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Commutative Distributive Groupoids

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A groupoid is called distributive if it satisfies the identities $x \, . \, yz = xy \, . \, xz$ and $yz \, . \, x = yx \, . \, zx$. The main purpose of the paper is to show that every finitely generated distributive groupoid has only a finite number of ideals.

Группоид называется дистрибутивным, если в нем выполнены тождества x.yz = xy.xz и yz.x = yx.zx. В статье доказывается, что любой конечно порожденный коммутативный дистрибутивный группоид имеет только конечное множество идеалов.

Grupoid se nazývá distributivní jestliže splňuje identity $x \, . \, yz = xy \, . \, xz$ a $yz \, . \, x = yx \, . \, zx$ Cílem článku je dokázat, že každý konečně generovaný komutativní distributivní grupoid má jen konečně mnoho ideálů.

In the present paper, some properties and classes of commutative distributive groupoids are studied. A special emphasis is laid on finitely generated and subdirectly irreducible groupoids.

I. Preliminaries

Let G be a groupoid and $a \in G$. We define two mappings R_a^G and L_a^G of G into G by $R_a^G(b) = ba$ and $L_a^G(b) = ab$ for every $b \in G$. We shall say that G is a cancellation (division) groupoid if these mappings are injective (surjective) for every $a \in G$. Further, G is called a quasigroup if it is both a cancellation and division groupoid. A congruence r of G is said to be normal if the corresponding factorgroupoid G/r is a cancellation groupoid. As it is easy to see, the diagonal congruence d_G of G is normal iff G is a cancellation groupoid. On the other hand, there exist quasigroups which have non-normal congruences.

Let G be a groupoid. A non-empty subset $I \subseteq G$ is called an ideal if $ab, ba \in G$, whenever $a \in I$ and $b \in G$. The following lemma is obvious.

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1.1. Lemma. Let K_j, $j \in \mathcal{J}$ be a non-empty set of ideals of a groupoid G and $A = \bigcup K_j$, $B = \bigcap K_j$. Then A is an ideal and B is an ideal, provided B is non-empty. Moreover, if $\mathcal{J}$ is finite then B is an ideal.

From 1.1 we see that the set $I(G)$ of all ideals of G is a distributive lattice containing the greatest element. If $I(G)$ contains only one element then we shall say that G is an i-simple groupoid. Clearly, i-simple groupoids are closed under homomorphic images. The following lemma is easy.

1.2. Lemma. Let r be a congruence of a groupoid G such that every class of r is either a one-element set or an i-simple subgroupoid of G. Then G is i-simple iff the factorgroupoid G/r is.

It is easy to see that every division groupoid is i-simple. The following lemma is clear.

1.3. Lemma. Let I be an ideal of a groupoid G. Define r by arb iff either $a, b \in I$ or $a = b$. Then r is a congruence of G.

An element z of a groupoid G is called a zero if $az = z = za$ for every $a \in G$. Clearly, z is a zero iff the one-element subset $\{z\}$ is an ideal of G. It is evident that every groupoid contains at most one zero.

Let G be a groupoid and z be an element such that $z \notin G$. We shall define a groupoid $G\{z\}$ as follows:
$G\{z\} = G \cup \{z\}$, G is a subgroupoid of $G\{z\}$ and z is the zero.

The following two lemmas are evident.

1.4. Lemma. The following conditions are equivalent for a groupoid variety $\mathscr{V}$:

(i) $\mathscr{V}$ contains all commutative idempotent semigroups.

(ii) The two-element groupoid $\{0,1\}$ with multiplication $0 . 0 = 0 . 1 = 1 . 0 = 0$, $1 . 1 = 1$ is contained in $\mathscr{V}$.

(iii) If $G \in \mathscr{V}$ and $z \notin G$ then $G\{z\} \in \mathscr{V}$.

(iv) $\mathscr{V}$ can be determined by a set of identities $t_j = s_j$ such that for every j, the two terms t_j and s_j contain the same variables.

1.5. Lemma. Let G be an i-simple groupoid and $z \notin G$. Then $G\{z\}$ is subdirectly irreducible iff G is.

2. Distributive Groupoids

A groupoid G is called

- distributive if it satisfies the identities $x . yz = xy . xz$ and $yz . x = yx . zx$,
- commutative if it satisfies the identity $xy = yx$,
- idempotent if it satisfies the identity $xx = x$,
- medial if satisfies the identity $xy . uv = xu . yv$,
- a B-groupoid if it satisfies the identity $x . yz = uv . w$.

The variety of all distributive (resp. commutative, idempotent, medial) groupoids will be denoted by $\mathscr{D}$ (resp. $\mathscr{C}$, $\mathscr{Q}$, $\mathscr{M}$). Further, we denote by $\mathscr{B}$ the variety of all B-groupoids. The following lemma is trivial.

2.1. Lemma. $\mathscr{M} \cap \mathscr{Q} \subseteq \mathscr{D}$, $\mathscr{B} \subseteq \mathscr{D} \cap \mathscr{M}$.

In the sequel, we shall use also the notation $\mathscr{A} = \mathscr{C} \cap \mathscr{Q} \cap \mathscr{M}$ and $\mathscr{E} = \mathscr{C} \cap \mathscr{Q} \cap \mathscr{D}$. If G is a groupoid then $Id\, G$ denotes the set of all idempotents of G.

2.2. Proposition. Let G be a distributive groupoid. Then:

(i) $Id\ G$ is an ideal of G and $a \,.\, bc$, $ab \,.\, c \in Id\ G$ for all $a, b, c \in G$.

(ii) $aa \,.\, a = aa \,.\, aa = a \,.\, aa$ for every $a \in G$.

(iii) The mapping f, defined by $f(x) = x \,.\, xx$, is a homomorphism of G onto $Id\ G$.

Proof. See [1], Proposition 1.2.

2.3. Proposition. The following conditions are equivalent for every groupoid G:

(i) G is distributive and $Id\ G$ contains exactly one element.

(ii) G is a semigroup with zero z and $a \,.\, bc = z$ for all $a, b, c \in G$.

(iii) G is a B-groupoid.

Proof. See [1], Proposition 1.3.

2.4. Proposition. Let G be a distributive groupoid and r be the congruence corresponding to $Id\ G$ in the sense of 1.3 Put $B(G) = G/r$. Then $B(G)$ is a B-groupoid and $Id\ G$ is an idempotent distributive groupoid. Moreover, G is isomorphic to a subdirect product of $B(G)$ and $Id\ G$.

Proof. See [1], Proposition 1.5.

2.5. Proposition. Let t and s be two groupoid terms such that both t and s have length at least three and the identity $t = s$ is valid in every idempotent distributive groupoid. Then every distributive groupoid satisfies $t = s$.

Proof. The result is an easy consequence of 2.4 and 2.3.

2.6. Lemma. Let G be a distributive groupoid and I be an ideal of $Id\ G$. Then I is an ideal of G.

Proof. Let $b \in I$ and $a \in G$. Then $ab = (ab \,.\, ab)ab = (aa \,.\, a)b \in I$, since $aa \,.\, a \in Id\ G$. The rest is similar.

2.7. Proposition. Let G be a distributive groupoid. The lattice $I(G)$ is isomorphic to a subdirect product of the lattices $I(Id\ G)$ and $I(B(G))$.

Proof. Let g be the canonical homomorphism of G onto $B(G)$ and $I, K \in I(G)$. It is clear that $(I \cap Id\ G) \cup (K \cap Id\ G) = (I \cup K) \cap Id\ G$, $g(I \cup K) = g(I) \cup g(K)$, $(I \cap Id\ G) \cap (K \cap Id\ G) = (I \cap K) \cap Id\ G$ and $g(I \cap K) \subseteq g(I) \cap g(K)$. Let $a \in I$, $b \in K$ and $g(a) = g(b)$. Then either $a = b \in I \cap K$ or $a, b \in Id\ G$. In the second case, $g(a) = g(b) = g(c)$, where $c \in I \cap K \cap Id\ G$ is arbitrary, and we see that $g(I \cap K) = g(I) \cap g(K)$ (the set $I \cap K \cap Id\ G$ is non-empty by 1.1). Hence the mapping $I \rightarrow \langle I \cap Id G, g(I)\rangle$ is a homomorphism of the lattice $I(G)$ into the cartesian product of the lattices $I(Id\ G)$ and $I(B(G))$. Now we prove that this homomorphism is injective. For, let $I \cap Id\ G = K \cap Id\ G$ and $g(I) = g(K)$. Then clearly every non-idempotent element from I is contained in K and conversely, and consequently $I = K$. Further, if L is an ideal of $B(G)$ and I is the inverse image of L then I is an ideal of G and $g(I) = L$. The rest is clear from 2.6.

2.8. Lemma. Let f be a homomorphism of a distributive groupoid G into a distributive groupoid H. Then f induces in a natural way two homomorphisms g and h of $Id\ G$ into $Id\ H$ and of $B(G)$ into $B(H)$, respectively. Moreover, if f is injective (surjective) then both g and h have the same property.

Proof. Easy.

2.9. Lemma. Let G be a commutative distributive groupoid and $a, b \in G$ be such that the set $K(a, b) = \{c \in G \mid ac = bc\}$ is not empty. Then $K(a, b)$ is an ideal.

Proof. Let $c \in K(a, b)$ and $d \in G$. Then $a . ac = a . bc = ab . ac = ba . bc = b . ac = b . bc$ and $a . dc = ad . ac = (a . ac)(d . ac) = (b . bc)(d . bc) = bd . bc = b . dc$.

2.10. Corollary. Let G be an i-simple commutative distributive groupoid. Then G is an idempotent cancellation groupoid and every congruence of G is normal.

Proof. It is enough to prove that G is a cancellation groupoid (G is idempotent by 2.2). For, let $a, b, c \in G$ and $ca = cb$. Then $c \in K(a, b)$ and $K(a, b) = G$, since it is an ideal. Hence $a = aa = ab = ba = bb = b$.

3. Distributive Cancellation Groupoids

3.1. Lemma. Let G be an idempotent distributive groupoid and $a \in G$ be such that L_a^G (R_a^G) is injective. Then there exists an idempotent distributive groupoid H with the following properties:

(i) G is a subgroupoid in H.

(ii) If $b \in G$ then $ac = b$ ($ca = b$) for some $c \in H$.

(iii) The mapping L_a^H (R_a^H) is an isomorphism of H onto G.

Proof. The mapping $g = L_a^G$ is an injective homomorphism of G into G. We can identify H with G and G with $g(G)$. The rest is clear.

3.2. Proposition. Let G be a distributive cancellation groupoid. Then G is idempotent and G is a subgroupoid of a distributive quasigroup Q such that Q generates the same groupoid variety as G.

Proof. This result is an easy consequence of 3.1.

3.3. Proposition. Let G be a subgroupoid of a distributive quasigroup Q such that G is contained in no proper subquasigroup of Q. Then every normal congruence of G can be extended to exactly one normal congruence of Q.

Proof. See [1], Proposition 1.7.

Let G be a distributive cancellation groupoid. A distributive quasigroup Q is called the q-envelope of G and denoted by $E(G)$ if G is a subgroupoid of Q and generates Q as a quasigroup. It follows from 3.2 and 3.3 that $E(G)$ exists and is determined up to G-isomorphism. Moreover G and $E(G)$ generate the same groupoid variety. Hence $E(G)$ is commutative (medial), provided G has the same property.

3.4. Proposition. Let P be a subquasigroup of a medial quasigroup Q. Then there exists a normal congruence r of Q such that P is one of the classes of r.

Proof. See [2], Theorem 43.

3.5. Corollary. Let P be a subquasigroup of a medial idempotent cancellation groupoid G. Then there exists a normal congruence r of G such that P is one of its classes.

Let G be a subdirectly irreducible groupoid. Then we denote by t_G the least non-trivial congruence of G. If G is trivial then we put $t_G = d_G$.

3.6. Lemma. Let G be a subdirectly irreducible idempotent distributive groupoid and $c, d \in G$ be such that $c \neq d$ and $c\, t_G\, d$. Let $a \in G$ be such that L_a^G (R_a^G) is injective and H be the groupoid constructed in 3.1. Then H is subdirectly irreducible and $c\, t_H d$.

Proof. By 3.1, the mapping $g = L_a^H$ is an isomporphism of H onto G. Hence H is subdirectly irreducible and $g^{-1}(c)\, t_H\, g^{-1}(d)$. Conesequently $c = a \,.\, g^{-1}(c)\, t_H\, a \,.\, g^{-1}(d) = d$.

3.7. Remark. Let k be a limit ordinal and G_i, $i < k$, be a chain of subgroupoids of a groupoid G such that $G = \bigcup G_i$. Suppose that each G_i is subdirectly irreducible and there exist different elements $c, d \in G_0$ such that $c\, t_{G_i}\, d$ for every $i < k$. Let r be a congruence of G such that $c\, r\, d$ is not true. Assume that $r \neq d_G$. Then $r \mid G_i$, for some $i < k$, is different from d_{G_i}, and so $c\, r \mid G_i\, d$. Thus $c\, r\, d$, a contradiction. Now we see that G is subdirectly irreducible and $c\, t_G\, d$.

3.8. Corollary. Let G be a subdirectly irreducible distributive cancellation groupoid. Then $E(G)$ is subdirectly irreducible as a groupoid.

Proof. It suffices to take into account 3.6, 3.7 and the proof of 3.2.

4. Distributive Quasigroups

The following proposition is proved in [3].

4.1. Proposition. Let Q be a distributive quasigroup and $ab \,.\, cd = ac \,.\, bd$ for some $a, b, c, d \in Q$. Then the subquasigroup generated by these four alements is medial.

4.2. Corollary. Every distributive quasigroup generated by at most three elements is medial.

The following lemma is easy and well-known.

4.3. Lemma. Let a normal congruence r and a congruence s of a quasigroup Q have a common class. Then $r = s$.

4.4. Proposition. Let Q be a distributive quasigroup and r be a congruence of Q such that at least one of the classes of r is a subquasigroup in Q. Then r is normal congruence.

Proof. (i) Let P be a subquasigroup of Q and P be a class of r. Consider some elements $a, b, c \in Q$ with $ab, ac \in P$. We show that $b\, r\, c$. For, let H be the subquasigroup of Q generated by a, b, c and $G = P \cap H$. Then H is a medial quasigroup and G is a subquasigroup in H (G is non-empty, since $ab, ac \in G$). By 3.4, there is a congruence s of H such that s is normal and G is one of its classes. However G is a class of $r \mid H$, and therefore $s = r \mid H$. Thus $b\, r\, c$, since s is normal. Similarly we show that $b\, r\, c$, whenever $a, b, c \in Q$ and ba, ca P.
(ii) There is a class A of r such that A is a subquasigroup of Q. Let B be an arbitrary class of r. We show that B is a subquasigroup of Q. If $a, b \in B$ then there is $c \in Q$ with $ac \in A$. We have $ac\, r\, bc$, and hence $ac, bc \in A$. Then $bc = ac \,.\, dc = ad \,.\, c$ for some $d \in Q$ with $dc \in A$. From this we get $b = ad$ and $d \in B$ (since $ac, dc \in A$, $a\, r\, d$ by (i)). Similarly $b = ea$ for some $e \in B$ and we have proved that B is a subquasigroup of Q.
(iii) Let $a, b, c \in Q$ and $ab\, r\, ac$. Then $ab, ac \in B$ for a class B of r. According to (ii), B is a subquasigroup in Q and $b\, r\, c$ by (i). Similarly if $ba\, r\, ca$.

A commutative idempotent groupoid G satisfying the identity $x \,.\, xy = y$ is called a Steiner quasigroup. It is clear that G is a quasigroup, every congruence of G is normal and every subgroupoid of G is a subquasigroup. Consider the following three-element groupoid S:

$S = \{1, 2, 3\}$, $1 . 1 = 1$, $1 . 2 = 3 = 2 . 1$, $1 . 3 = 2 = 3 . 1$, $2 . 2 = 2$, $2 . 3 = 1 = 3 . 2$, $3 . 3 = 3$.

One can easily see that this groupoid is a medial distributive Steiner quasigroup. Also the following lemma is easy.

4.5. Lemma. Any non-trivial Steiner quasigroup contains a subquasigroup isomorphic to the quasigroup S.

Let G be a groupoid. Then m_G (or only m) denotes the least congruence of G with $G/m_G \in \mathscr{M}$. Further, if $x \in G$ then $M(G, x)$ is the set of all $y \in G$ such that $ab . xc = ay . bc$ for some $a, b, c \in G$. It is clear that $x \in M(G, x)$.

4.6. Proposition. Let Q be a distributive quasigroup. Then:

(i) m is a normal congruence of Q and Q/m is a medial quasigroup.

(ii) Every class of m is a Steiner subquasigroup of Q.

(iii) If $x \in Q$ then the class of m determined by x is just the subquasigroup of Q generated by $M(Q, x)$.

Proof. Let $x \in Q$. Denote by Q_x the subquasigroup generated by $M(Q, x)$. It is proved in [3] that there exists a congruence r_x of Q satisfying the properties (i), (ii) of this proposition such that Q_x is one of its classes. It is obvious that $m \subseteq r_x$, and therefore every class of m is a subquasigroup. By 4.4 m is a normal congruence. Further, if $y \in M(Q, x)$ then $ab . xc = ay . bc$ for some $a, b, c \in Q$ and we have $ax . bc \; m \; ay . bc$, since Q/m is medial. However m is normal, and so $x \, m \, y$. We see that m and r_x have a common class, and hence $m = r_x$ by 4.3. The rest is clear.

Let R by the set of all rational numbers $a/2^b$, where a, b are integers. It is easy to see that R is a commutative and associative ring with unit. Moreover, R is a principal ideal domain. An abelian group $G(+)$ is a module over R iff the mapping $x \to x + x$ is a permutation of $G(+)$. The following proposition is easy and well-known.

4.7. Proposition. Let Q be a groupoid. Then Q is a commutative idempotent medial quasigroup iff there is an R-module $Q(+)$ such that $ab = 1/2(a + b)$ for all $a, b \in Q$. In this case, Q is subdirectly irreducible iff $Q(+)$ has the same property.

Let $p \geq 3$ be a prime and $n \geq 1$ be a natural number. Then $C(p^n)(+)$ denotes the cyclic group of order p^n. Further, $C(p^\infty)(+)$ is the quasicyclic Prüfer p-group. By $C(p^n, 1/2)$ and $C(p^\infty, 1/2)$ we denote the corresponding commutative idempotent medial quasigroups. The groups $C(p^n)(+)$, $C(p^\infty)(+)$ are the only subdirectly irreducible non-zero R-modules and we can formulate the following proposition.

4.8. Proposition. The one-element quasigroup and the quasigroups $C(p^n, 1/2)$, where $p \geq 3$ is a prime and $1 \leq n \leq \infty$ are the only subdirectly irreducible commutative idempotent medial quasigroups. Moreover, $C(p^i, 1/2)$ is isomorphic to $C(q^j, 1/2)$ iff $p = q$ and $i = j$.

The following lemma is an easy consequence of 4.8.

4.9. Lemma. The subdirectly irreducible commutative idempotent medial quasigroups are closed under subgroupoids and homomorphic images.

4.10. Proposition. Every congruence of a commutative distributive quasigroup is normal.

Proof. Use 2.10.

5. Subdirectly Irreducible Commutative Idempotent Distributive Groupoids

5.1. Proposition. Let G be a subdirectly irreducible commutative idempotent distributive groupoid. Then exactly one of the following conditions holds:
(i) G is a cancellation groupoid.
(ii) G contains a zero element z such that $H = \{a \in G \mid a \neq z\}$ is a subgroupoid of G, H is a cancellation groupoid and $G = H\{z\}$.

Proof. We can assume that G contains at least two elements. Then there are $a, b \in G$ such that $a \neq b$ and $a\,t\,b$, $t = t_G$. First let $K = K(a, b)$ be non-empty. With respect to 2.9, K is an ideal. If K contains at least two elements then $r \neq d_G$, where r is the congruence corresponding to K in the sense of 1.3. In that case we have $t \subseteq r$, $a, b \in K$ and $a = b$, a contradiction. Hence $K = \{z\}$ is a one-element set. Since K is an ideal, z is the zero. Let $c \in G$, $c \neq z$. Then L_c is an endomorphism of G and $t \subseteq s$, where s is the conqruence corresponding to L_c, provided L_c is not injective. Consequently $a\,s\,b$, $ca = cb$, $c \in K$ and $c = z$, a contradiction. Thus L_c is injective and the rest is clear as well as the case if K is empty.

Let $p \geq 3$ a prime and $1 \leq n \leq \infty$. Let z be an element not belonging to $C(p^n, 1/2)$. Then we put $C(p^n, 1/2, z) = C(p^n, 1/2)\,\{z\}$. Further we denote by $C(0)$ the one-element groupoid and by $C(0, z)$ the groupoid $C(0)\,\{z\}$, where $z \notin C(0)$.

5.2. Lemma. Let G be a groupoid containing no zero element. Let $z \notin G$ and suppose that $H = G\{z\}$ is subdirectly irreducible. Then G is subdirectly irreducible.

Proof. Put $t = t_H$ and $r = t \mid G$. It is clear that $r \subseteq s$ for every congruence s of G with $s \neq d_G$. Put $K = \{a \in G \mid a\,t\,z\}$ and suppose $r = d_G$. If K is empty then $t = d_H$, a contradiction, since H contains at least two elements. Thus K is non-empty and we see that K is an ideal of G. Since G contains no zero element, K contains at least two elements, a contradiction with $r = d_G$.

5.3. Lemma. No proper ideal of a cancellation groupoid is finite.

Proof. Let I be a proper finite ideal of a cancellation groupoid G. Then I is a subquasigroup of G. Take $a \in I$ and $b \in G$, $b \notin I$. Then $ab \in I$ and there is $c \in I$ with $ac = ab$. Hence $b = c \in I$, a contradiction.

5.4. Theorem. The only subdirectly irreducible commutative idempotent medial groupoids (up to isomorphism) are the following: The one-element groupoid $C(0)$, the semigroup $C(0, z)$, the quasigroups $C(p^n, 1/2)$ and the groupoids $C(p^n, 1/2, z)$, where $p \geq 3$ is a prime and $1 \leq n \leq \infty$. These groupoids are pair-wise non-isomorphic.

Proof. The fact that these groupoids are subdirectly irreducible commutative idempotent medial groupoids follows easily from 4.10, 4.8, 1.5 and 1.4. Conversely, let G be a subdirectly irreducible groupoid from $\mathscr{A}$. According to 5.1, 5.3 and 5.2, we can assume that G is a cancellation groupoid. The rest of the proof follows now from 3.8, 4.10 and 4.8.

5.5. Corollary. Subdirectly irreducible commutative idempotent medial groupoids are closed under subgroupoids and homomorphic images.

6. More About the Congruence m_G

We shall say that a groupoid G satisfies the condition (S) if every class of the congruence m_G is a Steiner subquasigroup of G. The following two lemmas are easy.

6.1. Lemma. The class of all groupoids satisfying (S) is closed under subgroupoids and cartesian products.

6.2. Lemma. Let a groupoid G satisfy (S) and $z \notin G$. Then $G\{z\}$ satisfies (S).

6.3. Proposition. Every distributive cancellation groupoid satisfies (S).

Proof. Apply 4.6, 3.2 and 6.1.

6.4. Proposition. Every commutative idempotent distributive groupoid satisfies the condition (S).

Proof. Let G be a commutative idempotent distributive groupoid. With respect to 6.1, we can assume that G is subdirectly irreducible. Then the result follows from 5.1, 6.3 and 6.2.

6.5. Theorem. Let G be a commutative distributive groupoid. Put $m = m_G$ and $r = m_{Id\ G}$. Then:

(i) If $a, b \in G$ then $a\ m\ b$ iff either $a = b$ or $a, b \in Id\ G$ and $a\ r\ b$.

(ii) Every class of m containing at least two elements is a Steiner subquasigroup of G.

Proof. Apply 2.1, 2.4 and 6.4.

6.6. Corollary. Let G be a commutative distributive groupoid such that G is not medial. Then G contains a subquasigroup isomorphic to the three-element quasigroup S.

Proof. The assertion is an immediate consequence of 6.5 and 4.5.

6.7. Corollary. Let G be a commutative distributive groupoid such that G/m is an i-simple groupoid. Then G is i-simple.

Proof. Use 6.5 and 1.2.

The following lemma can easily be verified.

6.8. Lemma. Let r be a congruence of a groupoid G such that every class of r is either a one-element set or an i-simple subgroupoid of G. Then the lattice $I(G)$ is canonically isomorphic to the lattice $I(G/r)$.

6.9. Corollary. Let G be a commutative distributive groupoid. The lattice $I(G)$ is canonically isomorphic to the lattice $I(G/m)$.

Proof. Apply 6.5 and 6.8.

7. Closed Subgroupoids of Distributive Groupoids

A subgroupoid H of a groupoid G is said to be closed if $ab \in H$ (resp. $ba \in H$) implies $b \in H$ for all $a \in H$ and $b \in G$. It is clear that the intersection of a set of closed subgroupoids is either empty or a closed subgroupoid. If $A \subseteq G$ is a subset then $cl(A)$ denotes the closed subgroupoid generated by A. We shall say that G satisfies the condition (M) if $cl(a, b, c, d)$ is a medial groupoid, whenever $a, b, c, d \in G$ and $ab \,.\, cd = ac \,.\, bd$. The following lemma is immediate.

7.1. Lemma. The class of groupoids satisfying (M) is closed under subgroupoids and cartesian products.

7.2. Lemma. Let a groupoid G satisfy (M) and $z \notin G$. Then $G\{z\}$ satisfies (M).

7.3. Corollary. Every commutative distributive groupoid satisfies (M).

Proof. With respect to 2.1, 2.4 and 7.1, we can assume that G is idempotent and subdirectly irreducible. In this case, the satement is an easy consequence of 5.1, 7.2, 4.1 and 3.2.

7.4. Theorem. Let G be a distributive groupoid. Then G satisfies (M), provided at least one of the following conditions holds:

(i) $Id\ G$ satisfies (M).

(ii) G is a cancellation groupoid.

(iii) G is commutative.

(iv) G is a division groupoid.

(v) Every subgroupoid of G is closed.

Proof. (i)follows from 7.1, 2.1 and 2.4, (ii) follows from 7.1, 3.2 and 4.1, (iii) follows from 7.3, (iv) is proved in [4] and (v) is proved in [5].

8. Epimorphisms in the Variety of Medial Groupoids

8.1. Lemma. Let G be a medial groupoid generated by two subgroupoids A, B and H be the cartesian product of A and B. Suppose that there is an element $a \in A \cap B$ such that $aB = B$ and $Aa = A$. Then G is a homomorphic image of the groupoid H.

Proof. Define a mapping f of H into G by $f(\langle x, y\rangle) = xy$. Since G is medial, f is a homomorphism. Let $b \in A$ be arbitrary. Then $b = ca$ for some $c \in A$, and so $b = f(\langle c,a\rangle)$. We have proved that $A \subseteq Im f$. Similarly $B \subseteq Im f$, and consequently $Im f = G$.

8.2. Lemma. Let Q be a quasigroup such that every congruence of the cartesian power $Q \times Q$ is normal and let f, g be two homomorphismus of Q into a medial groupoid Q. Suppose that $f \mid P = g \mid P$ for a subgroupoid P *of* Q such that Q is generated by P as a quasigroup. Then $f = g$.

Proof. We can assume that G is generated by $f(Q) \cup g(Q)$. It is evident that $f(Q)$, $g(Q)$ are division groupoids and $f(Q) \cap g(Q)$ is non-empty. By the hypothesis and 8.1, G is a quasigroup. Then both f and g are quasigroup homomorphisms and $f = g$, since they coincide on a generator set.

8.3. Corollary. Let Q be a medial quasigroup such that every congruence of

$Q \times Q$ is normal and $P \subseteq Q$ be a subgroupoid such that Q is generated by P as quasigroup. Then the inclusion $P \subseteq Q$ is an epimorphism in the variety $\mathscr{M}$.

8.4. Corollary. Let Q be a commutative idempotent medial quasigroup and P its subgroupoid which generates Q. Then the inclusion $P \subseteq Q$ is an epimorphism in $\mathscr{M}$ (and hence in $\mathscr{M} \cap \mathscr{D}$, $\mathscr{A}$, etc.).

Proof. Use 4.10 and 8.3.

Let $\mathscr{V}$ be a variety of groupoids. We shall say that $\mathscr{V}$ satisfies the condition (C) if $G \in \mathscr{V}$ is a cancellation groupoid, whenever G/m is a quasigroup.

8.5. Lemma. The variety of commutative distributive groupoids satisfies the condition (C).

Proof. Apply 6.5, 1.2 and 2.10.

8.6. Proposition. Let $\mathscr{V}$ be a variety satisfying (C) and $Q \in \mathscr{V}$ be a quasigroup such that every congruence of $Q \times Q/m$ is normal. Suppose that Q is generated as a quasigroup by a subgroupoid G. Then the inclusion $G \subseteq Q$ is an epimorphism in $\mathscr{V}$.

Proof. Let f, g be two homomorphism of Q into $H \in \mathscr{V}$ such that $f \mid G = g \mid G$. We can assume that H is generated by $Im\, f \cup Im\, g$. Similarly as in the proof of 8.2, we can show that H/m is a quasigroup. Then H is a cancellation groupoid and f, g are quasigroup homomorphisms. The rest is clear.

8.7. Corollary. Let G be a commutative distributive cancellation groupoid. Then the inclusion $G \subseteq E(G)$ is an epimorphism in the variety $\mathscr{C} \cap \mathscr{D}$.

Proof. Apply 8.5, 8.6 and 4.10.

8.8. Corollary. The following varieties have non-surjective epimorphisms:
$\mathscr{M}, \mathscr{M} \cap \mathscr{D}, \mathscr{M} \cap \mathscr{Q}, \mathscr{M} \cap \mathscr{C}, \mathscr{A}, \mathscr{C} \cap \mathscr{D}, \mathscr{E}, \mathscr{M} \cap \mathscr{C} \cap \mathscr{D}$.

Proof. With respect to 8.4, 8.7 and 3.2, it is enough to find a commutative idempotent medial cancellation groupoid G such that G is not a quasigroup. However, it is very easy. For example, consider the groupoid $G(\circ)$ consisting of all positive rational numbers with the operation $x \circ y = (x + y)/2$.

9. Free Distributive Groupoids

9.1. Proposition. Let G be a free distributive groupoid freely generated by a set X. Let $f : G \to Id\, G$ and $g : G \to B(G)$ be the canonical homomorphisms. Then $Id\, G$ (resp. $B(G)$) is a free groupoid in $\mathscr{D} \cap \mathscr{Q}$ ($\mathscr{B}$) and it is freely generated by $f(X)$ ($g(X)$). Moreover, $f(x) \neq f(y)$ and $g(x) \neq g(y)$, whenever $x, y \in X$ and $x \neq y$. Hence G, $Id\, G$ and $B(G)$ have the same rank.

Proof. Let $x, y \in G$, $x \neq y$ and $f(x) = f(y)$. Then $x \,.\, xx = y \,.\, yy$ and every distributive groupoid satisfies the identity $x \,.\, xx = y \,.\, yy$. Consequently every idempotent distributive groupoid has only one element, a contradiction. Let $g(x) = g(y)$. Then $x, y \in Id\, G$ and every distributive groupoid is idempotent, a contradiction. The rest is clear from 2.8.

9.2. Remark. Let X be a non-empty set, $A = X \cup (X \times X)$ and suppose that the intersection $X \cap (X \times X)$ is empty. Let 0 be an element such that $0 \notin A$ and put

$B = A \cup \{0\}$. Define a multiplication on B as follows: Let $a, b \in B$. Then $a \,.\, b = \langle a, b\rangle$ if $a, b \in X$ and $a \,.\, b = 0$ otherwise. Then the groupoid B is a free groupoid in the variety $\mathscr{B}$ and X is the unique set of free generators of B. Further, let Y be a set of the same cardinality as X, f be a biunique mapping of X onto Y and G be a free idempotent distributive groupoid freely generated by Y. We shall assume that the intersection $G \cap A$ is empty. Define $f(a) = a$, $f(\langle x, y\rangle) = f(x) \,.\, f(y)$ for all $a \in G$, $x, y \in X$ and put $F = G \cup A$. Then f is a mapping of F onto G and F is a partial groupoid. Let $a, b \in F$. We put $a \,.\, b = \langle a, b\rangle$ if $a, b \in X$ and $a \,.\, b = f(a) \,.\, f(b)$ otherwise. It is easy to see that F is a free distributive groupoid and X is its set of free generators.

9.3. Corollary. Let $n \geq 1$ and G be a B-groupoid which can be generated by n elements. Then G contains at most $n^2 + n + 1$ different elements.

9.3. Lemma. Let a variety $\mathscr{V}$ of groupoids be generated by a class $\mathscr{U}$ of groupoids. Suppose that every groupoid from $\mathscr{U}$ is a cancellation groupoid. Then every free groupoid of $\mathscr{V}$ is a cancellation groupoid.

Proof. Let $F \in \mathscr{V}$ be free and $uv = uw$ for some $u, v, w \in F$. Then every groupoid from $\mathscr{U}$ satisfies the identity $v = w$, and therefore every groupoid from $\mathscr{V}$ satisfies the identity. The rest is clear.

Let $\mathscr{P}$ be the class of all quasigroups. As it is well known, $\mathscr{P}$ is equivalent to a variety of algebras with three binary operations. We shall sometimes identify $\mathscr{P}$ with this variety. Let $\mathscr{V}$ be a variety of groupoids. Then $\mathscr{V}_q$ denotes the quasigroup variety generated by $\mathscr{V} \cap \mathscr{P}$. Further we shall say that $\mathscr{V}$ satisfies the condition (Q) if $\mathscr{V}$, as a groupoid variety, is generated by $\mathscr{V} \cap \mathscr{P}$.

9.4. Lemma. Let a groupoid variety $\mathscr{V}$ satisfy (Q) and $F \in \mathscr{V}$ be a free groupoid. Then F is a subgroupoid of a quasigroup $Q \in \mathscr{V}$.

Proof. As it is easy to see, there exist a quasigroup $Q \in \mathscr{V}$, a subgroupoid G of Q and a homomorphism f of G onto F. Let X be a set of free generators of F and g be a mapping of X into G such that $fg(x) = x$ for each $x \in X$. It is evident that F is isomorphic to the subgroupoid generated by $g(X)$.

9.5. Proposition. Let a groupoid variety $\mathscr{V}$ satisfy (Q) and $Q \in \mathscr{V}_q$ be a free quasigroup freely generated by a set X. Denote by F the subgroupoid of Q generated by X. Then F is free in $\mathscr{V}$ and X is a free basis of F.

Proof. Apply 9.4.

9.6. Theorem. The varieties $\mathscr{M} \cap \mathscr{Q}$, $\mathscr{E}$ and $\mathscr{A}$ satisfy the condition (Q).

Proof. First we show that $\mathscr{E}$ satisfies (Q). Denote by $\mathscr{V}$ the variety generated by all commutative distributive cancellation groupoids. It follows from 3.2 that $\mathscr{V}$ satisfies (Q). Further, let F be the set of all rational numbers $a/2^b$, where $0 \leq a \leq 2^b$, and $I = \{x \in F \mid x \neq 0\}$. Define an operation $\circ$ on F by $x \circ y = (x + y)/2$. It is obvious that $F(\circ) \in \mathscr{A}$ and I is an ideal of $F(\circ)$. The corresponding factorgroupoid belongs to $\mathscr{V}$ (since $F(\circ)$ is a cancellation groupoid) and is isomorphic to the groupoid defined in 1.4.(ii). According to 1.4 and 5.1, $\mathscr{V}$ contains every subdirectly irreducible commutative idempotent distributive groupoid, and hence $\mathscr{V} = \mathscr{E}$. Similarly we can prove that $\mathscr{A}$ satisfies (Q). For $\mathscr{M} \cap \mathscr{Q}$, the assertion follows from the main result of [6].

Let R be the set of all rational numbers which are equal to $2^{-k}a$ for some integer a and natural number k. Given a natural number $n \geq 1$, the cartesian power R^n is a commutative idempotent medial quasigroup with respect to the operation $\circ$ defined by $a \circ b =$ $= (a + b)/2$ (see 4.7). Denote by F_n the set of all $\langle a_1, \ldots, a_n \rangle \in R^n$ such that $a_1 \geq 0, \ldots,$ $a_n \geq 0$ and $a_1 + \ldots + a_n \leq 1$. In [7] there is proved that $F_n(\circ)$ is free in the variety $\mathscr{A}$ and the elements $e_0^n = \langle 0,0,\ldots,0 \rangle$, $e_1^n = \langle 1,0,\ldots,0 \rangle, \ldots, e_n^n = \langle 0,\ldots,0,1 \rangle$ are its free generators (see also 9.6 and 9.5). Put $F_{n,0} = \{\langle a_1,\ldots,a_n \rangle \in F_n \mid a_1 + \ldots + a_n = 1\}$ and for $i = 1, 2, \ldots, n$, $F_{n,i} = \{\langle a_1, \ldots, a_n \rangle \in F_n \mid a_i = 0\}$. Further, let Int F_n be the set of all $a \in F_n$ such that $a \notin F_{n,0} \cup F_{n,1} \cup \ldots \cup F_{n,n}$. It is an easy exercise to show that all these sets are subgroupoids of $F_n(\circ)$. Moreover, Int F_n, is an ideal of $F_n(\circ)$.

9.7. Lemma. The groupoid *Int* $F_n(\circ)$ is i-simple.

Proof. See [1], Corollary 3.2.

9.8. Lemma. Let H be an *i*-simple subgroupoid and I be an ideal of a groupoid G. Then $H \subseteq I$ or the intersection $H \cap I$ is empty. Moreover, if H is an ideal of G then $H \subseteq I$, and so H is the intersection of all ideals of G.

9.9. Corollary. Int $F_n(\circ)$ is the intersection of all ideals of $F_n(\circ)$.

Proof. Apply 9.7. and 9.8.

If G is a groupoid then we put *Int* $G = \bigcap I$, $I \in I(G)$. It is clear that *Int* G is an ideal of G, provided it is non-empty.

We denote by F_0 the one-element subgroupoid of F_1 containing the element 0. Clearly, F_0 is free of rank 1 in $\mathscr{A}$.

9.10. Proposition. Let $n \geq 0$. Then the lattice $I(F_n(\circ))$ is finite.

Proof. We use induction on n. If $n = 0$ then the assertion is obvious. Let $n \geq 1$ and $I \subseteq F_n(\circ)$ be an ideal. Then *Int* $F_n \subseteq I$ and $I =$ *Int* $F_n \cup \bigcup (I \cap F_{n,i})$, $i = 0,1,\ldots,n$. The intersection $I \cap F_{n,i}$ is empty or it is an ideal of $F_{n,i}$. However, as it is easy to see, all the groupoids $F_{n,i}$ are isomorphic to $F_{n-1}(\circ)$. The rest is easy.

10. Finitely Generated Distributive Groupoids

10.1. Theorem. Let G be a finitely generated commutative distributive groupoid. Then the lattice $I(G)$ of ideals of G is finite.

Proof. With respect to 9.3 and 2.7, we can assume that G is idempotent. Further, according to 6.9, we can restrict ourselves to the medial case. The rest is clear from 9.10.

A subgroupoid H of a groupoid G is called dense if $cl(H) = G$. The following lemma is obvious.

10.2. Lemma. Let H be a subgroupoid of a groupoid G. Then H is dense in G, provided at least one of the following conditions holds:

(i) H is a left (right) ideal of G.

(ii) G is a quasigroup and H generates G.

10.3. Proposition. Let H be a dense subgroupoid of a distributive groupoid G. Then every normal congruence of H can be extended to uniquely determined normal congruence of G.

Proof. The proof is in fact the same as that of Proposition 1.7 [1].

10.4. Lemma. Let G be a commutative distributive groupoid and $a \in Int\, G$, $b, c \in G$ be such that $ac = bc$. Then $a = ab$.

Proof. With regard to 2.9, $Int\, G \subseteq K(a, b)$. Hence $aa = ab$. However $Int\, G \subseteq$ $\subseteq Id\, G$, and so $aa = a$.

10.5. Proposition. Let G be a commutative distributive groupoid such that $Int\, G$ is non-empty. Then:

(i) $Int\, G$ is an ideal of G.

(ii) $Int\, G$ is a cancellation groupoid.

(iii) Every normal congruence of $Int\, G$ can be extended to a normal congruence of G.

Proof. Apply 10.2, 10.3 and 10.4.

10.6. Proposition. The following conditions are equivalent for a commutative distributive groupoid G:

(i) $Int\, G = G$.

(ii) G is i-simple.

(iii) Every congruence of G is normal.

Proof. See 2.10 and 1.3.

10.7. Lemma. Let f be a homomorphism of a groupoid G onto a groupoid H. Then $f(Int\, G) \subseteq Int\, H$. Moreover, if $Int\, G$ is not empty then $f(Int\, G) = Int\, H$.

Proof. Easy.

10.8. Lemma. Let G be a commutative distributive groupoid and f be the canonical homomorphism of G onto G/m. Then $f(Int\, G) = Int\, G/m$.

Proof. Apply 6.9.

10.9. Lemma. Let G be a distributive groupoid. Then $Int\, G = Int\, Id\, G$.

Proof. Use 2.6.

10.10. Proposition. Let G be a commutative distributive groupoid such that $Int\, G$ is non-empty. Then $Int\, G$ is an i-simple groupoid.

Proof. With respect to 1.2, 6.5 and 10.8, we can assume that G is medial. Let r be a congruence of $Int\, G$ and $ac\ r\ bc$ for some $a, b, c \in Int\, G$. If $x \in G$ then $ax, bx \in Int\, G$ and we put $K = \{x \in G \mid ax\ r\ bx\}$. Let $x \in K$ and $y \in G$. Then $a \,.\, yx = ay \,.\, ax\ r\ ay \,.\, bx =$ $= by \,.\, ax\ r\ by \,.\, bx = b \,.\, yx$ (since $ay, by \in Int\, G$ and G is commutative and medial). We have proved that K is an ideal, and so $Int\, K \subseteq K$. Consequently $aa\ r\ bb$. However $Int\, G \subseteq Id\, G$, and hence $a\ r\ b$. The rest is clear from 10.6.

10.11. Corollary. Let H be a subgroupoid of a commutative distributive groupoid G such that $Int\, G \subseteq H$. Suppose that $Int\, G$ is non-empty. Then $Int\, H = Int\, G$.

10.12. Theorem. Let G be a commutative distributive groupoid such that $Int\, G$ is non-empty. Then:

(i) $Int\, G$ is an ideal of G.

(ii) $Int\, G$ is an i-simple idempotent cancellation groupoid.

(iii) Every congruence of $Int\, G$ is normal.

(iv) Every congruence of $Int\, G$ can be extended to G.

Proof. See 10.5, 10.10, 2.10 and 10.3.

10.13. Proposition. Let G be a finitely generated commutative distributive groupoid. Then *Int* G is non-empty.

Proof. According to 10.8 and 10.9, we can assume that G is medial and idempotent. In this case, the result follows from 9.9 and 10.7.

10.14. Corollary. Let G be a finite commutative distributive groupoid. Then *Int* G is a quasigroup.

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