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On a Class of Groupoids

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For any groupoid G we can define a congruence t_G by $(a, b) \in t_G$ iff $ax = bx$ and $xa = xb$ for every $x \in G$. If G is a subdirectly irreducible groupoid with $t_G \neq \mathrm{id}_G$ then there exist two elements a, b in G such that $a \neq b$ and $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$. In the paper, groupoids having this property are called primitive and these primitive groupoids are investigated. Special attention is paid to regular primitive groupids.

Для всякого группоида G определяется конгруэнция t_G как $(a, b) \in t_G$, если $ax = bx$ и $xa = xb$ для всех $x \in G$. G называется примитивным, если в G существуют два элемента a, b так, что $a \neq b$ и $t_G = \{(a, b), (b, a)\} \cup \mathrm{i}_G$. В статье исследуются примитивные и в частности регулярные примитивные групподы.

V libovolném grupoidu G lze definovat kongruenci t_G předpisem $(a, b) \in t_G$ právě když $ax = bx$ a $xa = xb$ pro každé x z G. Je-li G subdirektně nerozložitelný a je-li $t_G \neq \mathrm{id}_G$, pak v G existují dva různé prvky a, b tak, že $t_G = \{(a, b), (b, a)\} \cup \mathrm{id}_G$. Grupoidy s touto vlastností jsou v článku nazývány primitivní a jsou studovány v rozmanitých situacích. Speciální pozornost je věnována regulárním primitivním grupoidům.

1. Preliminaries

Let G be a groupoid. For $a \in G$, define two transformations L_a and R_a of G by $L_a(x) = ax$ and $R_a(x) = xa$. The groupoid G is said to be a left (right) cancellation (division) groupoid if the transformations L_a (R_a) are injective (surjective).

Let r be a relation on G. Then r is called

- left stable if $(a, b) \in r$ implies $(ca, cb) \in r$ for every $c \in G$,
- right stable if $(a, b) \in r$ implies $(ac, bc) \in r$ for every $c \in G$,
- compatible if $(ac, bd) \in r$, provided $(a, b) \in r$ and $(c, d) \in r$,
- left cancellative if $(ca, cb) \in r$ implies $(a, b) \in r$,
- right cancellative if $(ac, bc) \in r$ implies $(a, b) \in r$.

Moreover, if r is an equivalence (congruence) then G/r is the corresponding factorset (factorgroupoid).

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For any subsets M, N of G, put $MN = \{xy \mid x \in M,\ y \in N\}$. A non-empty subset I of G is said to be an ideal if $GI \subseteq I$ and $IG \subseteq I$.

We denote by G^o the opposite groupoid, i.e. $G^o = G(\circ)$, where $x \circ y = yx$.

The groupoid G is said to be

– medial if it satisfies the identity $xy \,.\, uv = xu \,.\, yv$,
– left distributive if it satisfies the identity $x \,.\, yz = xy \,.\, xz$,
– a left unar if it satisfies the identity $xy = xz$,
– a Z-groupoid if it is both a left and right unar,
– injective if the operation of G is an injective mapping.

Let f be a mapping of a set M into N. Then $\ker f$ is the equivalence on M defined by $(a, b) \in \ker f$ iff $f(a) = f(b)$. Let i_M designate the identical transformation (relation) of M and card M the cardinal number corresponding to M.

2. Some Relations

Let G be a groupoid. Define a relation p_G on G by $(a, b) \in p$ iff $L_a = L_b$ (ie., $ax = bx$ for every $x \in G$). The groupoid G is said to be right faithful if $p = \mathrm{i}_G$. Clearly, every right cancellation groupoid is right faithful.

2.1 Lemma. (i) p_G is a right stable equivalence.
(ii) $p_G = \bigcap \ker R_x,\ x \in G$.
(iii) A block H of p_G is a right unar, provided it is a subgroupoid.

Proof. Obvious.

For every ordinal $0 \leqq \mathrm{a}$, define an equivalence $p_{G,\mathrm{a}}$ as follows: $p_0 = \mathrm{i}_G$; if $0 \leqq \mathrm{a}$ then $(a, b) \in p_{\mathrm{a}+1}$ iff $(ax, bx) \in p_{\mathrm{a}}$ for every x; if $0 < \mathrm{a}$ is limit then $p_{\mathrm{a}} = \bigcup p_{\mathrm{b}}$, $0 \leqq \mathrm{b} < \mathrm{a}$. It is obvious that $p_{\mathrm{c}} = p_{\mathrm{c}+1}$ for some ordinal c and we put $p_{G,\mathrm{c}} = \bar{p}_G$. Moreover, we denote by $\mathrm{lp}(G)$ the least ordinal d with $\bar{p} = p_{\mathrm{d}}$.

2.2 Lemma. (i) For every ordinal a, $p_{G,\mathrm{a}}$ is a right stable equivalence and $p_{G,\mathrm{a}} \subseteq$ $\subseteq p_{G,\mathrm{b}}$, whenever $\mathrm{a} \leqq \mathrm{b}$.
(ii) $\bar{p}_G$ is a right stable equivalence.
(iii) $\bar{p}_G \subseteq r$, provided r is a congruence of G such that G/r is right faithful.
(iv) If $\bar{p}_G$ is a congruence of G then $G/\bar{p}$ is right faithful.
(v) If $0 \leqq \mathrm{n}$ is natural and $a, b \in G$ then $(a, b) \in p_{G,\mathrm{n}}$ iff $((ax_1) \ldots) x_{\mathrm{n}} = ((bx_1) \ldots) x_{\mathrm{n}}$ for all $x_1, \ldots, x_{\mathrm{n}} \in G$.

Proof. Easy.

We shall say that G satisfies (C1p) if p_H is a congruence of H for every factor H of G.

2.3 Proposition. G satisfies (C1p), provided at least one of the following conditions holds:

(i) G is medial and $G = GG$.
(ii) G is a semigroup.
(iii) G is right distributive.
(iv) G is commutative.
(v) G is a right (left) unar.

Proof. Only (i) is not immediate. It suffices to show that p is left stable. For, let $a, b, x, y \in G$ and $(a, b) \in p$. Then $x = uv$ for some u, v and we can write $ya \,.\, x = ya \,.\, uv = yu \,.\, av = yu \,.\, bv = yb \,.\, uv = yb \,.\, x$.

2.4 Lemma. Let G satisfy (C1p). Then:
(i) For every ordinal number a, $p_{G,\mathrm{a}}$ is a congruence and $p_{\mathrm{a}+1}/p_{\mathrm{a}} = p_{G/p_{\mathrm{a}}}$.
(ii) $\bar{p}_G$ is a congruence and $G/\bar{p}$ is right faithful.

Proof. Easy.

Now, define an equivalence q_G on G by $(a, b) \in q$ iff $R_a = R_b$. Similarly as for p, we introduce the equivalences $q_{G,\mathrm{a}}$, $\bar{q}_G$, the ordinal number $\mathrm{lq}(G)$ and the condition (C1q). The groupoid G is said to be left faithful if it is both left and right faithful. Finally, G is said to satisfy (C1) if it satisfies both (C1p) and (C1q).

2.5 Proposition. G satisfies (C1), provided at least one of the following conditions holds:
(i) G is medial and $G = GG$.
(ii) G is a semigroup.
(iii) G is distributive.
(iv) G is commutative.
(v) G is a right (left) unar.

Proof. Apply 2.3.

Put $t_G = p_G \cap q_G$. G is said to be semifaithful if $t = \mathrm{i}_G$.

2.6 Lemma. (i) Every equivalence contained in t_G is a congruence of G.
(ii) t_G is a congruence.
(iii) If a block of t_G is a subgroupoid then it is a Z-groupoid.

Proof. Easy.

For every ordinal a, define $t_{G,\mathrm{a}}$ as follows: $t_0 = \mathrm{i}_G$; if $0 \leqq \mathrm{a}$ then $(a, b) \in t_{\mathrm{a}+1}$ iff $(xa, xb), (ax, bx) \in t_{\mathrm{a}}$ for every x; if $0 < \mathrm{a}$ is limit then $t_{\mathrm{a}} = \bigcup t_{\mathrm{b}}$, $0 \leqq \mathrm{b} < \mathrm{a}$. Put $\mathrm{l}(G) = \mathrm{c}$ and $\bar{t}_G = t_{\mathrm{c}}$, where c is the least ordinal with $t_{\mathrm{c}} = t_{\mathrm{c}+1}$.

2.7 Lemma. (i) For every ordinal a, $t_{G,\mathrm{a}}$ is a congruence, $t_{G,\mathrm{a}} \subseteq p_{G,\mathrm{a}} \cap q_{G,\mathrm{a}}$ and $t_{\mathrm{a}+1}/t_{\mathrm{a}} = t_{G/t_{\mathrm{a}}}$.
(ii) $\bar{t}_G$ is the least congruence of G such that the corresponding factor is semifaithful. Moreover, $\bar{t}_G \subseteq \bar{p}_G \cap \bar{q}_G$.

Proof. Easy.

2.8 Lemma. Let G satisfy (C1). Then:

(i) For all natural numbers $0 \leqq n, m$ with $1 \leqq n + m$, $p_n \cap q_m \subseteq t_{n+m-1}$.

(ii) $p_o \cap q_o = t_o$, where o is the first infinite ordinal.

(iii) If $\mathrm{lp}(G)$, $\mathrm{lq}(G) \leqq o$ then $\mathrm{l}(G) \leqq o$.

(iv) $\bar{t}_G = \bar{p}_G \cap \bar{q}_G$.

Proof. (i) By induction on (n, m). If either n = 0 or m = 0 then there is nothing to prove. Let $1 \leqq n, m$ and $a, b \in G$, $(a, b) \in p_n \cap q_m$. We have $(ax, bx) \in p_{n-1} \cap q_m$ for every x, and so $(ax, bx) \in t_{n+m-2}$. Similarly, $(xa, xb) \in t_{n+m-2}$.

(ii) and (iii). These assertions follow from (i).

(iv) One can show easily by induction on (a, b) that $p_a \cap q_b \subseteq \bar{t}$.

2.9 Example. Consider the following groupoid $G : G = \{a, b, c\}$, $aa = ba = b$, $ab = ac = bb = bc = cb = cc = c$, $ca = a$. It is easy to see that $p = \{(a,b), (b, a)\} \cup \mathrm{i}$, $q = \{(b, c), (c, b)\} \cup \mathrm{i}$, $q = \bar{q}$, $\bar{p} = G \times G$, $t = \mathrm{i} = \bar{t}$. Moreover, $t_0 = \bar{t} \neq \bar{p} \cap \bar{q} = q$, $\mathrm{l}(G) = 0$, $\mathrm{lq}(G) = 1$, $\mathrm{lp}(G) = 2$ and p, q are not congruences.

2.10 Lemma. Let f be a homomorphism of G onto a groupoid H.

(i) For every ordinal a, $f(p_{G,a}) \subseteq p_{H,a}$, $f(q_{G,a}) \subseteq q_{H,a}$ and $f(t_{G,a}) \subseteq t_{H,a}$.

(ii) $f(\bar{p}_G) \subseteq \bar{p}_H$, $f(\bar{q}_G) \subseteq \bar{q}_H$ and $f(\bar{t}_G) \subseteq \bar{t}_H$.

(iii) Let $p_{G,a} \subseteq \ker f$ for some ordinal a. Then $f(p_{G,a+b}) \subseteq p_{H,b}$ for every ordinal b.

(iv) Let $t_{G,a} \subseteq \ker f$ for some ordinal a. Then $f(t_{G,a+b}) \subseteq t_{H,b}$ for every ordinal b.

(v) Let $\ker f \subseteq p_{G,a}$ for some ordinal a. Then $f^{-1}(p_{H,b}) \subseteq p_{G,a+b}$ for every ordinal b.

(vi) Let $\ker f \subseteq t_{G,a}$ for some ordinal a. Then $f^{-1}(t_{H,b}) \subseteq t_{G,a+b}$ for every ordinal b.

Proof. Easy.

2.11 Lemma. Let f be a homomorphism of G onto H.

(i) If $\ker f \subseteq \bar{p}_G$ then $f(\bar{p}_G) = \bar{p}_H$ and $\mathrm{lp}(H) \leqq \mathrm{lp}(G)$.

(ii) If $\ker f \subseteq \bar{t}_G$ then $f(\bar{t}_G) = \bar{t}_H$ and $\mathrm{l}(H) \leqq \mathrm{l}(G)$.

Proof. Use 2.10.

2.12 Lemma. Let H be a subgroupoid of G. Then $p_{G,a} \mid H \subseteq p_{H,a}$, $q_{G,a} \mid H \subseteq q_{H,a}$ and $t_{G,a} \mid H \subseteq t_{H,a}$ for every ordinal a. Moreover, $\bar{p}_G \mid H \subseteq \bar{p}_H$, $\bar{q}_G \mid H \subseteq \bar{q}_H$ and $\bar{t}_G \mid H \subseteq \bar{t}_H$.

Proof. Easy.

2.13 Lemma. Let H be a semifaithful subgroupoid of G. Then H is isomorphic to a subgroupoid of $G/\bar{t}$.

Proof. This follows from 2.12.

2.14 Lemma. Let G_i, $i \in I$, be a non-empty system of groupoids. Put $G = \prod G_i$.

(i) If $0 \leqq n$ is natural and $a_i, b_i \in G_i$, $(a_i, b_i) \in p_{G_i,n}$, then $((a_i), (b_i)) \in p_{G,n}$.

(ii) If $\mathrm{lp}(G_i) \leqq n$ for every $i \in I$ and some natural n, then $\mathrm{lp}(G) \leqq n$.

(iii) $t_{G,n} = \prod t_{G_i,n}$ for every natural n.

(iv) If $\mathrm{l}(G_i) \leqq n$ for every $i \in I$ and some natural n, then $\mathrm{l}(G) \leqq n$.

(v) Suppose that the index set I is finite. Then $p_{G,a} = \prod p_{G_i,a}$ and $t_{G,a} = \prod t_{G_i,a}$ for every ordinal a.

(vi) Suppose that I is finite. Then $\mathrm{lp}(G) = \max \mathrm{lp}(G_i)$ and $\mathrm{l}(G) = \max \mathrm{l}(G_i)$.

Proof. Easy.

2.15 Lemma. (i) G is right faithful (faithful, semifaithful) iff $\mathrm{lp}(G) = 0$ ($\mathrm{lp}(G) = 0 = \mathrm{lq}(G)$, $\mathrm{l}(G) = 0$).

(ii) The class of right faithful (faithful, semifaithful) groupoids is closed under subdirect products.

Proof. Use 2.14.

2.16 Lemma. Let r be a congruence of G such that $r \cap t_G = \mathrm{i}_G$. Then $r \cap \bar{t}_G = \mathrm{i}_G$.

Proof. Suppose, on the contrary, that $r \cap \bar{t}_G \neq \mathrm{i}$. Then there is an ordinal a which is the least with $r \cap t_a \neq \mathrm{i}$. Obviously, a is not limit. Farther, there are $x, y \in G$ such that $x \neq y$ and $(x, y) \in t_a \cap r$. Then, $(xz, yz) \in t_{a-1} \cap r$ and $(zx, zy) \in t_{a-1} \cap r$ for every z. Consequently, $xz = yz$, $zx = zy$, $(x, y) \in t$, $t \cap r \neq \mathrm{i}$, a contradiction.

G is said to be torsion if $t_G = G \times G$.

2.17 Proposition. (i) G is torsion iff no factor of G is semifaithful.

(ii) The class of torsion groupoids is closed under subgroupoids, factorgroupoids and finite cartesian products.

Proof. Easy.

We shall say that G satisfies (C2) if card G = card G/p = card G/q.

2.18 Lemma. G satisfies (C2), provided L_a and R_b are surjective for some $a, b \in G$.

Proof. There is a transformation f of G with $L_a f = \mathrm{i}_G$. Put $k = gf$, where g is the natural mapping of G onto G/q. Then k is injective.

2.19 Lemma. Let G be a division groupoid such that t_G is left cancellative. Then G is a left quasigroup.

Proof. Let $ca = cb$. Then $(ca, cb) \in t$, $(a, b) \in t$. There are $x, y \in G$ with $a = ax$, $b = ay$. Then $(ax, ay) \in t$, $(x, y) \in t$ and $a = ax = ay = b$.

2.20 Example. Let $G(+)$ be the quasicyclic 2-group. For every $0 \leqq \mathrm{n}$, let A_n be the set of elements of order 2^n. Hence $A_0 = 0$, card $A_0 = 1$ and card $A_\mathrm{n} = 2^{\mathrm{n}-1}$ for $1 \leqq \mathrm{n}$. Farther, let B_n be the set of all elements of order at most 2^n. Then B_n is a subgroup and card $B_\mathrm{n} = 2^\mathrm{n}$. Take $a_\mathrm{i} \in A_\mathrm{i}$, $\mathrm{i} = 0, 1, 2, \ldots$, such that $2a_{\mathrm{j}+1} = a_\mathrm{j}$ for every $0 \leqq \mathrm{j}$. Define a transformation f of G by $f(x) = a_{\mathrm{i}+1}$, where $0 \leqq \mathrm{i}$ is such that $x \in A_\mathrm{i}$. Finally, let α be an element not belonging to G. Put $H = G \cup \{\alpha\}$ and define a multiplication on H as follows: $xy = 2x + 2y$, $x\alpha = f(x) = \alpha x$ and $\alpha\alpha = 0$ for all $x, y \in G$.

2.20.1 Lemma. (i) H is a commutative groupoid and G is a subgroupoid of H.

(ii) $f(2x) = 2f(x)$ for every $x \in G$, $x \neq 0$.

Proof. Obvious.

2.20.2 Lemma. The groupoid H is generated by α.

Proof. Let K be the subgroupoid generated by α and let $L = G \cap K$. Clearly, $0 \in L$, $f(x) \in L$ and $2x + 2y \in L$ for all $x, y \in G$. We are going to show by induction on n that $B_n \subseteq L$. For $n = 0$, $B_n = 0$ and $0 \in L$. Now, suppose that $0 \leqq n$ and $B_n \subseteq L$. We have $B_{n+1} = B_n \cup A_{n+1}$. Let $a \in A_n$ and $b = 2f^2(a)$. Then $b \in A_{n+1}$. Farther, $b + 2x = 2f^2(a) + 2x \in L$ for every $x \in B_n$. But $2B_n = B_{n-1}$, and so $C_{n-1} = b + B_{n-1} \subseteq L$. Similarly, $3b + 4x = 2f^2(a) + 2(b + 2x) \in L$ for every $x \in B_n$, $C_{n-2} = 3b + B_{n-2} \subseteq L, \ldots,$ $C_0 = (2^n - 1)\, b + B_0 \subseteq L$ and $C_{-1} = \{(2^{n+1} - 1)\, b\} = \{-b\} \subseteq L$. One can see easily that card $C_{-1} = 1 =$ card C_0, card $C_1 = 2, \ldots,$ card $C_{n-1} = 2^{n-1}$. Let $-1 \leqq i < j \leqq n - 1$ and $x \in C_i \cap C_j$. We have $x = (2^{n-i} - 1)\, b + y = 2^{n-j} - 1)\, b + z$, where $y, z \in B_j$. From this, $(2^{n-j} - 2^{n-i})\, b \in B_j$ and $0 = 2^j(2^{n-j} - 2^{n-i})\, b$. But $2^j\, 2^{n-i} b = 2^{n-i+j} = 0$, since $n + 1 \leqq n - i + j$. Therefore $0 = 2^n b$, a contradiction. We have proved that $C_i \cap C_j = \emptyset$ and consequently card $D = 2^n$, where $D = C_{-1} \cup C_0 \cup \ldots \cup C_{n-1}$. However, $D \subseteq L$, $D \subseteq A_{n+1}$ and card $D =$ card A_{n+1}. Thus $D = A_{n+1} \subseteq L$ and $B_{n+1} \subseteq L$.

2.20.3 Proposition. H is a commutative groupoid generated by one element and $l(H) = 1$. G is a subgroupoid of H, G is torsion and $l(G) = o$, where o is the first infinite ordinal.

3. Regular Groupoids

A groupoid G is said to be regular if $p_G = \ker R_x$ and $q_G = \ker L_x$ for every $x \in G$. It is said to be right (left) regular if $p_G = \ker R_x$ ($q_G = \ker L_x$).

3.1 Lemma. The class of (right) regular groupoids is closed under subgroupoids and cartesian products.

Proof. Obvious.

3.2 Lemma. A groupoid is a (right) cancellation groupoid iff it is (right) regular and (right) faithful.

Proof. Obvious.

3.3 Lemma. Let H be a subgroupoid of G.

(i) If G is right regular then $p_H = p_G \mid H$.

(ii) If G is regular then $t_H = t_G \mid H$.

Proof. Obvious.

3.4 Lemma. Let r be a right cancellative congruence of G such that $r \subseteq p_G$. Then $r = p_G$, $\mathrm{lp}(G) \leqq 1$ and G is right regular.

Proof. Easy.

3.5 Lemma. Let a division groupoid G possess a cancellative congruence $r \subseteq t_G$. Then G is a quasigroup.

Proof. By 3.4, G is regular and $r = p = q = t$. It is enough to show that G is faithful. For, let $a, b \in G$, $(a, b) \in q$. Then $a = ac$, $b = ad$ for some c, d, $(ac, ad) \in q$, $(c, d) \in q$ and $a = ac = ad = b$. Similarly the rest.

3.6 Lemma. Let G be a groupoid such that either G or G/t is regular. Let r be a congruence of G with $r \cap t_G = \mathrm{i}_G$. Suppose that $a, b \in G$, $a \neq b$, $(a, b) \in r$. Then at least one of the following assertions holds:

(i) $ax \neq bx$ and $xa \neq xb$ for every $x \in G$.
(ii) $(a, b) \in p_G$ and $xa \neq xb$ for every $x \in G$.
(iii) $(a, b) \in q_G$ and $ax \neq bx$ for every $x \in G$.

Proof. Let $ac = bc$ for some $c \in G$. If G is regular then $(a, b) \in p$. If G/t is regular then $(ax, bx) \in t \cap r$, $ax = bx$ for every x and $(a, b) \in p$. But $(a, b) \in r$ and $r \cap t = \mathrm{i}$. Thus $(a, b) \notin q$ and $xa \neq xb$ for every x.

3.7 Proposition. Let G be a regular groupoid such that card $GG \leqq \mathrm{n}$ for some natural $1 \leqq \mathrm{n}$. Then card $G/t \leqq \mathrm{n}^2$.

Proof. Let $b_1, \ldots, b_\mathrm{n} \in G$ be such that $GG \subseteq \{b_1, \ldots, b_\mathrm{n}\}$. For $\mathrm{i} = 1, \ldots, \mathrm{n}$, let $A_\mathrm{i} = \{x \mid xb_1 = b_\mathrm{i}\}$. Obviously A_i are blocks of p and there are no other blocks of p. Hence card $G/p \leqq \mathrm{n}$. Similarly, card $G/q \leqq \mathrm{n}$ and card $G/t \leqq \mathrm{n}^2$.

3.8 Corollary. Let G be a semifaithful regular groupoid with card $GG \leqq \mathrm{n}$ for some natural n. Then card $G \leqq \mathrm{n}^2$.

3.9 Lemma. Let G be a cancellation groupoid and f, g two tranformations of G. Put $x * y = f(x)\, g(y)$ for all $x, y \in G$. Then:

(i) $G(*)$ is a regular groupoid.
(ii) $p_{G(*)} = \ker f$, $q_{G(*)} = \ker g$.
(iii) $G(*)$ satisfies (C2) iff card $f(G) =$ card $G =$ card $g(G)$.
(iv) $G(*)$ is a division groupoid, provided G is and f, g are surjective.
(v) $G(*)$ is commutative, provided G is and $f = g$.

Proof. Obvious.

3.10 Proposition. The following conditions are equivalent:

(i) G is a regular groupoid satisfying (C2).
(ii) There exist a cancellation groupoid $G(\circ)$ and two surjective transformations f, g of G such that $xy = f(x) \circ g(y)$ for all $x, y \in G$.

Proof. (i) implies (ii). There are transformations k, h, f, g of G such that $(x, kf(x)) \in p$, $(x, hg(x)) \in q$ for every $x \in G$ and $(k(a), k(b)) \in p$ implies $a = b$, $(h(c), h(d)) \in q$ implies $c = d$ for all $a, b, c, d \in G$. Put $x \circ y = k(x)\, h(y)$. It is easy to check that $G(\circ)$ is a cancellation groupoid and $xy = f(x) \circ g(y)$ for all $x, y \in G$. Finally, $fk = \mathrm{i} = gh$.

(ii) implies (i). See 3.9.

3.11 Proposition. The following conditions are equivalent:

(i) G is a regular division groupoid.

(ii) There exist a loop $G(\circ)$ and two surjective transformations f, g of G such that $xy = f(x) \circ g(y)$ for all x, y.

Proof. Similar to that of 3.10.

3.12 Example. Let $G(+)$ be a vector space with an infinite countable basis $\{a_1, \dots\}$. Define two endomorphisms f and g of $G(+)$ by $f(a_1) = 0$, $f(a_i) = a_{i-1}$, $g(a_2) = 0$, $g(a_1) = a_2$, $g(a_3) = a_1$, $g(a_j) = a_{j-1}$ for all $2 \leqq i$, $4 \leqq j$. Clearly, f and g are surjective. Farther, put $xy = f(x) + g(y)$ for all $x, y \in G$. One may verify easily that G is a semifaithful regular division groupoid, $p = \ker f$, $q = \ker g$, $t = \bar{t} = \mathrm{i}_G$, $\bar{p} = p_o = G \times G = q_o = \bar{q}$ and $\mathrm{l}(G) = 0$, $\mathrm{lp}(G) = \mathrm{o} = \mathrm{lq}(G)$, where o is the first infinite ordinal. Moreover, p and q are not congruences of G.

We shall say that G satisfies (C3) (C4), (C4a)) if G/t_n is regular for every natural $0 \leqq n \leqq 1$ $(0 \leqq n, 1 \leqq n)$.

We shall say that G satisfies (C5) if every factor of G is regular.

3.13 Lemma. (i) The class of groupoids satisfying (C3) is closed under subgroupoids and cartesian products.

(ii) The class of groupoids satisfying (C4) is closed under subgroupoids and cartesian products.

(iii) The class of groupoids satisfying (C4a) is closed under cartesian products.

(iv) The class of groupoids satisfying (C5) is closed under factors.

(v) Every left (right) unar satisfies (C5).

Proof. Use 2.14.

3.14 Lemma. Let G/t_n be regular for some natural $0 \leqq n$ and let $a, b, c, d \in G$.

(i) If $ac = bc$ then $(a, b) \in p_{n+1}$.

(ii) If $da = db$ then $(a, b) \in q_{n+1}$.

(iii) If $ac = bc$, $da = db$ then $(a, b) \in t_{n+1}$.

Proof. Easy.

3.15 Proposition. Let G/t be regular and $\mathrm{lp}(G), \mathrm{lq}(G) \leqq 1$. Then G is regular and G satisfies (C3).

Proof. This is an easy consequence of 3.14.

3.16 Proposition. Let G satisfy (C4a). Then $\mathrm{l}(G) \leqq \mathrm{o}$ and $G/\bar{t}$ is regular and semifaithful.

Proof. Let $a, b \in G$, $(a, b) \in t_{o+1}$. For $c \in G$, $(ac, bc) \in t_o$, $(ca, cb) \in t_o$, and so $(ac, bc), (ca, cb) \in t_n$ for some $1 \leqq n$. Since G/t_n is regular, $(ax, bx), (xa, xb) \in t_n$ for every $x \in G$, $(a, b) \in t_{n+1}$ and $(a, b) \in t_o$. The rest is clear.

3.17 Lemma. Let G satisfy (C4). Then $t_{H,a} = t_{G,a} \mid H$ for every subgroupoid H of G and every ordinal a.

Proof. Easy (use 3.16).

3.18 Proposition. Let G satisfy (C4) and let H be a subgroupoid of G. Then $\mathrm{l}(H) \leqq \mathrm{l}(G) \leqq \mathrm{o}$.

Proof Apply 3.16 and 3.17.

3.19 Lemma. Let G be a right regular medial groupoid. Then $\mathrm{lp}(G) \leqq \mathrm{o}$ and $\bar{p}_G$ is the least right cancellative congruence of G. Moreover, p_G is a congruence and G/p is right regular.

Proof. First, we show that p is a congruence. It suffices to prove that p is left stable. We have $ca \,.\, aa = ca \,.\, ba = cb \,.\, aa$ for $a, b, c \in G$, $(a, b) \in p$. Since G is right regular, $(ca, cb) \in p$. Thus p is a congruence. Farther, if $a, b, c \in G$ and $(ac, bc) \in \in p$, then $ax \,.\, ca = ac \,.\, xa = bc \,.\, xa = bx \,.\, ca$, and so $(ax, bx) \in p$ for every $x \in G$. Thus G/p is right regular and the rest is easy.

3.20 Proposition. Every regular medial groupoid satisfies (C4).

Proof. By 3.19 (and its left hand form), p and q are congruences of G and G/p (G/q) is right (left) regular. We are going to show that G/p is left regular. Let $a, b, c \in G$ be such that $(ca, cb) \in p$. Then $cx \,.\, ax = ca \,.\, xx = cb \,.\, xx = cx \,.\, bx$ for every $x \in G$. Consequently, $(ax, bx) \in q$ and $ya \,.\, ax = yy \,.\, ax = yy \,.\, bx = = yb \,.\, yx$, $(ya, yb) \in p$. Similarly the other case and we have proved that G/p and G/q are regular. But $t = p \cap q$ and G/t is a subdirect product of G/p and G/q. By 3.1, G/t is regular.

3.21 Corollary. A medial groupoid G satisfies (C4a) iff G/t is regular.

3.22 Remark. By [1, Proposition 2.14], every medial division groupoid satisfies (C4a). On the other hand, there exist commutative medial division groupoids which are not regular. Thus (C4a) does not imply (C4).

3.23 Example. Consider the following groupoid $G : G = \{a, b, c\}$, $aa = ab = = ba = bb = cc = a$, $ac = bc = ca = cb = b$. It is easy to verify that G is commutative and satisfies (C5). Put $H = G \times G$ and denote by A the block of t_H containing (a, a). Clearly, $(x, y) \in A$ iff $x, y \in \{a, b\}$. Let $B = A \setminus \{(a, a)\}$ and $r = = (t_H \setminus (A \times A)) \cup (B \times B) \cup \mathrm{i}_H$. Then r is a congruence of H. However, $(c, c) \,.\, . (a, b) = (b, b)$, $(c, c)(a, c) = (b, a)$, $((b, b), (b, a)) \in r$, $(a, a)(a, b) = (a, a)$, $(a, a) \,.\, . (a, c) = (a, b)$. Hence H/r is not regular, H does not satisfy (C5) and the class of groupoids satisfying (C5) is not closed under cartesian products.

3.24 Remark. Every cancellation groupoid satisfies (C4) and can be imbedded into a simple cancellation groupoid. On the other hand, there are cancellation groupoids not satisfying (C5). Thus the class of groupoids satisfying (C5) is not closed under subgroupoids and (C4) does not imply (C5).

4. Primitive Groupoids

Let G be a groupoid. We shall say that G is primitive if there are two elements $a, b \in G$ such that $a \neq b$ and $t_G = \{(a, b), (b, a)\} \cup i_G$. Farthermore, we shall say that G is strongly primitive if $a, b \in GG$ and superprimitive if $a, b \in Gx \cap xG$ for every $x \in G$.

4.1 Lemma. A groupoid G is primitive iff $t_G \neq i_G$ is a minimal congruence. In this case, card G = card $G/t + 1$, $1 \leqq l(G)$, t_G is a minimal equivalence and $r \cap t_G = i_G$ for every congruence r with $t_G \nsubseteq r$.

Proof. Obvious.

4.2 Lemma. A primitive groupoid G is subdirectly irreducible iff t_G is the least non-trivial congruence of G.

Proof. Obvious.

4.3 Proposition. Let G be a subdirectly irreducible groupoid. Then just one of the following cases takes place:

(i) G is semifaithful.
(ii) G is a two-element Z-groupoid.
(iii) G is strongly primitive.

Proof. Suppose that $t \neq i_G$. According to 2.6(i), $t = \{(a, b), (b, a)\} \cup i_G$ for some $a, b \in G$, $a \neq b$. Therefore G is primitive. Let $2 \leqq$ card GG and $r = (GG \times GG) \cup i_G$. Then $r \neq i$ is a congruence and $r \cap t \neq i$. Thus $(a, b) \in r$ and $a, b \in GG$.

4.4 Proposition. Let G be a subdirectly irreducible (regular) groupoid satisfying (C1). Then just one of the following cases takes place:

(i) G is a left faithful (left cancellation) groupoid.
(ii) G is a right faithful (right cancellation) groupoid.
(iii) G is a two-element Z-groupoid.
(iv) G is strongly primitive.

Proof. Use the equality $t = p \cap q$ and 4.3.

4.5 Proposition. Every factorgroupoid of a groupoid G is semifaithful iff no factorgroupoid is primitive.

Proof. Use 4.3 and 2.15.

4.6 Proposition. Let G be a regular division groupoid such that G is not semifaithful. Then there is a congruence $r \subseteq t_G$ such that G/r is primitive.

Proof. There is a block A of t containing at least two elements. Let $a \in A$, $B = A \setminus \{a\}$ and $r = (t \setminus (A \times A)) \cup (B \times B) \cup i_G$. Then $r \subseteq t$ is a congruence. Put $H = G/r$ and denote by f the natural homomorphism of G onto H. Let $x, y \in G$ be such that $(f(x), f(y)) \in t_H$. Then $(xz, yz), (zx, zy) \in r$ for every z. On the other hand, $a = xu$, $(a, yu) \in r$, $xu = yu$ and $(x, y) \in p$. Similarly, $(x, y) \in q$, $(x, y) \in t$ and the rest is clear.

4.7 Lemma. Every primitive division groupoid is infinite and superprimitive.

Proof. Obvious.

4.8 Lemma. Let G be a regular primitive groupoid such that G/t is left regular. Suppose that $xy = a \neq b = xz$ for some $x, y, z \in G$ with $(a, b) \in t_G$. Then card $G/p \leqq 2$.

Proof. Since $(xy, xz) \in t$ and G/t is left regular, $(uy, uz) \in t$ for every $u \in G$. If $vy = vz$ for some v, then $(y, z) \in q$ and $a = xy = xz = b$, a contradiction. Hence $uy \neq uz$ and $\{uy, uz\} = \{a, b\}$. Thus $uy = a, b$ and $|G/p| \leqq 2$, since G is regular.

4.9 Proposition. Let G be a regular superprimitive groupoid. If G/t is left (right) regular then G/t is a Z-groupoid, G/t is regular and G contains at most five elements.

Proof. Let G/t be left regular. By 4.8, p has at most two blocks, say A and B (possibly $A = B$). Let $a, b \in G$ be such that $a \neq b$, $(a, b) \in t$ and let $c \in G$. There are $d, e \in G$ with $dc = a$ and $ec = b$. Then $(d, e) \in p$, $d \in A$, $e \in B$. Now, $Ac = \{a\}$, $Bc = \{b\}$ and $Gc = \{a, b\}$. We have proved that $GG = \{a, b\}$ and consequently G/t is a Z-groupoid. In particular, G/t is regular and card $G/q \leqq 2$ by the right hand form of 4.8.

4.10 Corollary. Let G be a regular primitive division groupoid. Then G/t is neither left nor right regular.

4.11 Corollary. (i) No superprimitive groupoid containing at least six elements satisfies (C3).

(ii) No primitive division groupoid satisfies (C3).

4.12 Proposition. Let G be a division groupoid.

(i) If G is regular and not semifaithful then G is a subdirect product of its primitive factors.

(ii) If G satisfies (C5) then every factorgroupoid of G is semifaithful.

Proof. Apply 4.6 and 4.11.

4.13 Proposition. The following conditions are equivalent:

(i) G is a regular primitive groupoid satisfying (C2).

(ii) There are a cancellation groupoid $G(\circ)$ and surjective transformations f, g of G such that $xy = f(x) \circ g(y)$ for all $x, y \in G$ and $\ker f \cap \ker g = \{(a, b), (b, a)\} \cup \cup \mathrm{i}_G$ for some $a, b \in G$, $a \neq b$. (In this case, G satisfies (C3) iff for every $x \in G$, either $a \notin x \circ G$ or $b \notin x \circ G$ and either $a \notin G \circ x$ or $b \notin G \circ x$.

Proof. Apply 3.10.

4.14 Proposition. The following conditions are equivalent:

(i) G is a regular primitive division groupoid.
(ii) There exist a loop $G(\circ)$ and surjective transformations f, g of G such that $xy = f(x) \circ g(y)$ for all $x, y \in G$ and $\ker f \cap \ker g = \{(a, b), (b, a)\} \cup \mathrm{i}_G$ for some $a, b \in G$, $a \neq b$.

Proof. Apply 3.11.

4.15 Example. Let $G(+)$ be the quasicyclic 2-group. Put $xy = 2x + y$ for all $x, y \in G$. Then G is a regular medial division groupoid, G is not a quasigroup and G satisfies (C5). Moreover, every congruence of G is left cancellative.

4.16 Lemma. Let G be a regular primitive groupoid, $a, b \in G$, $a \neq b$, $(a, b) \in t_G$. Suppose that G/t is not semifaithful. Then there is $c \in G$ such that either $cG \subseteq \{a, b\}$ or $Gc \subseteq \{a, b\}$. Moreover, either card $G/q \leqq 2$ or card $G/p \leqq 2$.

Proof. There are $c, d \in G$ such that $(c, d) \notin t$ and $(cx, dx), (xc, xd) \in t$ for every x. We can assume that $(c, d) \notin p$. Then $cx \neq dx$, and so $cx \in \{a, b\}$. The rest is clear.

4.17 Proposition. Let G be a superprimitive regular groupoid. Then G/t is a Z-groupoid, provided at least one of the following conditions holds:
(i) G/t is left regular.
(ii) G/t is right regular.
(iii) G/t is not semifaithful.

Proof. (i) and (ii). See 4.9.
(iii) With respect to 4.16, we can assume that card $G/q \leqq 2$. Let A and B be the only blocks of q (possibly $A = B$). Let $a, b \in G$, $a \neq b$, $(a, b) \in t$. For $x \in G$, there exist $y, z \in G$ with $xy = a$, $xz = b$. We have $(y, z) \notin q$, $y \in A$, $z \in B$, $xA = \{a\}$, $xB = \{b\}$. Therefore $GG \subseteq \{a, b\}$.

4.18 Corollary. Let G be a superprimitive regular groupoid. Then either $\mathrm{l}(G) = 1$ or $\mathrm{l}(G) = 2$ and G/t is a Z-groupoid.

4.20 Lemma. Let G be a primitive groupoid such that G is not strongly primitive. Then G/t is semifaithful and $\mathrm{l}(G) = 1$.

Proof. Easy.

4.21 Example. Consider the following groupoid $G : G = \{a, b, c, d\}$, $aa = ab = ba = bb = c$, $ac = ad = bc = bd = d$, $ca = cb = dc = dd = a$, $cc = cd = da = db = b$. Then G is strongly primitive and regular and G/t is right but not left regular. Moreover, $G = GG$ and $\mathrm{l}(G) = 2$.

4.22 Lemma. Let G be a regular primitive groupoid, $a, b \in G$, $a \neq b$, $(a, b) \in t_G$. Let H be a subgroupoid of G such that $a, b \in H$. Then H is regular and primitive.

Proof. Obvious.

5. Primitive Groupoids

5.1 Lemma. The following conditions are equivalent:

(i) G/t is a Z-groupoid.
(ii) GG is contained in a block of t_G.
(iii) G is torsion and $\mathrm{l}(G) \leqq 2$.

Proof. Easy.

5.2 Lemma. Let G be a groupoid such that G/t is a Z-groupoid. Then:

(i) G satisfies (C1) and (C4a).
(ii) G is medial.
(iii) Either $\mathrm{l}(G) = 2$ or $\mathrm{l}(G) \leqq 1$ and G is a Z-groupoid.

Proof. Easy.

5.3 Proposition. Let G be a primitive groupoid such that G/t is a Z-groupoid. Let $a, b \in G$, $a \neq b$, $(a, b) \in t_G$. Then:

(i) G is subdirectly irreducible and $GG \subseteq \{a, b\}$.
(ii) Either G is strongly primitive and $GG = \{a, b\}$ or G is a two-element Z-groupoid.
(iii) Every proper factorgroupoid of G is regular.

Proof. If G is a Z-groupoid then G contains just 2 elements. Suppose that G is not a Z-groupoid. Then $2 \leqq \operatorname{card} GG$. But GG is contained in a block of t, and so $GG = \{a, b\}$. Finally, let $r \neq \mathrm{i}_G$ be a congruence of G. There are $x, y \in G$ with $x \neq y$, $(x, y) \in r$. If $\{x, y\} = \{a, b\}$, then $t \subseteq r$. Let $x \notin \{a, b\}$. Then $(x, y) \notin t$ and either $(x, y) \notin p$ or $(x, y) \notin q$. In particular, either $xz \neq yz$ or $zx \neq zy$ for some z and consequently $(a, b) \in r$, $t \subseteq r$.

5.4 Lemma. Let $G(*)$ and $G(\circ)$ be two regular groupoids such that $p_{G(*)} = p_{G(\circ)}$ and $q_{G(*)} = q_{G(\circ)}$. Suppose that $G * G$, $G \circ G \subseteq \{a, b\}$ and $c * d = c \circ d$ for some $a, b, c, d \in G$. Then $G(*) = G(\circ)$.

Proof. Let $c * d = a$. If $e \in G$ then either $c * e = a$, and hence $(d, e) \in q$, $a = c \circ d = c \circ e$, or $c * e = b$, $(d, e) \notin q$, $a = c \circ d \neq c \circ e = b$. Thus $c * x = c \circ x$ for every x. The rest is similar.

Consider the following groupoids: $A(0) = \{a, b\}$, $aa = ab = ba = bb = a$; $A(1) = \{a, b, c\}$, $aa = ab = ba = bb = cc = a$, $ac = bc = ca = cb = b$; $A(2) = \{a, b, c\}$, $aa = ab = ba = bb = ac = bc = a$, $ca = cb = cc = b$; $A(3) = \{a, b, c\}$, $aa = ab = ba = bb = ca = cb = a$, $ac = bc = cc = b$; $A(4) = \{a, b, c, d\}$, $aa = ab = ba = bb = ac = bc = cd = dd = a$, $ad = bd = ca = cb = cc = da = db = dc = b$; $A(5) = \{a, b, c, d\}$, $aa = ab = ba = bb = ca = cb = dc = dd = a$, $ac = ad = bc = bd = cc = cd = da = db = b$; $A(6) = \{a, b, c, d\}$, $aa = ab = ba = bb = ac = bc = cd = da = db = dc = a$, $ad = bd = ca = cb = cc = dd = b$; $A(7) = \{a, b, c, d, e\}$, $aa = ab = ba = bb =$

$ac = bc = cd = ce = da = db = dc = ed = ee = a$, $ad = ae = bd = be = ca = cb = cc = dd = de = ea = eb = ec = b$.

5.5 Lemma. (i) $A(0), \ldots, A(7)$ are primitive regular groupoids.

(ii) $A(0)/t, \ldots, A(7)/t$ are Z-groupoids.

(iii) $A(2), A(3)$ are strongly primitive and $A(1), A(4), A(5), A(6), A(7)$ are superprimitive.

(iv) $A(0), \ldots, A(7)$ are pair-wise non-isomorphic.

Proof. Easy.

5.6 Proposition. Let G be a regular primitive groupoid such that G/t is a Z-groupoid. Then:

(i) G is isomorphic to exactly one of the groupoids $A(0), \ldots, A(7)$.

(ii) G is subdirectly irreducible and contains at most 5 elements.

(iii) G satisfies (C5).

Proof. First, let $a, b \in G$, $a \neq b$, $(a, b) \in t$. Then $GG \subseteq \{a, b\}$, and so card G/p, card $G/q \leqq 2$, since G is regular. From this, card $G \leqq 5$. Now, assume that $G = \{a, b, c, d, e\}$ contains five elements. We have $GG = \{a, b\}$, and hence we can assume that $aa = a$. Then $ab = ba = bb = a$. If $p \subseteq q$ then $p = t$ and p has four blocks, a contradiction. Thus $p \nsubseteq q$, similarly $q \nsubseteq p$ and both p and q have exactly two blocks, say A, B of p and C, D of q. Then $A \cap C = \{a, b\}$, card $A \cap D$, card $B \cap C$, card $B \cap D \leqq 1$. Consequently, card $A = 3$ and card $B = 2$. Without loss of generality, we can assume that $A = \{a, b, c\}$ and $B = \{d, e\}$. Then either $C = \{a, b, d\}$, $D = \{c, e\}$ or $C = \{a, b, e\}$, $D = \{c, d\}$. The rest is now clear from 5.4 and 5.5. Similarly if card $G \leqq 4$.

5.7 Proposition. Let G be a regular superprimitive groupoid such that G/t is either left or right regular. Then G is isomorphic to exactly one of the groupoids $A(1), A(4), A(5), A(6), A(7)$.

Proof. Apply 4.9, 5.5 and 5.6.

6. Primitive Groupoids

6.1 Proposition. Let G be a regular primitive groupoid such that G/t is a left unar. Then just one of the following cases takes place:

(i) G is a left unar.

(ii) G is isomorphic to $A(3)$.

(iii) G is isomorphic to one of the groupoids $A(4), A(5), A(6), A(7)$.

Proof. Let $a, b \in G$, $a \neq b$, $(a, b) \in t$. Farther, suppose that G is not a left unar. Then $cd \neq ce$ for some $c, d, e \in G$. However, $(xy, xz) \in t$ for all $x, y, z \in G$ and G is regular. Consequently, $\{xd, xe\} = \{a, b\}$ for every x. Finally, $(xd, xy) \in t$ and

$xd \in \{a, b\}$. Hence $xy \in \{a, b\}$ for all $x, y \in G$, $GG \subseteq \{a, b\}$ and G/t is a Z-groupoid. The rest follows from 5.5 and 5.6.

6.2 Proposition. The following conditions are equivalent:

(i) G is a primitive left unar.

(ii) There exist a transformation f of G and elements $a, b \in G$ such that $a \neq b$, $\ker f = \{(a, b), (b, a)\} \cup \mathrm{i}_G$ and $xy = f(x)$ for all $x, y \in G$.

Proof. Easy.

6.3 Corollary. Let G be a primitive left unar. Then G/t is either a right cancellation groupoid or a primitive groupoid.

6.4 Corollary. Let G be a primitive left unar. Then:

(i) $\mathrm{l}(G) < \mathrm{o}$ iff G/t_n is a right cancellation groupoid for some $1 \leqq \mathrm{n} < \mathrm{o}$.

(ii) $\mathrm{o} \leqq \mathrm{l}(G)$ iff $\mathrm{l}(G) = \mathrm{o}$ iff G/t_n is primitive for every $0 \leqq \mathrm{n} < \mathrm{o}$.

Let G be a left unar and $f(x) = xx$ for every $x \in G$. We shall say that G is quasicyclic if there exists an element $a \in G$ such that for every x, $f^{\mathrm{n}}(x) = f^{\mathrm{m}}(a)$ for some $1 \leqq \mathrm{n}, \mathrm{m}$.

Consider the following left unars: $B(\mathrm{n}) = \{a_0, a_1, \dots, a_n\}$, $1 \leqq \mathrm{n}$, $f(a_0) = a_0$, $f(a_i) = a_{i-1}$ for $1 \leqq \mathrm{i} \leqq \mathrm{n}$; $B(\S) = \{a_0, a_1, \dots\}$, $f(a_0) = a_0$, $f(a_i) = a_{i-1}$ for $1 \leqq \mathrm{i}$; $C(\mathrm{n}, \mathrm{m}) = \{a_1, \dots, a_n, b_1, \dots, b_m, c_1, c_2, \dots\}$, $1 \leqq \mathrm{n}, \mathrm{m}$, $f(a_1) = c_1 = f(b_1)$, $f(a_i) = a_{i-1}, f(b_j) = b_{j-1}, f(c_k) = c_{k+1}$ for $2 \leqq \mathrm{i} \leqq \mathrm{n}$, $2 \leqq \mathrm{j} \leqq \mathrm{m}$, $1 \leqq \mathrm{k}$; $C(\mathrm{n}, \S) = \{a_1, \dots, a_n, b_1, b_2, \dots, c_1, c_2, \dots\}$, $1 \leqq \mathrm{n}$, $f(a_1) = c_1 = f(b_1)$, $f(a_i) = a_{i-1}$, $f(b_j) = b_{j-1}$, $f(c_k) = c_{k+1}$ for $2 \leqq \mathrm{i} \leqq \mathrm{n}$, $2 \leqq \mathrm{j}$, $1 \leqq \mathrm{k}$; $C(\S, \S) = \{a_1, a_2, \dots, b_1, b_2, \dots, c_1, c_2, \dots\}$, $f(a_1) = c_1 = f(b_1)$, $f(a_i) = a_{i-1}$, $f(b_j) = b_{j-1}$, $f(c_k = c_{k+1}$ for $2 \leqq \mathrm{i}, \mathrm{j}$, $1 \leqq \mathrm{k}$; $E(\mathrm{n}, \mathrm{m}) = \{a_0, \dots, a_n, b_1, \dots, b_m\}$, $1 \leqq \mathrm{n}, \mathrm{m}$, $f(a_0) = a_n$, $f(b_1) = a_0$, $f(a_i) = a_{i-1}$, $f(b_j) = b_{j-1}$ for $1 \leqq \mathrm{i} \leqq \mathrm{n}$, $2 \leqq \mathrm{j} \leqq \mathrm{m}$; $E(\mathrm{n}, \S) = \{a_0, \dots, a_n, b_1, b_2, \dots\}$, $1 \leqq \mathrm{n}$, $f(a_0) = a_n$, $f(b_1) = a_0$, $f(a_i) = a_{i-1}$, $f(b_j) = b_{j-1}$ for $1 \leqq \mathrm{i} \leqq \mathrm{n}$, $2 \leqq \mathrm{j}$.

6.5 Lemma. (i) $B(\mathrm{n})$, $B(\S)$, $C(\mathrm{n}, \mathrm{m})$, $C(\mathrm{n}, \S)$, $C(\S, \S)$, $E(\mathrm{n}, \mathrm{m})$, $E(\mathrm{n}, \S)$ are quasicyclic primitive left unars.

(ii) $B(\mathrm{n})$ is torsion, subdirectly irreducible and $\mathrm{l}(B(\mathrm{n})) = \mathrm{n}$.

(iii) $B(\S)$ is torsion, subdirectly irreducible and $\mathrm{l}(B(\S)) = \mathrm{o}$.

(iv) $C(\mathrm{n}, \mathrm{m})$ is not subdirectly irreducible and $\mathrm{l}(C(\mathrm{n}, \mathrm{m})) = \min(\mathrm{n}, \mathrm{m})$.

(v) $C(\mathrm{n}, \S)$ is not subdirectly irreducible and $\mathrm{l}(C(\mathrm{n}, \S)) = \mathrm{n}$.

(vi) $C(\S, \S)$ is not subdirectly irreducible and $\mathrm{l}(C(\S, \S)) = \mathrm{o}$.

(vii) $E(\mathrm{n}, \mathrm{m})$ is not subdirectly irreducible and $\mathrm{l}(E(\mathrm{n}, \mathrm{m})) = \mathrm{m}$.

(viii) $E(\mathrm{n}, \S)$ is not subdirectly irreducible and $\mathrm{l}(E(\mathrm{n}, \S)) = \mathrm{o}$.

Proof. Obvious.

6.6 Proposition. Let G be a quasicyclic primitive left unar. Then G is isomorphic to one of the groupoids $B(\mathrm{n})$, $B(\S)$, $C(\mathrm{n}, \mathrm{m})$, $C(\mathrm{n}, \S)$, $C(\S, \S)$, $E(\mathrm{n}, \mathrm{m})$, $E(\mathrm{n}, \S)$.

Proof. Easy.

6.7 Proposition. Let G be a subdirectly irreducible primitive left unar. Then G is isomorphic to $B(\mathrm{n})$ for some $1 \leqq \mathrm{n} \leqq \S$.

Proof. Easy.

6.8 Proposition. Let G be a primitive left unar and $a, b \in G$, $a \neq b$, $(a, b) \in t_G$ Then there exist two subsets H, K of G such that:

(i) $H \cup K = G$ and $H \cap K = \emptyset$.
(ii) $a, b \in H$, H is a subgroupoid of G and H is a quasicyclic primitive left unar.
(iii) Either $K = \emptyset$ or K is a subgroupoid of G and K is a right cancellation groupoid.

Proof. Easy.

7. Technical Results

Let G be a groupoid and $a \in G$. Define a relation ϱ_a on G by $(x, y) \in \varrho_a$ iff $xy = a$. Let $f : \varrho_a \to \{0, 1\}$ be a mapping. Consider the following conditions:

(1) If $x, y \in G$, $(x, y) \in t_G$, $x \neq y$, then there exists $z \in G$ such that either $xz = a$ and $f(x, z) \neq f(y, z)$ or $zx = a$ and $f(z, x) \neq f(z, y)$.
(2) For every $x \in G$, there exist $y, z, u, v \in G$ with $xy = xz = a = ux = vx$ and $f(x, y) \neq f(x, z)$, $f(u, x) \neq f(v, x)$.
(3) If $x, y, z \in G$, $xz = yz$ and either $xz \neq a$ or $xz = a$ and $f(x, z) = f(y, z)$ then $(x, y) \in p_G$ and $f(x, v) = f(y, v)$ for every $v \in G$ with $xv = a$.
(4) If $x, y, z \in G$, $zx = zy$ and either $zx \neq a$ or $zx = a$ and $f(z, x) = f(z, y)$ then $(x, y) \in q_G$ and $f(v, x) = f(v, y)$ for every $v \in G$ with $vx = a$.

We shall say that the element a satisfies (α) $((\beta), (\gamma), (\delta))$ if there exists a mapping $f : \varrho_a \to \{0, 1\}$ satisfying (1) ((1), (2); (1), (3), (4); (1), (2), (3), (4)).

We shall say that G satisfies (C6) ((C7), (C8), (C9)) if G contains an element satisfying (α) $((\beta), (\gamma), (\delta))$.

7.1 Lemma. Let G be a groupoid, $a \in G$ and $f : \varrho_a \to \{0, 1\}$

(i) If f satisfies (1) and G is not semifaithful then f is surjective.
(ii) If $a \notin GG$ then f satisfies (1) iff G is semifaithful.
(iii) If G is injective then card $\varrho_a \leqq 1$ and f is not surjective.
(iv) If f satisfies (2) then G is neither a left nor a right cancellation groupoid.

Proof. Obvious.

7.2 Lemma. Let a groupoid G satisfy (C6). The following statements are equivalent:

(i) There exist $a \in G$ and a surjective mapping $f : \varrho_a \to \{0, 1\}$ satisfying (1).
(ii) G is not injective.

Proof. Easy.

7.3 Lemma. (i) Every element from a semifaithful groupoid satisfies (α).
(ii) Every primitive groupoid satisfies (C6).
(iii) Every Z-groupoid satisfies (C6).
(iv) A Z-groupoid satisfies (C7) iff it is non-trivial.

Proof. (i) and (ii). These are obvious. (iii) and (iv). There is $a \in G$ such that $xy = a$ for all x, y. Hence $\varrho_a = G \times G$. Define f by $f(x, y) = 0$ if $x \neq y$ and $f(x, x) = 1$. The rest is clear.

7.4 Lemma. Let G be a regular groupoid, $a, b, c, d \in G$ and $bc = a = db$. Denote by B, C, D the blocks of t_G, q_G, p_G containing b, c, d, resp. If a satisfies (α) then card $B \leqq 2^m$, where m = card C + card D.

Proof. Let $f : \varrho_a \to \{0, 1\}$ be a mapping satisfying (1). If $x, y \in G$ and $bx = a = yb$, then $x \in C$, $y \in D$, since G is regular. Let P and Q designate the set of all subsets of C and D, resp. Define a mapping $g : B \to P \times Q$ as follows: For $x \in B$, $g(x) = (M, N)$, where $M = \{y \in D \mid f(y, x) = 0\}$ and $N = \{z \in C \mid f(x, z) = 0\}$. Since f satisfies (1), g is injective and the rest is clear.

7.5 Example. Let G be an infinite countable commutative loop containing an element a with $aa \neq 1$, 1 being the unit of G. As is easy to see, there is a surjective transformation f of G having the following properties:
(i) $f(1) = 1 = f(a)$ and $f(x) \neq 1$ for every $1, a \neq x \in G$.
(ii) For every $1 \neq x \in G$, there are at least 17 different elements $y \in G$ with $f(y) = x$.

Now, put $x \circ y = f(x) f(y)$. Then $G(\circ)$ is a commutative regular division groupoid. Using 7.4, one can show easily that $G(\circ)$ does not satisfy (C6).

7.6 Lemma. Let G be a right division groupoid such that card $B \leqq 2^{\mathrm{card} A}$, whenever A is a block of p_G and B of t_G. Then every element from G satisfies (α).

Proof. Let $a \in G$ and let P be the set of all ordered pairs (A, B), where $A \in G/p$, $B \in G/t$ and $AB = \{a\}$. Obviously, $\varrho_a = \bigcup (A \times B)$, $(A, B) \in P$. Moreover, if (A, B), $(C, D) \in P$ and $(A \times B) \cap (C \times D) \neq \emptyset$, then $A = C$ and $B = D$. Now, let $(A, B) \in P$. According to the hypothesis, there is an injective mapping g of B into Q, Q being the set of all subsets of A. We shall define a mapping $f : A \times B \to \{0, 1\}$ by $f(x, y) = 0$ if $x \in g(y)$ and $f(x, y) = 1$ otherwise. The rest is clear.

7.7 Lemma. Let G be a division groupoid such that $2 \leqq$ card A = card B for any two blocks A and B of t_G. Then every element from G satisfies (β).

Proof. Let $A, B \in G/t$ be such that $AB = \{a\}$, $a \in G$. There is a biunique mapping $g : A \to B$. Define $f : A \times B \to \{0, 1\}$ by $f(x, g(x)) = 0$ and $f(x, y) = 1$ if $y \neq g(x)$. The rest is clear.

7.8 Lemma. Let G be a left unar and $a \in GG$. Then a satisfies (α) iff there is $b \in G$ such that the following two conditions hold:
(i) $a = bc$ for some $c \in G$.

(ii) card $B \leqq 2^{\mathrm{card} A}$, whenever B is a block of t_G and A is the block of t_G containing b.

Proof. Easy.

7.9 Lemma. Let G be a left unar and $g(x) = xx$ for every $x \in G$. Then G satisfies (C6) iff there exists an element $a \in G$ such that card $B \leqq 2^{\mathrm{card} A}$, whenever B is a block of ker g and A is the block of ker g containing a.

Proof. Use 7.8.

7.10. Lemma. Every finite left unar satisfies (C6).

Proof. This is an easy consequence of 7.9.

7.11 Lemma. Let G be a semifaithful division groupoid such that $2 \leqq$ card A, card B for every block A of p_G and every block B of q_G. Then every element from G satisfies (β).

Proof. Let $a \in G$, A be a block of p, B of q and $AB = \{a\}$. Let $A = C \cup D$, $C \cap D = \emptyset$, $B = K \cup L$, $K \cap L = \emptyset$, $C, D, K, L \neq \emptyset$. Define $f : A \times B \to \{0, 1\}$ by $f(C \times K) = 0 = f(D \times L)$, $f(C \times L) = 1 = f(D \times K)$. The rest is clear.

8. Technical Results

8.1 Lemma. Let $a \in G$ satisfy (γ), $f : \varrho_a \to \{0, 1\}$ be the corresponding mapping and $x, y, u, v \in G$.

(i) If $xz = yz \neq a \neq ux = uy$ then $x = y$.

(ii) If $xz = yz = a = ux = uy$, $f(x, z) = f(y, z)$ and $f(u, x) = f(u, y)$ then $x = y$.

Proof. Obvious.

8.2 Lemma. Every element from a regular semifaithful groupoid satisfies (γ).

Proof. Obvious.

A groupoid G is said to be semiinjective if $xy = uv$ implies $(x, u) \in p_G$ and $(y, v) \in q_G$ for all $x, y, u, v \in G$.

8.3 Lemma. Let G be a regular semifaithful groupoid such that G is not semiinjective. Then there are $a \in G$ and $f : \varrho_a \to \{0, 1\}$ such that f satisfies (1), (3), (4) and f is surjective.

Proof. There are $x, y, u, v \in G$ such that $xy = a = uv$ and $(x, u) \notin p$. Let $A, B \in G/p$, $C, D \in G/q$ be such that $x \in A$, $u \in B$, $y \in C$, $v \in D$. Put $f(A \times C) = 0$ and $f(B \times D) = 1$. The rest is clear.

8.4 Lemma. Let G be a semiinjective groupoid, $a \in G$ and let $f : \varrho_a \to \{0, 1\}$ be a surjective mapping satisfying (3) and (4). Then G is either a left or a right unar.

Proof. Since G is semiinjective, $\varrho_a = A \times B$, where A is a block of p and B of q. Farther, $f(x, y) \neq f(u, v)$ for some $x, u \in A$, $y, v \in B$. Suppose that $xz \neq a$ for some $z \in G$. We have $xz = uz \neq a$ and $f(x, y) = f(u, y)$ by (3). Consequently, $f(u, y) \neq f(u, v)$ and $Gy = \{a\}$. Since G is semiinjective, $p = G \times G$ and G is a right unar. Similarly the rest.

8.5 Lemma. A Z-groupoid satisfies (C8) iff it contains at most 4 elements.

Proof. Let G be a Z-groupoid, $a, x, y \in G$, $a = xy$. Suppose that G satisfies (C8). Then there are $b \in G$ and a mapping $f : \varrho_b \to \{0, 1\}$ satisfying (1), (3), (4). Obviously, $b = a$ and $\varrho_a = G \times G$. Define two equivalences r and s by $(x, y) \in r$ iff $f(x, z) = f(y, z)$ and $(u, v) \in s$ iff $f(z, u) = f(z, v)$ for every $z \in G$. It is easy to show that $r \cap s = \mathrm{id}_G$ and card $G/r \leqq 2$, card $G/s \leqq 2$.

8.6 Lemma. Let G be a left unar. Then G satisfies (C8) iff at least one of the following assertions holds:

(i) G is a Z-groupoid containing at most 4 elements.
(ii) G is a right cancellation groupoid.
(iii) G is primitive.

Proof. It is easy to verify that G satisfies (C8), provided at least one of (i), (ii), (iii) is fulfilled. Hence, assume that G satisfies (C8). There are $a \in G$ and $f : \varrho_a \to \{0, 1\}$ satisfying (1), (3), (4). Farther, put $g(x) = xx$ for every $x \in G$. If G is a Z-groupoid then 8.5 may be applied. Suppose that G is not a Z-groupoid. Then $g(b) \neq a$ for some $b \in G$. If G is semifaithful then g is injective. Let $g(c) = g(d)$ for some $c \neq d$. If $x, y, z \in G$, $g(x) = a$, then $f(x, y) = f(x, z)$ by (4) (we have $bz = by = g(b) \neq a$). Now, by (1), there exists $e \in G$ such that $ce = a = de$ and $f(c, e) \neq f(d, e)$. In particular, $g(c) = a = g(d)$. On the other hand, if $x \in G$ and $g(x) = a$, then either $f(x, e) = f(c, e)$ or $f(x, e) = f(d, e)$, and so either $x = c$ or $x = d$ (use 8.1).

8.7 Lemma. Let G be a regular groupoid satisfying (C8). Then G is either semifaithful or a left (right) unar.

Proof. There is $a \in G$ satisfying (γ). Let $f : \varrho_a \to \{0, 1\}$ be the corresponding mapping. Suppose that G is not semifaithful. Then $(b, c) \in t$ for some $b \neq c$. By 8.1, $bx = a = cx$ for every $x \in G$ (the other case is similar). Then, for all x, y, $bx = cx = by = cy = a$. Since G is regular, $q = G \times G$ and G is a left unar.

8.8 Lemma. Let G be a regular semifaithful groupoid. The following conditions are equivalent:

(i) There are $a \in G$ and $f : \varrho_a \to \{0, 1\}$ satisfying (1), (3), (4) such that f is surjective.
(ii) G is not semiinjective.

Proof. Apply 8.3, 8.4, 8.6.

9. Technical Results

9.1 Lemma. Let G be a primitive groupoid, $a, b \in G$, $a \neq b$, $(a, b) \in t_G$. Put $H = G/t$ and $c = k(a)$, where k is the natural homomorphism of G onto H. Then:

(i) c satisfies (α) in H.
(ii) c satisfies (β), provided G is superprimitive.

(iii) c satisfies (γ), provided G is regular.

(iv) c satisfies (δ), provided G is regular and superprimitive.

Proof. Define $f : \varrho_c \to \{0, 1\}$ as follows: Let $x, y \in H$, $xy = c$ and $d, e \in G$, $k(d) = x$, $k(e) = y$. Then $(de, a) \in t$ and either $de = a$ or $de = b$. We put $f(x, y) = 0$ if $de = a$ and $f(x, y) = 1$ in the opposite case.

9.2 Construction. Let H be a groupoid, $a \in H$, $s = t_H$ and $f : \varrho_a \to \{0, 1\}$. Farther, let $b \notin H$ and $G = H \cup \{b\}$. We shall define a groupoid $G(*)$ as follows: $x * y = xy$ for all $x, y \in H$ with $xy \neq a$; $x * y = a$ for all $x, y \in H$ with $xy = a$ and $f(x, y) = 0$; $x * y = b$ for all $x, y \in H$ with $xy = a$ and $f(x, y) = 1$; $x * b = = x * a$ and $b * x = a * x$ for every $x \in H$; $b * b = a * a$. Obviously, $a \neq b$ and $(a, b) \in t$, $t = t_{G(*)}$. Put $k(x) = x$ for every $x \in H$ and $k(b) = a$. Then k is a homomorphism of $G(*)$ onto H.

9.2.1 Lemma. (i) $G(*)$ is primitive, provided f satisfies (1).

(ii) $G(*)$ is strongly primitive, provided f is surjective and satisfies (1).

(iii) $G(*)$ is superprimitive, provided f satisfies (1), (2).

(iv) $G(*)$ is regular, provided f satisfies (3), (4).

(v) $G(*)$ is a division groupoid, provided H is and f satisfies (2).

(vi) If $G(*)$ is primitive then H is isomorphic to $G(*)/t$.

Proof. Easy.

10. Main Results

10.1 Theorem. Let H be a groupoid. Then:

(i) H is isomorphic to G/t for a primitive groupoid G iff H satisfies (C6).

(ii) H is isomorphic to G/t for a strongly primitive groupoid G iff H satisfies (C6) and H is not injective.

(iii) H is isomorphic to G/t for a superprimitive groupoid G iff H satisfies (C7).

(iv) H is isomorphic to G/t for a regular primitive groupoid G iff H satisfies (C8).

(v) H is isomorphic to G/t for a regular strongly primitive groupoid G iff H satisfies (C8) and either H is not semifaithful or H is not semiinjective.

(vi) H is isomorphic to G/t for a regular superprimitive groupoid G iff H satisfies (C9).

(vii) H is isomorphic to G/t for a primitive division groupoid G iff H is a division groupoid satisfying (C7).

(viii) H is isomorphic to G/t for a regular primitive division groupoid G iff H is a division groupoid satisfying (C9).

Proof. Apply 7.1(i), (iii), 8.8, 9.1 and 9.2.

10.2 Theorem. (i) A groupoid G satisfies (C6), provided at least one of the following conditions holds:

(ia) G is semifaithful.

(ib) G is a right (left) division groupoid and card $B \leqq 2^{\mathrm{card} A}$, whenever A is a block of p_G (q_G) and B of t_G.

(ic) G is a left (right) unar and there exists a block A of t_G such that card $B \leqq$ $\leqq 2^{\text{card}A}$ for every block B of t_G.

(id) G is a finite left (right) unar.

(ie) G is primitive.

(ii) A groupoid G satisfies (C7), provided at least one of the following conditions holds:

(iia) G is a non-trivial Z-groupoid.

(iib) G is a division groupoid and $2 \leqq$ card $A =$ card B for any two blocks A, B of t_G.

(iic) G is a semifaithful division groupoid and $2 \leqq$ card A, $2 \leqq$ card B for every block A of p_G and B of q_G.

(iii) A groupoid G satisfies (C8), provided at least one of the following conditions holds:

(iiia) G is semifaithful and regular.

(iiib) G is a primitive left (right) unar.

Proof. Apply 7.3, 7.6, 7.7, 7.9, 7.10, 7.11, 8.2, 8.6.

10.3 Theorem. Let G be a primitive groupoid satisfying (C3). Then at least one of the following assertions holds:

(i) G/t is regular semifaithful and $\mathrm{l}(G) = 1$.

(ii) G/t is a Z-groupoid and either G is isomorphic to $A(0)$ and $\mathrm{l}(G) = 1$ or G is isomorphic to one of the groupoids $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, $A(6)$, $A(7)$ and $\mathrm{l}(G) = 2$.

(iii) G is a left unar, G/t is primitive and $3 \leqq \mathrm{l}(G) \leqq \mathrm{o}$.

(iv) G is a right unar, G/t is primitive and $3 \leqq \mathrm{l}(G) \leqq \mathrm{o}$.

Proof. Apply 9.1, 8.7, 8.6, 6.1, 5.6.

10.4 Corollary. Let G be a primitive groupoid satisfying (C3). Then:

(i) G satisfies (C4) and $1 \leqq \mathrm{l}(G) \leqq \mathrm{o}$.

(ii) Either $\mathrm{l}(G) = 1$ or G satisfies (C5).

(iii) If G is subdirectly irreducible then G is strongly primitive and either $\mathrm{l}(G) = 1$ or G is isomorphic to one of the groupoids $A(1), \ldots, A(7)$, $B(\mathrm{n})$, $B(\mathrm{n})^{\circ}$, $3 \leqq$ $\leqq \mathrm{n} \leqq \S$.

(iv) If G is torsion then G is isomorphic to one of the groupoids $A(0), \ldots, A(7)$, $B(\mathrm{n})$, $B(\mathrm{n})^{\circ}$, $3 \leqq \mathrm{n} \leqq \S$.

(v) If G is superprimitive then G is isomorphic to one of the groupoids $A(1)$, $A(4)$, $A(5)$, $A(6)$, $A(7)$.

Proof. Apply 10.2, 5.7 and 6.7.

Reference

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