. In each step, the coordinates $R, \eta$ are calculated from $x, y$. The value $\mathscr{S}_{n-1}(x, y)$ is obtained by the interpolation from the four neighbouring values supplied from the table $\mathscr{S}_{n-1}(IR, I\eta)$. Within the area (15), the computed values $\boldsymbol{S}_n(x, y)$ are all combined and their sum denoted by $\boldsymbol{ST}_n(IR, I\eta)$ is stored for the next use. The terminal values $\boldsymbol{S}_n(Iy)$ are stored as well. The employed integrational method must allow for the variable step $\Delta x$. I have used the simple Euler's method, but more precise results might be obtained by applying the Runge-Kutta methods.

v) The source term $\mathscr{S}_n(IR, I\eta)$ is calculated for all the table positions $(IR, I\eta)$ according to the expression (19). The numerical evaluation of the integral follows the Simpson's rule for tables where $I\vartheta$, $I\varphi$ are the integrational variables. The relation between them and the remaining angles in Fig. 1 is given by (8). The integrand in (13) is calculated according to the transformed formula

$$\mathscr{C}(-\varphi)\,\boldsymbol{M}(IR, I\vartheta)\,\mathscr{C}(\psi)\,\boldsymbol{ST}_n(IR, I\eta)\sin\vartheta/w(IR, I\eta) \tag{19}$$

with respect to the summation of $\boldsymbol{S}_n(x, y)$ hidden in $\boldsymbol{ST}_n(IR, I\eta)$. Finally, one is added to the counter of scatterings and the condition $n \leqq n_{max}$ is tested. If this is fulfilled, the control is transferred to step iv), in the opposite case to vi).

vi) The brightness distribution $I(Iy) = \sum_{n=1}^{n_{max}} I_n(Iy)$, the polarisation degree $p(Iy)$ and the position angle of the linearly polarized component of the emerging beam $\chi(Iy)$ are easily obtained from the stored values of $\boldsymbol{S}(Iy) = \sum_{n=1}^{n_{max}} \boldsymbol{S}_n(Iy)$ as follows:

degree of linear polarisation $\quad p_l = \sqrt{(Q^2 + U^2)}/I$

degree of circular polarisation $\quad p_c = V/I$

and the position angle

$$\chi = 180°/\pi \operatorname{arctg}(U/Q) \quad \text{for} \quad Q \neq 0$$

and

$$\chi = 45° \operatorname{sign} U \quad \text{for} \quad Q = 0$$

($\chi$ is measured conterclockwise from the radial straight line passing through the point of emergence in question, if looking to the projection of the nebula on the celestial sphere).

If any other values are demanded and stored, e.g. the radial distribution of the absorbed monochromatic energy, it might prove useful to check the results at the given moment.

vii) If required, the whole calculation process i)–vi) can be repeated for another wavelength, i.e. for a new set of appropriate input parameters.

## 5. Discussion

The described method of solving the radiation transfer in the circumstellar dust envelopes with spherical symmetry provides two sorts of resulting information:

a) the surface distribution of the brightness, of the stage of polarisation or of other characteristics of emergent radiation, and
b) the radiation field within the nebula (and, if required, the temperature distribution of the dust).

However, even if the knowledge of the inner radiation field is not needed, it must be calculated and stored. Thus the method is memory and time requiring to a large extent.

Let us denote the number of possible values of indices by the prefix $N$ as well as the indices themselves by the prefix $I$. If only the last values of $\mathbf{ST}_n$ and $\mathscr{S}_n$ are stored, they require $2 \,.\, (4 \,.\, NI\eta \,.\, NIy)$ places in the operational memory together. By analogy, the tables $y(Iy)$, $\mathbf{S}_n(Iy)$ require $NIy + 4 \,.\, NIy \,.\, n_{max}$ and the tables $\mathbf{M}(IR, I\vartheta)$ $4 \,.\, .\, NIR \,.\, NI\vartheta$ places. The longest computational time elapses by integrating the source terms, since this integral is calculated $n_{max}$-times in $NIy \,.\, NI\eta$ points. Each such evaluation mainly consists of the interpolations, for instance for the steps $\Delta\vartheta = \Delta\varphi = 5°$ there are $4 \,.\, [180°/\Delta\vartheta] \,.\, [360°/\Delta\varphi] \simeq 11\,000$ entries in the interpolating procedure. Thus, the Henyey's method is time consuming even for the most simple spherical geometry of the envelope.

There are some specific features of the developed method which have a substantial influence on the results. At first, the accuracy of $\mathscr{S}_n(IR, I\eta)$ depends mainly on the weight $w(IR, I\eta)$. However, the weighing function varies with the position in the nebula. Since the computational procedure runs through the same pattern of the integration points for any $n$, the errors should be amplified more quickly with the increasing $n$ at places where $w(IR, I\eta) \simeq 0$ than at others. I think, this difficulty is

in principle unavoidable, but the errors may be decreased by refining the division in $x$ and $y$. Secondly, a serious problem occurs if constructing a reliable set of integration lines. It is obvious that its arrangement differs in case of a different course of the absorption coefficient $\varkappa(R)$.

The last problem deals with the number of significant terms in the expansion (2). Starting from a certain $n_0$, the terms in the converging series (2) begins to decrease, but the $n_0$ may be determined here by a simply way without solving the set (3). If the dust particles scatter mainly forward, the radially going beams are the most intensive. Assuming the uniform density of particles of one only type, the attenuation of light satisfies the relation (6) and the quantity $1/\varkappa$ has the physical meaning of the mean free path of photons between the subsequent encounters with the dust grains. The number of scatterings is then

$$n_0 \simeq R/(1/\varkappa) = \varkappa R = \tau\,, \tag{20}$$

what is the optical equivalent of $R$. Since for the forward scattering the relation holds

$$I \simeq I_0\,.\,a\,, \tag{21}$$

where $I_0$, $I$ are the intensities of the incident and scattered beam respectively, and $a \in \langle 0, 1\rangle$ is the monochromatic albedo, the relations

$$I(R_0) \simeq I_*\,.\,a^{n_0} \tag{22}$$

and simultaneously from (6)

$$I(R_0) \simeq I_*\,.\,\exp(-\tau) \tag{23}$$

are valid after $n_0$ scatterings. The upper estimation $R_0$ of the radius of the sphere where scattered light dominates follows therefore from

$$R_0 \simeq \frac{n_0}{\varkappa} \ln a\,. \tag{24}$$

Thus, in case of the high optical thickness of the envelope, the radius $R_0$ specifies the volume, in which practically all the energy of radiation of the central star (mainly ultraviolet and visible) is absorbed and transformed to the thermal radiation of the heated dust. The application of Henyey's method can limit the extent of this "extensive infrared source" and it may also determine the temperature distribution within it, although the scattered radiation of the star does not emerge (i.e. $I(Iy) = 0$ for all $Iy$). In the light of the above considerations it is also beyond doubt that the Henyey's method would be of little use in case of optically thick medium with dielectric particles ($a \to 1$), since the significant number of scatterings would be very high.

If the envelope has other geometry than spherical, the Henyey's method becomes substantially complicated due to immense number of necessary integration warps, i.e. the sets of parallel integration straight lines. Every integration point needs its

own "bunch" of integration lines which belong in fact to different warps, and neither any class of equivalence of the warps, nor any other similarity among them can be found without the help of symmetry of the nebula model. The storage requirements make such an application practically impossible.

## 6. Summary

The Henyey's method solving the transfer of polarized radiation in the models of circumstellar dust envelopes is developed. This method proves to be efficient in case of the spherical model, but it is not flexible enough to be applied to models of other geometries.

## Appendix

The correct approach to the solution of the set of equations (10) requires the proof of the convergence of the series (2) written for each component of the Stokes vector.

Let us consider the radiation field in the envelope as stationary, i.e. not time dependent. The central star provides the nebula with exactly the same portion of monochromatic energy per unit time as the nebula outshines provided, for the time being, that the monochromatic albedo is $a = 1$. Integrating the $n$-th term of expansion (2) for $I$ over all the directions and over the whole volume $\tilde{V}$ of the nebula, a quantity proportional to the energy $\tilde{U}_n$ of the $n$-times scattered radiation contained in the nebula results in

$$\tilde{U}_n = \int_{\tilde{V}} \tilde{u}_n \, \mathrm{d}\tilde{V} = \int_{\tilde{V}} \left( \frac{1}{c} \int_{4\pi} I_n \, \mathrm{d}\omega \right) \mathrm{d}\tilde{V},$$

where $\tilde{u}$ is density of radiative energy.

Since the total amount of radiative energy $\tilde{U} = \sum_{n=1}^{\infty} \tilde{U}_n$ contained in the nebula is finite, the following series

$$\sum_{n=1}^{\infty} \frac{1}{c} \int_{\tilde{V}} \int_{4\pi} I_n \, \mathrm{d}\omega \, \mathrm{d}\tilde{V}$$

converges absolutely ($I_n \geqq 0$). According to the theorem of the mean value, there exists a number $M_n < \infty$, so that for a given point P and a given direction $\mathbf{p}$ in the nebula the relation

$$\int_{\tilde{V}} \int_{4\pi} I_n \, \mathrm{d}\omega \, \mathrm{d}\tilde{V} = M_n I_n(\mathrm{P}, \mathbf{p})$$

is valid. The limit $M = \min_{(\mathrm{P},\mathbf{p});n} M_n > 0$ over all admissible pairs $(\mathrm{P}, \mathbf{p})$ and $n \in \langle 1, \infty)$ cannot be zero, because in the opposite case $\lim_{n \to \infty} (\min_{(\mathbf{p})} M_n) = 0$ sets in at some points

and therefore the intensity would rise to infinity at these points. Hence the upper estimate of $I_n(\mathrm{P}, \mathbf{p})$ is $\tilde{U}_n/M$ and the series

$$\frac{1}{M} \sum_{n=1}^{\infty} \tilde{U}_n$$

majorizes the original series $\sum_{n=1}^{\infty} I_n$.

The main condition for the convergence is the validity of energy conservation of the monochromatic radiation during the pure scattering processes (since $a = 1$), i.e. for each dust particle the amount of energy scattered per unit time into the full solid angle equals precisely to the amount of energy of incident radiation, which has passed through the area of the efficient cross-section of the particle. This is exactly what the normalizing condition either (4) or (5) mean, even if $a < 1$. As these conditions are always fulfilled for any particle of an non-luminiscent material, there is no need to verify the convergence of the Henyey's method.

Since the components of the Stokes vector always have the property

$$I^2 \geqq Q^2 + U^2 + V^2$$

and therefore

$$I \geqq |Q| \quad I \geqq |U| \quad I \geqq |V|,$$

the series $1/M \sum_{n=1}^{\infty} \tilde{U}_n$ majorizes the series $\sum_{n=1}^{\infty} Q_n$, $\sum_{n=1}^{\infty} U_n$, $\sum_{n=1}^{\infty} V_n$ as well.

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