Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 19 (1978), No. 2, 61--62

Persistent URL: `http://dml.cz/dmlcz/142422`

1978 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 19, NO. 2, Pag. 61–62

A Note on WA-Quasigroups

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Received 10 March 1977

In the note, there is proved that every WA-quasigroup satisfies the identity $xy \cdot zx = xz \cdot yx$.

Доказывается, что в любой WA-квазигруппе выполняется тождество xy.zx = xz.yx.

V poznámce se dokazuje, že každá WA-kvazigrupa splňuje identitu $xy \cdot zx = xz \cdot yx$.

A groupoid is called a WA-groupoid if it satisfies the identities $xx \cdot yz = xy \cdot xz$ and $yz \cdot xx = yx \cdot zx$. It is the purpose of this note to show that every cancellation WA-groupoid satisfies the identity $xy \cdot zx = xz \cdot yx$.

Lemma 1. Every WA-groupoid satisfies the identity

$((yy \cdot yx)(zx \cdot zx))((xy \cdot zx)(xx \cdot zx)) = ((yy \cdot yx)(zx \cdot zx))((xz \cdot yx)(xx \cdot zx))$.

Proof. Let G be a WA-groupoid and $a, b, c \in G$. Then

$$\begin{aligned}
&((cc \cdot ca)(ba \cdot ba))((ac \cdot ba)(aa \cdot ba)) = ((cc \cdot ca)(ba \cdot ba))((ac \cdot aa)(ba \cdot ba)) = \\
&= ((cc \cdot ca)(ba \cdot ba))((aa \cdot ca)(ba \cdot ba)) = ((cc \cdot ca)(aa \cdot ca))((ba \cdot ba)(ba \cdot ba)) = \\
&= ((cc \cdot aa)(ca \cdot ca))((ba \cdot ba)(ba \cdot ba)) = ((ca \cdot ca)(ca \cdot ca))((ba \cdot ba)(ba \cdot ba)) = \\
&= ((ca \cdot ca)(ca \cdot ca))((bb \cdot aa)(ba \cdot ba)) = ((ca \cdot ca)(ca \cdot ca))((bb \cdot aa)(bb \cdot aa)) = \\
&= ((ca \cdot ca)(ca \cdot ca))((bb \cdot bb)(aa \cdot aa)) = ((ca \cdot ca)(bb \cdot bb))((ca \cdot ca)(aa \cdot aa)) = \\
&= ((ca \cdot bb)(ca \cdot bb))((ca \cdot ca)(aa \cdot aa)) = ((ca \cdot bb)(ca \cdot bb))((ca \cdot aa)(ca \cdot aa)) = \\
&= ((cb \cdot ab)(ca \cdot bb))((ca \cdot aa)(ca \cdot aa)) = ((cb \cdot ab)(cb \cdot ab))((ca \cdot aa)(ca \cdot aa)) = \\
&= ((cb \cdot cb)(ab \cdot ab))((ca \cdot aa)(ca \cdot aa)) = ((cb \cdot cb)(ca \cdot aa))((ab \cdot ab)(ca \cdot aa)) = \\
&= ((cb \cdot ca)(cb \cdot aa))((ab \cdot ab)(ca \cdot aa)) = ((cc \cdot ba)(cb \cdot aa))((ab \cdot ab)(ca \cdot aa)) = \\
&= ((cc \cdot ba)(ca \cdot ba))((ab \cdot ab)(ca \cdot aa)) = ((cc \cdot ca)(ba \cdot ba))((ab \cdot ab)(ca \cdot aa)) = \\
&= ((cc \cdot ca)(ba \cdot ba))((ab \cdot ca)(ab \cdot aa)) = ((cc \cdot ca)(ba \cdot ba))((ab \cdot ca)(aa \cdot ba)).
\end{aligned}$$

Corollary 2. Every cancellation WA-groupoid satisfies the identity $xy \cdot zx = xz \cdot yx$.

If Q is a commutative Moufang loop then $N(Q)$ is the nucleus of Q and a mapping f of Q into Q is said to be nuclear if $a^{-1} \cdot f(a) \in N(Q)$ for each $a \in Q$.

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Theorem 3. The following conditions are equivalent for every quasigroup Q:

(i) Q is a WA-quasigroup.

(ii) There exist a commutative Moufang loop $Q(\circ)$, its automorphisms f, g and $q \in Q$ such that $fg = gf$, fg^{-1} is a nuclear mapping and $ab = (f(a) \circ g(b)) \circ q$ for all $a, b \in Q$.

Proof. Apply Corollary 2 and [3] Theorem 1.

A groupoid satisfying the identity $xy \cdot uv = xu \cdot yv$ is called medial. A WA-groupoid is called primitive if it is commutative and satisfies the identity $xx \cdot xy = yy \cdot yy$.

Theorem 4. Let Q be a WA-quasigroup. Then there exists a normal congruence r of Q such that Q/r is a primitive WA-quasigroup. Moreover, if a class A a of r is a subquasigroup of Q then A is a medial quasigroup.

Proof. Apply Corollary 2 and [2] Proposition 5.

Corollary 5. Every simple WA-quasigroup is either primitive or medial.

A quasigroup is said to be trimedial if every its subquasigroup generated by at most three elements is medial.

Theorem 6. The following conditions are equivalent for every quasigroup Q:

(i) Q is trimedial.

(ii) Q is a WA-quasigroup and Q satisfies the identity $(x \cdot xx)(yz) = (xy)(xx \cdot z)$.

(iii) Q satisfies the identity $((xx \cdot yz)(vw \cdot uu))((p \cdot pp)(st)) = ((xy \cdot xz)(vu \cdot wu))((ps)(pp \cdot t))$.

Proof. Apply Corollary 2 and [3] Theorem 2.

References

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[3] T. Kepka: Structure of triabelian quasigroups, Comment. Math. Univ. Carolinae 17, 1976, 229–240.