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Under an Inductivity Condition the Stalks of the Covering Space of a Presheaf are Isomorphic (Inductive and Projective Modifications of Closurations of Presheaves)

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Under suitable conditions the stalks of the covering space of a presheaf $\mathscr{S} = \{X_U|\varrho_{UV}|\ X\}$ over a topological space X are isomorphic. A topology of uniform convergence can be then defined in any set Γ_U of all continuous sections in the covering space of $\mathscr{S}$ over an open set $U \subset X$ by which the natural maps p_U, sending any $a \in X_U$ onto its corresponding section $\hat{a} \in \Gamma_U$ are continuous. The conditions put on the presheaf are of inductive character. From this reason inductive closurations of presheaves are studied and also the dual notion, projective closurations are dealt with and shown to behave dually.

При одном индуктивном условии фибры накрывающего пространства предпучка изоморфны. Индуктивные и проективные модификации предпучков. — При удобных условиях фибры накрывающего пространства предпучка $\mathscr{S} = \{X_U|\varrho_{UV}|\ X\}$ над топологическим пространством изоморфны. Тогда может быть введена топология равномерной сходимости в каждом множестве Γ_U всех непрерывных резов в накрытии от $\mathscr{S}$ над произвольным открытым множеством, при которой естественные отображения p_U, которые отображают всякое $a \in X_U$ на его кореспондирующий рез $\hat{a} \in \Gamma_U$, непрерывны. Требуемые условия для предпучка имеют индуктивный характер. Поэтому индуктивные топологизации предпучков вместе с дуальным понятием проективных топологизаций здесь изучены и показано что они ведут себе дуально.

Za vhodných podmínek jsou fibry nakrytí předsvazku $\mathscr{S} = \{X_U|\varrho_{UV}|\ X\}$ nad topologickým prostorem X izomorfní. V každé množině Γ_U všech spojitých řezů v nakrytí předsvazku $\mathscr{S}$ nad otevřenou množinou U se pak dá zavést topologie stejnoměrné konvergence, při které jsou přirozená zobrazení p_U zobrazující každé $a \in X_U$ na odpovídající řez $\hat{a} \in \Gamma_U$ spojitá. Na svazku požadované podmínky jsou induktivní povahy. Z této příčiny se studují induktivní uzávěrování předsvazků a je též rozebrán duální pojem projektivního uzávěrování a ukázáno, že se chová duálně.

Introduction

When studying in [2] different topologies in the set A_U of those continuous sections in the covering space of a presheaf $\mathscr{S} = \{X_U|\varrho_{UV}|\ X\}$ that naturally cor-

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respond to the elements of X_U, such as that of pointwise or uniform convergence, we can see that it is difficult to find a nice way of defining the topology of uniform convergence on "big" sets that consist of more than finitely many points, which is just the case of topology of uniform convergence. There would be a way if we knew how to bring over neighborhoods from one stalk to another; then we would know which neighborhoods in different stalks are of the same size. Unfortunately we have not always such a bringover handy to be used. It is shown in this paper that if a condition of inductivity is fulfilled then the stalks are isomorphic, which is just what we need. It is shown then that the topology of uniform convergence can be defined in A_U in a natural way and that the natural map $p_U : X_U \to A_U$ is continuous in this topology, where $p_U(a) = \hat{a}$ for $a \in X_U$, and $\hat{a}(x)$ is the germ of a over x.

In [3] we have dealt with the question of when there is a closure t in the covering space of $\mathscr{S}$ such that all the natural maps p_U be homeomorphisms in the topology of pointwise convergence in A_U, and such that A_U be just the set of all continuous sections over U. It is shown there [3, 4.3.6, 4.3.7, 4.3.9] that if the presheaf fulfils again a condition of inductivity then there is even a topology with the mentioned properties. From this reason inductivity of closure collections is studied in the second section. It is shown that there are some inductive modifications from below to any closuration of a presheaf, and some conditions for the inductivity of the given closuration follow. A spacial case, when the "choice" of the "stars" of a set consists of all open sets containing it, was solved by Z. Frolík in [1]. However, in [3] and in the first section of this paper we need some more general choices, which is the reason of why we study the problem in a more general setting in the second section.

It turns out in [3] that some conditions of projectivity are needful for studying the topology of uniform convergence on compact sets and also that of uniform convergence. Also the possibility likewise to treat projectivity as we have done it with inductivity leads us to showing that a dual machinery gives us the corresponding dual results for projective modifications of closurations. This is done in the last section. Projective modifications for "choices" of covers consisting of all these were also fully solved by Z. Frolík in [1]. Also in [4] some special cases of the fourth section were dealt with.

1. Preparatory Notions

The set of all open subsets of a topological space X is denoted by $\mathscr{B}(X)$.

1.1. Let $\mathscr{F} = \{S_U | \varrho_{UV} | X\}$ be a presheaf of sets over X. A closuration of $\mathscr{F}$ is a family $\mu = \{\tau_U \mid U \in \mathscr{B}(X)\}$ (shortly $\mu = \{\tau_U\}$) such that for every $U \in \mathscr{B}(X)$ τ_U is a closure in S_U; μ is called compatible if every $\varrho_{UV} : (S_U, \tau_U) \to (S_V, \tau_V)$ is continuous.

1.2. Let t, t' be two closures in a set Y. If t is finer than t', we write $t \leqq t'$. If $\mu = \{\tau_U\}$, $\mu' = \{\tau'_U\}$ are two closurations of $\mathscr{F}$ we write $\mu \leqq \mu'$ if $\tau_U \leqq \tau'_U$ for every $U \in \mathscr{B}(X)$. Let $\mathscr{M}$ be a nonempty set of closures in Y. The finest (coarsest) closure in Y

coarser (finer) than each $t \in \mathcal{M}$ is denoted by $\varinjlim \{t \mid t \in \mathcal{M}\}/\varprojlim \{t \mid t \in \mathcal{M}\}$ – (shortly $\varinjlim \mathcal{M}/\varprojlim \mathcal{M}$). Clearly $\varprojlim \mathcal{M} \leqq t \leqq \varinjlim \mathcal{M}$ for each $t \in \mathcal{M}$.

1.3. A category is called inductive if every presheaf from it has the inductive limit there.

1.4. If $\mathcal{S} = \{S_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ is a presheaf, $B \subset A$, we set $\mathcal{S}_B = \{S_\gamma | \varrho_{\gamma\delta} | \langle B \leqq \rangle\}$.

1.5. Let $\mathcal{S} = \{S_U | \varrho_{UV} | X\}$ be a presheaf over a topological space X from an inductive category. If $x \in X$, we set $O(x) = \{U \subset X \mid U \text{ open}, \ x \in U\}$, $\mathcal{S}_x = \{S_U | \varrho_{UV} | \langle O(x) \leqq \}$, where for $U, V \in O(x)$ we have $U \leqq V$ iff $V \subset U$; then $\langle I_x | \{\xi_{Ux} | U \in O(x)\}\rangle$ (here $\xi_{Ux} : S_U \to I_x = \varinjlim \mathcal{S}_x$ are the natural maps) is called stalk over x.

2. Homeomorphness of Stalks of a Presheaf

2.1. Lemma. Let $\mathcal{S} = \{X_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be a presheaf from an inductive category, such that (1): There is a confinal set $B \subset A$ such that for every $b \in B$ there is a right-directed set $S(b) \subset A$ with $S(b) \subset S(b')$ if $b \leqq b'$.

(2): For every $a \in A$ there is $b = s(a) \in B$ such that $\varinjlim \mathcal{S}_{S(b')} = \langle \varrho_{ab'}(X_a) \mid \{\varrho_{cb'} \mid c \in S(b')\}\rangle$ for all $b' \in B$, $b' \geqq b$.

Then $D = \cup\{S(b) \mid b \in B\}$ is right-directed and if we set $\varinjlim \mathcal{S} = \langle I \mid \{\xi_\alpha \mid \alpha \in A\}\rangle$, $\varinjlim \mathcal{S}_D = \langle K \mid \{s_d \mid d \in D\}\rangle$, then there is an isomorphism f between K and I such that $fs_d = \xi_d$ for all $d \in D$.

Proof. For $b \in B$ let $\varinjlim \mathcal{S}_{S(b)} = \langle Z_b \mid \{f_{cb} \mid c \in S(b)\}\rangle$. If $b \leqq b'$ then, as $S(b) \subset S(b')$, there is a unique map $g_{bb'} : Z_b \to Z_{b'}$ such that $f_{cb'} = g_{bb'}f_{cb}$ for all $c \in S(b)$. If $b, b', b'' \in B$, $b \leqq b' \leqq b''$, $c \in S(b)$ then $g_{b'b''}g_{bb'}f_{cb} = g_{b'b''}f_{cb'} = f_{cb''}$. As $g_{bb''} : Z_b \to Z_{b''}$ is the unique map with $g_{bb''}f_{cb} = f_{cb''}$ for all $c \in S(b)$, we have $g_{b'b''}g_{bb'} = g_{bb''}$ hence $\mathcal{T} = \{Z_b | g_{bb'} | \langle B \leqq \rangle\}$ is a presheaf for B is right directed being confinal in A.

By virtue of (2) we may assume that for each $b, b' \in B$, $c \in S(b)$, $b \leqq b'$ we have $f_{cb} = \varrho_{cb}$, $Z_b \subset X_b$ and $g_{bb'} = \varrho_{bb'}/Z_b$. Indeed, if $a \in A$, $b = s(a) \in B$, $B' = \{b' \in B \mid b' \geqq b\}$ then by (2), B' fulfils (1), (2) of our lemma, and $Z_{b'} = \varrho_{ab'}(X_a) \subset X_{b'}$, $f_{cb'} = \varrho_{cb'}$ for all $b' \in B'$, $c \in S(b')$. Further, as $f_{cb'} = \varrho_{cb'}$ for all $c \in S(b')$ and as $\varrho_{b''b'}\varrho_{cb'} = \varrho_{cb''}$ for all $c \in S(b')$, the uniqueness of $g_{b'b''}$ yields $g_{b'b''} = \varrho_{b'b''}/Z_{b'}$.

Now we shall show that $\varinjlim \mathcal{T} = \langle T \mid \{k_b \mid b \in B\}\rangle$ is isomorphic $\varinjlim \mathcal{S}$. Firstly, for each $b \in B$, $\{\xi_c : X_c \to I \mid c \in S(b)\}$ is a fan between $\mathcal{S}_{S(b)}$ and I so there is a unique $h_b : Z_b \to I$ with (*): $h_b f_{cb} = \xi_c$ for all $c \in S(b)$ (since $f_{cb} = \varrho_{cb}$, $\xi_b \varrho_{cb} = \xi_c$ for all $c \in S(b)$, we have $h_b = \xi_b/Z_b$), secondly, for any $b, b' \in B$, $b \leqq b'$, $c \in S(b)$ we have $h_{b'}g_{bb'}f_{cb} = h_{b'}f_{cb'} = \xi_c$ wherefore the uniqueness of h_b possessing the property (*) yields that $h_{b'}g_{bb'} = h_b$; thirdly, by (2), for $a \in A$ there is $b \geqq a$, $b \in B$ with $Z_b = \varrho_{ab}(X_a)$, whence we have a map $k_b\varrho_{ab} : X_a \to T$. Recall that for $b, b' \in B$, $b \leqq b'$, we have $g_{bb}' = \varrho_{bb}'/Z_b$. Thus if $b' \geqq a$, $b' \in B$, then there is $b'' \in B$, $b'' \geqq b, b'$, and we have $k_b\varrho_{ab} = k_{b''}g_{bb''}\varrho_{ab} = k_{b'}g_{b'b''}g_{bb'}\varrho_{ab} = k_{b''}g_{b'b''}\varrho_{bb'}\varrho_{ab} = k_{b'}\varrho_{ab'}$ so we may set $(**)$: $l_a = k_b\varrho_{ab}$ because it does not depend on b over which we carry it. If $a, a' \in A$, $a \leqq a'$, then there is $b \in B$, $b \geqq a, a'$, and $l_{a'}\varrho_{aa'} = k_b\varrho_{a'b}\varrho_{aa'} = k_b\varrho_{ab} = l_a$. Altogether we have $h_b : Z_b \to I$ with $h_{b'}g_{bb'} = h_b$ for all b $b' \in B$, $b \leqq b'$, and $l_a : X_a \to T$ with $l_{a'}\varrho_{aa'} = l_a$ for all $a, a' \in A$, $a \leqq a'$. Thus there is a unique $i : I \to T$ with $l_a = i\xi_a$ for all $a \in A$ and $j : T \to I$ with $h_b = jk_b$ for all $b \in B$. We shall show that ji is identity on I. As identity on I is the unique map $f : I \to I$ posessing the property $f\xi_a = \xi_a$ for all $a \in A$, it is enough to show that $ji\xi_a = \xi_a$ for all $a \in A$. We have $ji\xi_a = jl_a = jk_b\varrho_{ab} = h_b\varrho_{ab}$ and as $h_b = \xi_b/Z_b$, we get $h_b\varrho_{ab} = \xi_b\varrho_{ab} = \xi_a$ as desired. Likewise ij is identity on T. Indeed, if $b \in B$, we have $ijk_b = ih_b = i(\xi_b/Z_b) = l_b/Z_b$. By $(**)$, if $b' \in B$ is large enough, we have $l_b/Z_b = k_{b'}(\varrho_{bb'}/Z_b) = k_b$ which, by the same argument as above, says that ij is identity on T. Thus T is isomorphic to I.

Now we shall show that $\varinjlim \mathscr{T}$ is isomorphic to $\varinjlim \mathscr{S}_D = \langle K \mid \{s_d \mid d \in D\rangle$. Firstly, since for every $b \in B$ the family $\{s_d : X_d \to K \mid d \in S(b)\}$ is a fan between $\mathscr{S}_{S(b)}$ and K, there is a unique $t_b : Z_b \to K$ with $t_bf_{cb} = s_c$ for all $c \in S(b)$. If $b, b' \in B$, $b \leqq b'$, then $t_{b'}g_{bb'}f_{cb} = t_{b'}f_{cb'} = s_c$ for any $c \in S(b)$ and, as t_b is the only map which being composed with any $f_{cb'}$, $c \in S(b)$ yields $s_{c'}$, we get $t_b = t_{b'}g_{bb'}$. On the other hand, for $d \in D$ we have the maps $l_d : X_d \to T$, with $l_{d'}, \varrho_{dd'} = l_{d'}$ whenever $d, d' \in D$, $d \leqq d'$, found above. Thus there are $p : K \to T$, $q : T \to K$ with $ps_d = l_d$, $qk_b = t_b$ for all $d \in D$, $b \in B$. If $d \in D$ then $d \in S(b)$ for a $b \in B$, and we have $qps_d = ql_d = qk_b\varrho_{db} = t_b\varrho_{db} = t_bf_{db} = s_d$ showing that qp is identity on K. To show that pq is identity on T, it is enough to prove that $pqk_b = k_b$ for any $b \in B$. We have $pqk_b = pt_b$, and for all $c \in S(b)$ we have $pt_bf_{cb} = ps_c = l_c = k_b\varrho_{cb} = k_bf_{cb}$ since $\varrho_{cb} = f_{cb}$ when $c \in S(b)$. This shows that $pt_b = k_b$ as desired.

Finaly, we set $f = jp : K \to I$. If $d \in D$ then there is $b \in B$ with $d \in S(b)$, and by $(*)$, $fs_d = jps_d = jl_d = jk_b\varrho_{db} = h_bf_{db} = \xi_d$. The proof is thereby finished.

2.2. Notation. Let X be a connected topological space, $M, N \subset X$, let $\mathscr{R}(M, N)$ be the set of all filters $\mathscr{B}$ consisting of connected open sets such that $M \cup N \subset B$ when $B \in \mathscr{B}$. If $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{R}(M, N)$, let $\mathscr{B}_1 \leqq \mathscr{B}_2$ if $\mathscr{B}_2$ majorizes $\mathscr{B}_1$ (meaning that for any $B_1 \in \mathscr{B}_1$ there is $B_2 \in \mathscr{B}_2$ with $B_2 \subset B_1$). The Maximality Principle readily yiels for any $\mathscr{B} \in \mathscr{R}(M, N)$ a maximal $\mathscr{D} \in \mathscr{R}(M, N)$ with $\mathscr{B} \leqq \mathscr{D}$. The maximal filters in $\mathscr{R}(M, N)$ shall be called branches between M and N. The set of all these is denoted by $\mathscr{B}(M, N)$; if N is a point $\{x\}$, we shortly write $\mathscr{B}(M, x)$. If also $M = \{y\}$, we write $\mathscr{B}(y, x)$. As $\mathscr{R}(M, N) = \mathscr{R}(N, M)$, we have $\mathscr{B}(M, N) = \mathscr{B}(N, M)$. If $M \subset L, \mathscr{B} \in \mathscr{B}(M, N)$, we set $\mathscr{B}(L) = \{B \in \mathscr{B} \mid L \subset B\}$. Every $\mathscr{B}(L)$ can be completed to a $\mathscr{B}_1 \in \mathscr{B}(L, N)$. Again we wirte $\mathscr{B}(x)$ instead of $\mathscr{B}(\{x\})$.

2.3. Lemma. Let $\mathscr{S} = \{X_U, \varrho_{UV} \mid X\}$ be a presheaf from an inductive category, let X be connected and locally connected.

For any open connected $U, V \subset X$ with $\overline{V} \subset U$, any $x, y \in X$ with $x \in V$, and any $\mathscr{B} \in \mathscr{B}(x, y)$ let us have

$$\varinjlim \mathscr{S}_{\mathscr{B}(V)} = \langle \varrho_{UV}(X_U) \mid \{\varrho_{WV} \mid W \in \mathscr{B}(V)\}\rangle . \qquad (*)$$

Then for every $x, y \in X$ there is an isomorphism $h_{xy} : I_x \to I_y$ between the stalks $\langle I_x \mid \{\xi_{Ux} \mid U \in O(x)\}\rangle = \varinjlim \mathscr{S}_x$ and $\langle I_y \mid \{\xi_{Uy} \mid U \in O(y)\}\rangle = \varinjlim \mathscr{S}_y$, such that for any open $U \subset X$ with $x, y \in U$ we have $h_{xy}\xi_{Ux} = \xi_{Uy}$.

Proof. We use lemma 2.1 to $\mathscr{S}_x$ with the set of all connected open nbds of x as B. If $\mathscr{B} \in \mathscr{B}(x, y)$, $V \in \mathscr{B}$, we set $S(V) = \mathscr{B}(V)$; then the condition (1) of 2.2 is fulfilled, and also the condition (2) of 2.1 because of (*). Let $D = U\{\mathscr{B}(V) \mid V \in \mathscr{B}\}$, $\varinjlim \mathscr{S}_D = \langle K \mid \{s_V \mid V \in D\}\rangle$. Since D is confinal in $\mathscr{B}$ we can for $V \in \mathscr{B}$ find a $W \in D$ with $W \subset V$ and set $r_V = s_W\varrho_{VW} : X_V \to K$. It is easy to show that r_V does not depend on the choice of W. Also it is easy to see that $\langle K \mid \{r_V \mid V \in \mathscr{B}\}\rangle = \varinjlim \mathscr{S}_{\mathscr{B}}$. By 2.2, there is an isomorphism $f : K \to I_x$ with $fs_V = \xi_{Vx}$ for all $V \in D$. If $U \subset X$ is open, $x, y \in U$ then $U \in \mathscr{B}$ and there is $W \in D$ with $r_U = s_W\varrho_{UW}$ whence $fr_U = fs_W\varrho_{UW} = \xi_{Wx}\varrho_{UW} = \xi_{Ux}$. Likewise there is an isomorphism $g : K \to I_y$ such that $gr_U = \xi_{Uy}$ for all open $U \subset X$ with $x, y \in U$. Setting $h_{xy} = gf^{-1}$ we have for open $U \subset X$ with $x, y \in U : h_{xy}\xi_{Ux} = gf^{-1}\xi_{Ux} = gr_U = \xi_{Uy}$ and we are done.

2.4. Remark. Let $\mathscr{S} = \{(X_U, t_U) |\varrho_{UV}| X\}$ be a presheaf from the category of topological spaces such that the conditions of 2.3 are fulfilled. For open $U \subset X$ let $A_U = \{\hat{a} \mid a \in X_U\}$, where $\hat{a}(x) = \xi_{Ux}(a)$ for $x \in U$. The homeomorphisms h_{xy} between the stalks (I_x, t_x), (I_y, t_y) enable us to bring over open nbds of elements from one stalk to another within connected sets, and thus define the topology of uniform convergence in A_U, for now we know what it means that two nbds in different stalks are of the same size. Namely, if U is open and connected, $a \in X_U$, $x \in U$, and if W is a t-nbd of $\hat{a}(x)$, we set $O(\hat{a}, W) = \{\hat{b} \in A_U \mid \hat{b}(y) \in h_{xy}(W) \text{ for all } y \in U\}$. Letting W run through the set of all t_x-nbds of $\hat{a}(x)$ and doing it for all $\hat{a} \in A_U$, we get a topology s_U (which may be called the topology of uniform convergence). Uf U is not connected then — as X is locally connected — its components are open; we projectively define s_U in A_U by the maps $\{r_{UV} : A_U \to A_V \mid V \in \mathscr{C}(U)\}$, where $\mathscr{C}(U)$ is the set of all components of U, and $r_{UV}(\hat{a}) = \hat{a}/V$. While in [2] there were difficulties with the continuity of the natural map $p_U : (X_U, t_U) \to (A_U, s_U)$ which sends $a \in X_U$ onto $\hat{a} \in A_U$, in our setting we have

2.5. Proposition. Under the conditions of 2.4, the map p_U is continuous.

Proof. Given $U \subset X$ open and connected, $a \in X_U$, and an s_U-nbd $O = O(\hat{a}, W)$ of $\hat{a}$, then from $h_{xy}\xi_{Ux} = \xi_{Uy}$ for any $x, y \in U$ we get $p_U(W) \subset 0$. If U is not connected it is enough to show that $r_{UV}p_U$ is continuous for every $V \in \mathscr{C}(U)$; but $r_{UV}p_U = p_V\varrho_{UV}$ and both p_V, ϱ_{UV} are continuous.

3. Inductive and Semiinductive Modifications

If $\mathscr{M}$ is a family of subsets of a set Y, we set $\bigcap\mathscr{M} = \bigcap\{M \mid M \in \mathscr{M}\}$.

3.1. Definition. If X is a topological space, $U \in \mathscr{B}(X)$, then a star of U is a set $\mathscr{S} \subset \mathscr{B}(X)$ such that $U \subset \bigcap\mathscr{S}$. The set of all stars of U is denoted by $\sigma(U)$. Let $\mu = \{\tau_U\}$ be a closuration of a presheaf $\mathscr{F} = \{S_U|\varrho_{UV}| X\}$ (see 1.1), $U \subset X$ open.

If $\mathscr{S} \in \sigma(U)$, we have a set of maps $\Delta(\mathscr{S}) = \{\varrho_{VU} \mid V \in \mathscr{S}\}$ of the closure spaces (S_V, τ_V), $V \in \mathscr{S}$ into S_U (the closure τ_U is not considered now). The closure inductively generated in S_U by the maps from $\Delta(\mathscr{S})$ is denoted by $\tau_U(\mathscr{S})$.

3.2. Definition. Let $U, V \in \mathscr{B}(X)$, $V \subset U$, $\mathscr{S}_1 \in \sigma(U)$, $\mathscr{S}_2 \in \sigma(V)$. We say that $\mathscr{S}_1$ refines $\mathscr{S}_2$ ($\mathscr{S}_1 \leqq \mathscr{S}_2$), if for every $M \in \mathscr{S}_1$ there is $N \in \mathscr{S}_2$ such that $N \subset M$. If moreover N can be found such that $\varrho_{MN} : (S_M \tau_M) \to (S_N \tau_N)$ is continuous, we say $\mathscr{S}_1$ strongly refines $\mathscr{S}_2$ ($\mathscr{S}_1 \lesssim \mathscr{S}_2$).

3.3. Proposition. Let $\mathscr{S}_1 \lesssim \mathscr{S}_2$. Then the map $\varrho_{UV} : (S_U, \tau_U(\mathscr{S}_1)) \to (S_V, \tau_V(\mathscr{S}_2))$ is continuous.

Proof. We take the following commutative diagram for any $M \in \mathscr{S}_1$ and $N \in \mathscr{S}_2$ such that $N \subset M$ and that $\varrho_{MN} : (S_M \tau_M) \to (S_N \tau_N)$ is continuous.

$$\begin{array}{ccc} (S_U, \tau_U(\mathscr{S}_1)) & \xleftarrow{\varrho_{MU}} & (S_M, \tau_M) \\ \downarrow{\scriptstyle \varrho_{UV}} & & \downarrow{\scriptstyle \varrho_{MN}} \\ (S_V, \tau_V(\mathscr{S}_2)) & \xleftarrow{\varrho_{NV}} & (S_N, \tau_N) \end{array}$$

Here ϱ_{UV} is continuous if so is $\varrho_{UV}\varrho_{MU}$ for each $M \in \mathscr{S}_1$. As $\varrho_{UV}\varrho_{MU} = \varrho_{NV}\varrho_{MN}$ and both ϱ_{VN}, ϱ_{MN} are continuous, we are done.

3.4. Definition. A choice of stars is a map $s : \{U \to s(U) \subset \sigma(U) \mid U \in \mathscr{B}(X)\}$ with $s(U) \neq \emptyset$ for all U's. A closuration $\mu = \{\tau_U\}$ of $\mathscr{F}$ is called s-semiinductive (s-inductive) if (3.5) $\tau_U(\mathscr{S}) \geqq \tau_U$ ($\tau_U(\mathscr{S}) = \tau_U$) for every $U \in \mathscr{B}(X)$, $\mathscr{S} \in s(U)$.

The following two propositions are clear:

3.6. Proposition. If $U \in \mathscr{B}(X)$, $\mathscr{S} \in s(U)$, then $\tau_U(\mathscr{S}) \geqq \tau_U$ iff the following condition is fulfilled: If (P, t) is any closure space and $f : (S_U, \tau_U) \to (P, t)$ any map, then the continuity of $f\varrho_{VU}$ for all $V \in \mathscr{S}$ yields that of f.

3.7. Proposition. Let $U \in \mathscr{B}(X)$, $\mathscr{S} \in \sigma(U)$. TFAE:

a) $\tau_U(\mathscr{S}) \leqq \tau$;

b) If (P, t) and f are as in 3.6, then the continuity of f yields that of $f\varrho_{VU}$ for each $V \in \mathscr{S}$.

c) $\varrho_{VU} : (S_V \tau_V) \to (S_U, \tau_U)$ is continuous for each $V \in \mathscr{S}$.

If μ is s-inductive, it need not be compatible, but we have

3.8. Proposition. Each of the following properties of s yields the compatibility of the s-inductive closuration μ:

a) If $V \subset U$ then there are $\mathscr{S} \in s(V)$, $\tilde{\mathscr{S}} \in s(U)$ such that $\tilde{\mathscr{S}} \lesssim \mathscr{S}$ (see 1.2).

b) If $V \subset U$ then there is $\mathscr{S} \in s(V)$ with $U \in \mathscr{S}$.

Proof. Let $\mu = \{\tau_U\}$ be s-inductive, $V \subset U$. (a): Let $\mathscr{S} \in s(V)$, $\tilde{\mathscr{S}} \in s(U)$, $\tilde{\mathscr{S}} \lesssim \mathscr{S}$. As $\tau_U = \tau_U(\tilde{\mathscr{S}})$, $\tau_V = \tau_V(\mathscr{S})$, (a) follows from 3.3. (b): We take $\mathscr{S} \in s(V)$ with $U \in \mathscr{S}$. As $\tau_V = \tau_V(\mathscr{S})$ and $\varrho_{UV} : (S_U \tau_U) \to (S_V, \tau_V(\mathscr{S}))$ is continuous, we are done.

3.9. Remark. Clearly if μ is s-semiinductive and compatible then it is s-inductive since by 3.7c, a, the compatibility yields $\tau_U(\mathscr{S}) \leqq \tau_U$ for $\mathscr{S} \in s(U)$, and the s-semiinductivity yields $\tau_U \leqq \tau_U(\mathscr{S})$. So if any of the conditions of 3.8 is fulfilled then μ is s-inductive iff it is compatible and s-semiinductive.

3.10. Proposition. Let Ω be a nonempty set of closurations of a presheaf $\mathscr{F}$, let μ^Ω be its supremum, i.e. $\mu^\Omega = \{\tau_U^\Omega\}$, where $\tau_U^\Omega = \varinjlim \{\tau_U^\nu \mid \nu = \{\tau_U^\nu\}, \nu \in \Omega\}$.

(a) If every $\nu \in \Omega$ is compatible then μ^Ω is.

(b) If every $\nu \in \Omega$ is s-semiinductive then μ^Ω is.

(c) If every $\nu \in \Omega$ is s-inductive then μ^Ω is.

Proof. To prove (b) it is enough by 3.6 to show the following: "Let $U \in \mathscr{B}(X)$, $\mathscr{S} \in s(U)$, let (P, t) be any closure space, $f : (S_U, \tau_U^\Omega) \to (P, t)$ a map. Then the continuity of $f\varrho_{VU}$ for all $V \in \mathscr{S}$ yields that of f". Let us look at the following commutative diagram for $\mathscr{S} \in s(U)$, $V \in \mathscr{S}$, $\nu \in \Omega$:

$$\begin{tikzcd}
(S_V, \tau_V^\nu) \arrow[r, "\varrho_{VU}^\nu"] \arrow[d, "i_V^\nu"] & (S_U \tau_U^\nu) \arrow[d, "i_U^\nu"] & \\
(S_V \tau_V^\Omega) \arrow[r, "\varrho_{VU}"] & (S_U \tau_U^\Omega) \arrow[r, "f"] & (P, t)
\end{tikzcd}$$

Here f is continuous iff for each $\nu \in \Omega f i_U^\nu$ is. Let $\nu \in \Omega$. Both $i_V^\nu, f\varrho_{VU}$ are continuous for each $V \in \mathscr{S}$, so for every $V \in \mathscr{S}$, $f\varrho_{VU} i_V^\nu = f i_U^\nu \varrho_{VU}^\nu$ is. But $\nu = \{\tau_U^\nu\}$ is s-semiinductive, thus $f i_U^\nu$ is continuous for each $\nu \in \Omega$, hence f is.

(c) By 3.6, 3.7a, b, it is enough to show the following: "Let $U \in \mathscr{B}(X)$, $\mathscr{S} \in s(U)$, let (P, t) be a closure space and $f : (S_U, \tau_U^\Omega) \to (P, t)$ a map. Then f is continuous iff for every $V \in \mathscr{S}$ $f\varrho_{VU} : (S_V \tau_V^\Omega) \to (S_U, \tau_U^\Omega) \to (P, t)$ is." The "if" part has just been proven. Now, let f be continuous. We can see from the above diagram that $f\varrho_{VU}$ is continuous iff for any $\nu \in \Omega\, f\varrho_{VU} i_V^\nu$ is. But it is just $f i_U^\nu \varrho_{VU}^\nu$. As f and i_U^ν are continuous for every ν, $f i_V^\nu$ is, too, and the s-inductivity of ν yields the continuity of $f i_U^\nu \varrho_{VU}^\nu$. Likewise (a) can be proven.

The part of the following theorem concerning μ_s^I if $s(U) = \sigma(U)$ is due to Z. Frolík, [1, p. 58, 59].

3.11. Theorem. Let μ be a closuration of $\mathscr{F}$, s a choice. Then there is a closuration μ_s^I and μ_s^{SI} of $\mathscr{F}$ such that

a) $\mu_s^I \leqq \mu_s^{SI} \leqq \mu$.

b) μ_s^I is s-inductive (hence compatible if (a) or (b) of 3.8 is fulfilled), μ_s^{SI} is s-semiinductive.

c) If $\mu^1(\mu^2)$ is an s-inductive (s-semiinductive) closuration of $\mathscr{F}$ such that $\mu^1 \leqq \mu(\mu^2 \leqq \mu)$ then $\mu^1 \leqq \mu_s^I(\mu^2 \leqq \mu_s^{SI})$.

Proof. If $\Omega^I(\mu)$ is the set of all s-inductive closurations of $\mathscr{F}$ finer than μ, we put $\mu_s^I = \{\tau_{U,s}^I\}$ where $\tau_{U,s}^I = \varinjlim \{\tau_U^\nu \mid \nu = \{\tau_U^\nu\},\ \nu \in \Omega^I(\mu)\}$. By 3.10, μ_s^I is s-inductive. Likewise we make μ_s^{SI}.

3.12. Definition. $\mu_s^I(\mu_s^{SI})$ is called s-inductive (s-semiinductive) modification of μ.

3.13. Proposition. Let μ be a closuration of $\mathscr{F}$, s a choice. For $U \in \mathscr{B}(X)$ let

$$\tau_{U,s}^\square = \varinjlim \{\tau_U(\mathscr{S}) \mid \mathscr{S} \in s(U)\}\,, \quad \mu_s^\square = \{\tau_{U,s}^\square\}\,. \tag{3.14}$$

Then $\mu_s^{SI} \leqq \mu_s^\square$. If μ is s-semiinductive (s-inductive) then $\mu \leqq \mu_s^\square(\mu = \mu_s^\square)$. Suppose moreover the following condition **C**: "For every U there is $\mathscr{S} \in s(U)$ such that for every $V \in \mathscr{S}$ the map $\varrho_{VU} : (S_V \tau_V) \to (S_U, \tau_U)$ is continuous." Then $\mu_s^\square \leqq \mu$. (**C** is fulfilled namely if μ is compatible or if $\{U\} \in s(U)$ for each U).

Proof. Let $U \in \mathscr{B}(X)$, $\mathscr{S} \in s(U)$. If $\mu_s^{SI} = \{\tau_{U,s}^{SI}\}$ then $\tau_{U,s}^{SI} \leqq \tau_{U,s}^{SI}(\mathscr{S}) \leqq \tau_U(\mathscr{S})$, hence by 3.14, $\tau_{U,s}^{SI} \leqq \tau_{U,s}^\square$ for all U so $\mu_s^{SI} \leqq \mu_s^\square$. If μ is s-semiinductive (s-inductive), then for each $\mathscr{S} \in s(U)$ we get $\tau_U \leqq \tau_U(\mathscr{S})$ $(\tau_U = \tau_U(\mathscr{S}))$. Thus $\mu \leqq \mu_s^\square(\mu = \mu_s^\square)$. If **C** holds then $\mu_s^\square \leqq \mu$ follows from 3.7a, c and 3.14.

3.15. Proposition. If $\mu = \mu_s^\square$ then $\mu = \mu_s^{SI}$. If moreover μ is compatible, then $\mu_s^I = \mu$ iff $\mu = \mu_s^\square$.

Proof. If $\mu = \mu_s^\square$ then by 3.14, $\tau_U = \tau_{Us}^\square \leqq \tau_U(\mathscr{S})$ for any $U \in \mathscr{B}(X)$, $\mathscr{S} \in s(U)$, hence μ is s-semiinductive and thus $\mu = \mu_s^{SI}$. If μ is moreover compatible, then having already been shown to be s-semiinductive, it is also s-inductive, by 3.9, hence $\mu = \mu_s^I$. On the other hand, if $\mu = \mu_s^I$ then μ is s-inductive hence $\mu_s^\square = \mu$ by 3.14.

3.16. Lemma. Let s fulfil the following: **Q**: "For every $U, V \in \mathscr{B}(X)$, $V \subset U$ and any $\mathscr{S}' \in s(V)$ there is $\mathscr{S} \in s(U)$ with $\mathscr{S} \lesssim \mathscr{S}'$." Then $\mu_s^\square$ is compatible if μ is. (**Q** holds if μ is compatible and **Q** is fulfilled with $\leqq$ instead of $\lesssim$ – see 3.2).

Proof. For open $U, V \subset X$, $V \subset U$, $\mathscr{S}' \in s(V)$, $\mathscr{S} \in s(U)$, $\mathscr{S} \lesssim \mathscr{S}'$ let us take the following commutative diagram, with identical i_U, i_V:

$$\begin{tikzcd}
(S_U, \tau_{U,s}^\square) \arrow[r, "i_U"] \arrow[d, "\varrho_{UV}"] & (S_U, \tau_U(\mathscr{S})) \arrow[d, "\varrho_{UV}"] \\
(S_V, \tau_{V,s}^\square) \arrow[r, "i_U"] & (S_V, \tau_V(\mathscr{S}))
\end{tikzcd}$$

Here ϱ_{UV} on the left hand side is continuous iff $i_V\varrho_{UV}$ is for any $\mathscr{S}' \in s(V)$, by 3.14. But $i_V\varrho_{UV} = \varrho_{UV}i_U$. Here i_U is continuous by 3.14, and ϱ_{UV} on the right hand side by 3.3.

3.17. Theorem. Let s fulfil **Q** and μ be compatible. Then $\mu_s^{SI} = \mu_s^I$ and they both are compatible. Moreover, μ_s^I can be reached by letting the operator $\nu \to \nu_s^\square$ work upon μ for enough times.

Proof. Set $\mu^1 = \mu$. Let α be any ordinal and let us have already made μ^β for all $\beta < \alpha$, with $\mu_s^{SI} \leqq \mu^\beta \leqq \mu$. If there is $\alpha - 1$, we set $\mu^\alpha = (\mu^{\alpha-1})_s^\square$ – which is by 3.16 compatible, and $\mu_s^{SI} \leqq (\mu_s^{SI})_s^\square \leqq \mu^\alpha \leqq \mu^{\alpha-1} \leqq \mu$ by 3.11a, 3.13, 3.14, hence $\mu_s^{SI} \leqq \mu^\alpha \leqq \mu$. If there is not $\alpha - 1$, we set $\mu^\alpha = \underline{\lim}\,\{\mu^\beta \mid \beta < \alpha\}$. Again $\mu_s^{SI} \leqq \mu^\alpha \leqq \mu$ and μ is compatible by 3.10a. For α large enough (say, if $\operatorname{card} \alpha > \operatorname{card} \{\nu \mid \nu$ is a closuration of $\mathscr{F}$, $\mu_s^{SI} \leqq \nu \leqq \mu\}$) we have $\mu^\alpha = (\mu^\alpha)_s^\square$. As μ^α is compatible, we get from 3.15 that μ^α is s-inductive. As $\mu_s^I \leqq \mu_s^{SI} \leqq \mu^\alpha \to \mu$, we get $\mu^\alpha = \mu_s^I = \mu_s^{SI}$.

4. Projective Modifications

In the foregoing section inductive modifications have been dealt with. Here we show for completeness that the projective ones can be treated likewise.

4.1. Definition. If X is a topological space then the set of all open covers of $U \in \mathscr{B}(X)$ is denoted by $\mathscr{C}(U)$. Let $\mu = \{\tau_U\}$ be a closuration of a presheaf $\mathscr{S} = \{S_U | \varrho_{UV} | X\}$, $U \in \mathscr{B}(X)$, $\mathscr{V} \in \mathscr{C}(U)$. The closure projectively defined in S_U by the set of maps $\Delta(\mathscr{V}) = \{\varrho_{UV} : S_U \to (S_V, \tau_V) \mid V \in \mathscr{V}\}$ is denoted by $\tau_U(\mathscr{V})$.

4.2. Definition. Let $U, V \in \mathscr{B}(X)$, $V \subset U$, $\mathscr{V}_1 \in \mathscr{C}(U)$, $\mathscr{V}_2 \in \mathscr{C}(V)$. We say $\mathscr{V}_2$ refines $\mathscr{V}_1 (\mathscr{V}_2 \leqq \mathscr{V}_1)$ if for every $M \in \mathscr{V}_2$ there is $N \in \mathscr{V}_1$ such that $M \subset N$. If moreover N can be found that $\varrho_{NM} : (S_N \tau_N) \to (S_M \tau_M)$ be continuous, we say $\mathscr{V}_2$ strongly refines $\mathscr{V}_1 (\mathscr{V}_2 \lesssim \mathscr{V}_1)$.

4.3. Proposition. Let $U, V \in \mathscr{B}(X)$, $V \subset U$, $\mathscr{V}_1 \in \mathscr{C}(U)$, $\mathscr{V}_2 \in \mathscr{C}(V)$, $\mathscr{V}_2 \lesssim \mathscr{V}_1$. Then $\varrho_{UV} : (S_U, \tau_U(\mathscr{V}_1)) \to (S_V, \tau_V(\mathscr{V}_2))$ is continuous.

Proof. The same as that of 3.3 for inductive case, only we use the projective definition of $\tau_V(\mathscr{V}_2)$.

For the part of the following definition concerning projective closurations see Z. Frolík, [1, p.58, 59].

4.4. Definition. A choice of covers is a map $c : \{U \to c(U) \subset \mathscr{C}(U) \mid U \in \mathscr{B}(X)\}$ with $c(U) \neq \emptyset$ for all U's; $\mu = \{\tau_U\}$ is called c-semiprojective (c-projective) if $\tau_U(\mathscr{V}) \leqq \tau_U(\tau_U(\mathscr{V}) = \tau_U)$ for every $U \in \mathscr{B}(X)$, $\mathscr{V} \in c(U)$.

The following two propositions are clear:

4.5. Proposition. If $U \in \mathscr{B}(X)$, $\mathscr{V} \in \mathscr{C}(U)$, then $\tau_U(\mathscr{V}) \leqq \tau_U$ iff the following condition is fulfilled: "If (P, t) is a closure space and $f : (P, t) \to (S_U, \tau_U)$ a map, then the continuity of $\varrho_{UV} f$ for all $V \in \mathscr{V}$ yields that of f".

4.6. Proposition. Let $U \in \mathscr{B}(X)$, $\mathscr{V} \in \mathscr{C}(U)$. TFAE:

a) $\tau_U(\mathscr{V}) \geqq \tau_U$.

b) If (P, t) and f are as in 4.5, then the continuity of f yields that of $\varrho_{UV} f$ for every $V \in \mathscr{V}$.

c) $\varrho_{UV} : (S_U \tau_U) \to (S_V \tau_V)$ is continuous for each $V \in \mathscr{V}$.

If μ is c-projective it need not be compatible, but we have

4.7. Proposition. Each of the following properties of c yields the compatibility of the c-projective closuration μ:

a) If $V \subset U$ then there is $\mathscr{V} \in c(U)$, $\tilde{\mathscr{V}} \in c(V)$ with $\tilde{\mathscr{V}} \lesssim \mathscr{V}$ (see 4.2).

b) If $V \subset U$ then there is $\mathscr{V} \in c(U)$ with $V \in \mathscr{V}$.

Proof. Since $\tau_U = \tau_U(\mathscr{V})$, $\tau_V = \tau_V(\tilde{\mathscr{V}})$, (a) follows from 4.3. To prove (b), we take $\mathscr{V} \in c(U)$ with $V \in \mathscr{V}$. As $\tau_U = \tau_U(\mathscr{V})$ and $\varrho_{UV} : (S_U, \tau_U(\mathscr{V})) \to (S_V, \tau_V)$ is continuous, we are done.

4.8. Remark. μ is s-projective if it is compatible and s-semiprojective, since by 4.6c, a, the compatibility yields $\tau_U \leqq \tau_U(\mathscr{V})$ for any $\mathscr{V} \in c(U)$, and c-semiprojectivity yields $\tau_U(\mathscr{V}) \leqq \tau_U$. So if any of the conditions of 4.7 is fulfilled then μ is c-projective iff it is compatible and c-semiprojective.

4.9. Proposition. Let Ω be a nonempty set of closurations of a presheaf $\mathscr{F}$, μ_Ω its infimum, i.e. $\mu_\Omega = \{\tau_{U,\Omega}\}$, where $\tau_{U,\Omega} = \varliminf \{\tau_U^\nu \mid \nu = \{\tau_U^\nu\}, \nu \in \Omega\}$ – see 1.2. If each $\nu \in \Omega$ is compatible (c-semiprojective, c-projective), then μ_Ω is.

Proof. As in 3.10, only we use the properties of projectively defined closure.

The statement of the following theorem for μ_c^P with $c(U) = \mathscr{C}(U)$ is due to Z. Frolík [1, p. 58, 59].

4.10. Theorem. Let μ be a closuration of $\mathscr{F}$, c a choice. Then there is a closuration μ_c^P and μ_c^{SP} such that

a) $\mu \leqq \mu_c^{SP} \leqq \mu_c^P$,

b) μ_c^P is c-projective (and if the condition (a) or (b) of 4.7 holds compatible), μ_c^{SP} is c-semiprojective.

c) If $\mu^1(\mu^2)$ is c-projective (c-semiprojective) closuration of $\mathscr{F}$ and $\mu \leqq \leqq \mu^1(\mu \leqq \mu^2)$ then $\mu_c^P \leqq \mu^1(\mu_c^{SP} \leqq \mu^2)$.

Proof. Easy from 4.9.

4.11. Definition. $\mu_c^P(\mu_c^{SP})$ is called c-projective (c-semiprojective) modification of μ.

4.12. Proposition. Given a choice c, we set

$$\tau_{U,c}^* = \varliminf \{\tau_U(\mathscr{V}) \mid \mathscr{V} \in c(U)\}\,, \quad \mu_c^* = \{\tau_{U,c}^* \mid U \in \mathscr{B}(X)\}\,. \tag{4.13}$$

Then $\mu_c^* \leqq \mu_c^{SP}$. If μ is c-semiprojective then $\mu_c^* \leqq \mu$. Suppose moreover the following condition **D**: "For every $U \in \mathscr{B}(X)$ there is $\mathscr{V} \in c(U)$ such that for every $V \in \mathscr{V}$ the map $\varrho_{UV} : (S_U \tau_U) \to (S_V \tau_V)$ is continuous." Then $\mu \leqq \mu_c^*$. (**D** holds namely if μ is compatible or if $U \in c(U)$).

Proof. If μ is c-semiprojective then $\tau_U(\mathscr{V}) \leqq \tau_U$ for any $\mathscr{V} \in c(U)$ so $\mu_c^* \leqq \mu$. This yields $\mu_c^* \leqq (\mu_c^{SP})_c^* \leqq \mu_c^{SP}$. If **D** holds then $\mu \leqq \mu_c^*$ by 4.6c, a and 4.13.

4.14. Proposition. If $\mu = \mu_c^*$ then $\mu = \mu_c^{SP}$. If μ is compatible then $\mu_c^P = \mu$ iff $\mu = \mu_c^*$.

Proof. If $\mu = \mu_c^*$ then by 4.13, $\tau_U = \tau_{U,s}^* \geqq \tau_U(\mathscr{V})$ for any $U \in \mathscr{B}(X)$, $\mathscr{V} \in c(U)$, hence μ is c-semiprojective and thus $\mu = \mu_c^{SP}$. If μ is compatible, then having already been shown to be c-semiprojective, it is also c-projective by 4.8, hence $\mu = \mu_c^P$. On the other hand, if $\mu = \mu_c^P$ then μ is c-projective hence $\mu_c^* = \mu$ by 4.13.

4.15. Lemma. Let μ and c fulfil the following condition **R**: "If $U, V \in \mathscr{B}(X)$, $V \subset U$ then for every $\mathscr{V} \in c(U)$ there is $\mathscr{W} \in c(V)$ with $\mathscr{W} \lesssim \mathscr{V}$ (see 4.2, it namely holds if μ is compatible and **R** holds only with $\leqq$). Then μ_c^* is compatible.

Proof. Take the following commutative diagram for $\mathscr{V} \in c(U)$, $\mathscr{W} \in c(V)$ with $\mathscr{W} \lesssim \mathscr{V}$, and with identical i_U, i_V:

$$\begin{tikzcd}
(S_U, \tau_{U,c}^*) \arrow[d, "\varrho_{UV}"] & (S_U, \tau_U(\mathscr{V})) \arrow[l, dashed, "i_U"'] \arrow[d, "\varrho_{UV}"] \\
(S_V, \tau_{V,c}^*) & (S_V, \tau_V(\mathscr{W})) \arrow[l, "i_V"']
\end{tikzcd}$$

Here ϱ_{UV} is continuous iff $\varrho_{UV} i_U$ is for any $\mathscr{V} \in c(U)$. But $\varrho_{UV} i_U = i_V \varrho_{UV}$, where both maps on the right hand side are continuous by 4.3, 4.13.

4.16. Proposition. If μ is compatible and c fulfils **R** then $\mu_c^{SP} = \mu_c^P$, and they both are compatible. Further, μ_c^P can be reached by letting the operator $v \to v_c^*$ work upon μ for enough times.

Proof. Set $\mu^1 = \mu$. Let us have made a compatible μ^β for each ordinal $\beta < \alpha$ with $\mu \leqq \mu^\beta \leqq \mu_c^{SP}$. If there is $\alpha - 1$, we set $\mu^\alpha = (\mu^{\alpha-1})_c^*$, if there is not $\alpha - 1$, we set $\mu^\alpha = \varinjlim \{\mu^\beta \mid \beta < \alpha\}$. In the both cases μ^α is compatible, by 4.15 in the former case, by 4.9 in the latter, and $\mu \leqq \mu^\alpha \leqq \mu_c^{SP}$. For α large enough (say, if card $\alpha \geqq$ $\geqq$ card $\{v \mid \mu \leqq v \leqq \mu_s^P\}$) we have $\mu^\alpha = (\mu^\alpha)_c^*$ and by 4.14, $\mu^\alpha = (\mu^\alpha)_c^P$ whence μ^α is s-projective. As $\mu \leqq \mu^\alpha \leqq \mu_c^{SP} \leqq \mu_c^P$, we have $\mu^\alpha = \mu_c^P = \mu_c^{SP}$ and we're done.

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