Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 18 (1977), No. 2, 23--34

Persistent URL: http://dml.cz/dmlcz/142402

1977 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 18, NO. 2, Pag. 23–34

The Infinite Minimal Rich Monoid

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Received 29 January 1976

The question is studied, which monoids of unary operations are rich enough for the corresponding categories of algebras to contain any algebraic category. The paper presents an example of the infinite monoid minimal in the class of all monoids with this richness-property, ordered by factorization.

В статье исследуются моноиды унарных операций обладающих следующим свойством «богатости»: соответствующая категория унарных алгебр содержит любую алгебраическую категорию. Показывается пример бесконечного моноида минимального в классе всех моноидов обладающих этим свойством, упорядоченным факторизацией.

Článek se zabývá otázkou bohatosti monoidů unárních operací z hlediska vnořování odpovídajících kategorií algeber. Monoid je nazýván bohatým, jestliže příslušná kategorie algeber obsahuje každou algebraickou kategorii. Je uveden příklad nekonečného monoidu minimálního ve třídě všech bohatých monoidů uspořádané faktorizací.

0. Introduction and conventions

In the present paper the question is studied, which monoids of unary operations are rich enough for the corresponding categories of algebras to contain any algebraic category.

First, let us recall some notions. A category C is said to be *algebraic* if there exists a full embedding of C into some category of algebras and all their homomorphisms. A category is said to be *binding* ([2]) if every algebraic category can be embedded into it. A small category c, e.g. monoid, is said to be *rich* ([1], [3]) if the functor category Set^c is binding, otherwise c is called *poor*.

It has been shown in [4] that each cardinal number greater than four is the cardinality of some rich monoid. Nevertheless, a question arises whether infinite rich monoids bring anything essentially new, i.e. whether the property of richness of monoids is not already always somehow based on some substantially finite feature of their structure. The „inheriting" of richness of monoids from their factor-monoids suggests us the natural exact formulation of this question, viz., whether

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every infinite rich monoid has a finite rich factormonoid. There is given a negative answer to this question in this paper by showing an example of an infinite rich monoid (moreover: with only two generators), whose every proper factormonoid is poor, i.e. of the infinite monoid minimal in the class of all rich monoids ordered by factorization.

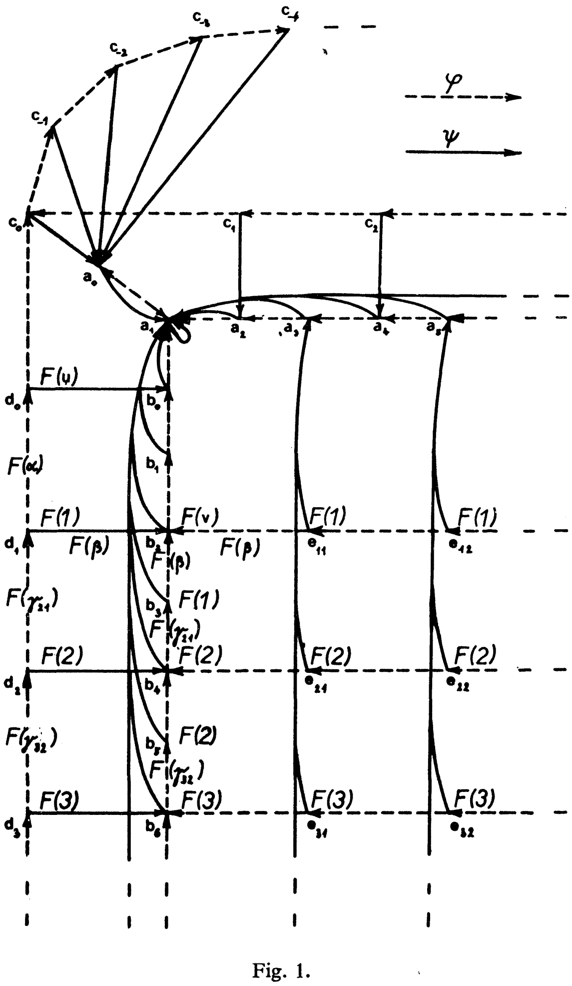

Fig. 1.

If $Q_1, \ldots, Q_k$ are some equalities of words in the alphabet $\{\varphi, \psi\}$, then $M_{Q_1, \ldots, Q_k}$ denotes the monoid with two generators φ, ψ fulfilling the identities $Q_1, \ldots, Q_k$. The identity $\varphi^2\psi = \psi\varphi$ is denoted by P. We do not distinguish between monoids and one-object categories as well as between the functor category

$SetM_{Q_1,\ldots,Q_k}$ and the category of algebras with two unary operations φ, ψ fulfilling the identities $Q_1, \ldots, Q_k$.

In these terms, the aim of this paper is to show that the monoid $M = M_{P,\psi^3=\psi^2=\varphi^2\psi^2}$ is an infinite minimal rich monoid.

If C is a category, then C^o, C^m, $C(a, b)$ denote the classes of all of its objects, morphisms, morphisms from a to b $(a, b \in C^o)$, respectively. Z denotes the set of all integers, N the set of all positive integers. The restriction of the mapping φ on the set X is denoted by $\varphi|X$, the set difference of sets X, Y by $X \setminus Y$. The disjoint union of sets is denoted by $\vee$.

I. Richness

1.1 Proposition: The monoid $M = M_{P,\psi^3=\psi^2=\varphi^2\psi^2}$ is rich.

Proof: Define the small category $\mathcal{J}$ by
$\mathcal{J}^o = \{u, v\} \vee N$, $\mathcal{J}(i+1, i) = \{\gamma_{i+1,i}\}$ for all $i \in N$, $\mathcal{J}(1, u) = \{\alpha\}$, $\mathcal{J}(1, v) = \{\beta\}$, $\mathcal{J}$ is thin and its only morphisms are identities and compositions of morphisms $\alpha, \beta, \gamma_{i+1,i}$ for $i \in N$ if they make sense.

Since $\mathcal{J}$ is rich according to [3], it is sufficient for us to construct a full embedding

$$\Phi : Set^J \to Set^M ,$$

Denote (see Fig. 1)

$$\overline{X} = \{c_i; i \in Z\} \vee \{a_j; j \geq 0\} \vee \{b_k; k \geq 0\} \vee \{d_l; l \geq 0\} \vee \vee \{e_{m,n};\ m, n \in N\}$$

(i.e., $\overline{X}$ is considered as a set of mutually different elements).

For $F \in (Set^J)^o$ define the mapping $\Phi_F : \overline{X} \to Set^o$ by

$$(1)\qquad \begin{aligned}
\Phi_F(c_i) &= \{c_i\} && \text{for} \quad i \in Z ,\\
\Phi_F(a_j) &= \{a_j\} && \text{for} \quad j \geq 0 ,\\
\Phi_F(b_k) &= \{b_k\} && \text{for} \quad k = 0, 1,\\
\Phi_F(d_o) &= F(u) ,\\
\Phi_F(d_l) &= F(l) && \text{for} \quad l \in N ,\\
\Phi_F(b_2) &= F(v) ,\\
\Phi_F(b_3) &= F(1) ,\\
\Phi_F(b_{2i}) &= \Phi_F(b_{2i+1}) = F(i) && \text{for} \quad i \geq 2,\\
\Phi_F(e_{m,n}) &= F(m) && \text{for} \quad m, n \in N ,
\end{aligned}$$

(Φ_F is, in fact, a collection of sets with the index set $\overline{X}$), and put

$$\Phi(F) = (X, \varphi, \psi) , \qquad \text{where} \qquad X = \bigvee_{\bar{x} \in \overline{X}} \Phi_F(\bar{x}) ,$$

and the operations φ, ψ are defined by

$\varphi(c_i) = c_{i-1}$ for $i \in Z$,
$\varphi(a_i) = a_{i-1}$ for $i \in N$,

$\varphi(a_0) = \varphi(b_0) = a_1$,
$\varphi(b_1) = b_0$,
$\varphi(\Phi_F(b_{j+1})) \subseteq \Phi_F(b_j)$ for $j \in N$, $\varphi(\Phi_F(b_2)) = \{b_1\}$,
$\varphi \mid \Phi_F(b_3) = F(\beta)$,
$\varphi \mid \Phi_F(b_{2i}) = F(\gamma_{i,i-1})$ for $i \geq 2$,
$\varphi \mid \Phi_F(b_{2i+1}) = id_{F(i)}$ for $i \geq 2$,
$\varphi(\Phi_F(d_0)) = \{c_0\}$,
$\varphi(\Phi_F(d_{j+1})) \subseteq \Phi_F(d_j)$ for $j \geq 0$, $\varphi \mid \Phi_F(d_1) = F(\alpha)$,
$\varphi \mid \Phi_F(d_i) = F(\gamma_{i,i-1})$ for $i \geq 2$,
$\varphi(\Phi_F(e_{i,1})) \subseteq \Phi_F(b_{2i})$ for $i \in N$,
$\varphi(\Phi_F(e_{i,j})) \subseteq \Phi_F(e_{i,j-1})$ for $i \in N$, $j \geq 2$,
$\varphi \mid \Phi_F(e_{1,1}) = F(\beta)$,
$\varphi \mid \Phi_F(e_{i,j}) = id_{F(i)}$ for $i, j \in N$, $i \,.\, j \neq 1$,
$\psi(c_i) = a_0$ for $i \leq 0$,
$\psi(c_i) = a_{2i}$ for $i \in N$,
$\psi(a_i) = a_1$ for $i \geq 0$,
$\psi(\Phi_F(d_0)) = \{b_0\}$
$\psi(\Phi_F(b_i)) = \{a_1\}$ for $i \geq 0$,
$\psi(\Phi_F(d_i)) \subseteq \Phi_F(b_{2i})$ for $i \in N$, $\psi \mid \Phi_F(d_1) = F(\beta)$,
$\psi \mid \Phi_F(d_i) = id_{F(i)}$ for $i \geq 2$,
$\psi(\Phi_F(e_{i,j})) = \{a_{2j+1}\}$ for $i, j \in N$.

It is easy to verify that $\Phi(F) \in Set^M$.

If $F' \in (Set^J)^o$, $\tau \in (Set^J)^m$, $\tau : F \to F'$, $\tau = \{\tau^j ; j \in \mathcal{J}^o\}$, $\Phi(F) = (X, \varphi, \psi)$, $\Phi(F') = (X', \varphi', \psi')$, define the mapping
$\Phi(\tau) : X \to X'$ by
$\Phi(\tau) \mid S = id_S$ where $S \subseteq X$, $S = \{a_i ; i \geq 0\} \vee \{b_0, b_1\} \vee \{c_k ; k \in Z\} \subseteq \bar{X}$, and $\Phi(\tau)(\Phi_F(\bar{x})) \subseteq \Phi_{F'}(\bar{x})$, $\Phi(\tau) \mid \Phi_F(\bar{x}) = \tau^j$ for $\bar{x} \in \bar{X} \setminus S$,
where $j \in \mathcal{J}^o$ is by (1) uniquelly determined by the condition

$$\Phi_F(\bar{x}) = F(j) .$$

The mapping $\Phi(\tau)$ is obviously a homomorphism of algebras (X, φ, ψ), (X', φ', ψ'). Now, we shall prove that the functor Φ, which is evidently 1—1, is also full.

Let $f \in (Set^M)^m$, $f : \Phi(F) \to \Phi(F')$, $\Phi(F) = (X, \varphi, \psi) = A$, $\Phi(F') = (X', \varphi', \psi') = A'$.

(*i*) The point a_1 is the only fix point of the operation ψ, hence $f(a_1) = a_1$.

(*ii*) We have $f(a_0) = f\varphi(a_1) = \varphi' f(a_1) = a_0$.

(*iii*) We have $\psi(c_0) = a_0$, $f(a_0) = a_0$, thus $\psi' f(c_0) = a_0$. Hence $f(c_0) = c_{-n}$ for some integer $n \geq 0$. With regard to the commutating of f and φ^{2n+2} we have either $f(c_{2n+2}) = c_{2n+2}$ or $f(c_{2n+2}) \in \Phi_{F'}(d_{n+1})$, and either $f(\Phi_F(d_n)) \subseteq \Phi_{F'}(d_0)$ or $f(\Phi_F(d_n)) = \{c_1\}$.

(iv) If $f(c_{2n+2}) \in \Phi_{F'}(d_{n+1})$, then

$f(a_{4n+4}) \in \Phi_{F'}(b_{2n+2})$ (commutating of f and ψ),

$f\psi(e_{1,2n+1}) = f(a_{4n+3}) \in \Phi_{F'}(b_{2n+1})$ (commutating of f and φ).

But $(\psi')^{-1}(\Phi_{F'}(b_{2n+1})) = \varnothing$, so that the formula

$f\psi(e_{1,2n+1}) = \psi' f(e_{1,2n+1})$

cannot hold, which contradicts f being a homomorphism. Hence,

(2) $f(c_{2n+2}) = c_{2n+2}$,

$f(a_{4n+4}) = a_{4n+4}$ (commutating of f and ψ).

(v) If $f(\Phi_F(d_n)) = \{c_1\}$, then

$f(\Phi_F(b_{2n})) = \{a_2\}$ (commutating of f and ψ). ,

$f(\Phi_F(e_{n+1,2n+2})) = \{a_{2n+6}\}$ (commutating of f and φ^{2n+4}),

$f(a_{4n+5}) = a_1$ (commutating of f and ψ),

$f(a_{4n+4}) = a_0$ (commutating of f and φ).

According to (2) and the condition $n \geq 0$ the case (v) cannot take place, so that we have

$f(\Phi_F(d_n)) \subseteq \Phi_{F'}(d_0)$.

(vi) If $n \neq 0$, then

$f(\Phi_F(b_{2n})) = \{b_0\}$ (commutating of f and ψ),

$f(\Phi_F(e_{n,2n})) = \subseteq \Phi_{F'}(b_{2n})$ (commutating of f and φ^{2n}),

(3) $f(a_{4n+1}) = a_1$ (commutating of f and ψ).

But from (2) we obtain

(4) $f(a_{4n+1}) = a_{2n+1}$ (commutating of f and φ^3).

From (3) and (4) follows $n = 1$, which implies

$f(c_n) = c_n$ for all $n \in Z$,

$f(\Phi_F(d_i)) \subseteq \Phi_{F'}(d_i)$ for all $i \geq 0$,

$f(a_i) = a_i$, $f(\Phi_F(b_i)) \subseteq \Phi_F'(b_i)$ for all $i \geq 0$,

$f(\Phi_F(e_{i,j})) \subseteq \Phi_{F'}(e_{i,j})$ for all $i, j \in N$.

For $j \in \mathcal{J}^o$ define the mapping $\tau^j : F(j) \rightarrow F'(j)$ by

$\tau^u = f \mid \Phi_F(d_0)$, $\tau^v = f \mid \Phi_F(b_2)$,

$\tau^i = f \mid \Phi_F(d_i)$ for all $i \in N$.

Then $\tau = \{\tau\ ;\ j \in \mathcal{J}^o\}$ is a natural transformation from the functor F to the functor F' and we have $\Phi(\tau) = f$. The proof is concluded.

2. Minimality

In the first two lemmas we formulate some properties of the monoid M_P occasionally used in the sequel.

2.1. Lemma: Let m, n be positive integers. Then for the elements of the monoid M_P we have $\psi^m \varphi^n = \varphi^{n.2^m} \psi^m$.

Proof: Induction after m.

2.2. Lemma: Each element $e \in M_P$ can be uniquelly written in the form $e = \varphi^B \psi^A$. If $e = \varphi^{b_k} \psi^{a_k} \ldots \varphi^{b_1} \psi^{a_1}$, then $A = \sum_{i=1}^{k} a_i$, $B = \sum_{j=1}^{k} (b_j \,.\, 2^{\sum_{i=j+1}^{k} a_i})$; ($A$, B, a_i, b_i be non-negative integers).

Proof: Uniqueness is obvious, existence by induction after k.

2.3. Lemma: Let $(X, \varphi, \psi) \in (Set^{M_P})^\circ$.

a) Let K be a φ-component of the algebra (X, φ, ψ). Then there exists a φ-component L of (X, φ, ψ) such that $\psi(K) \subseteq L$ (i.e., ψ preserves φ-connectedness).

b) Let $K_0 \subseteq K$ be a φ-cycle. Then L has a non-empty cyclic part L_0, $\psi(K_0) \subseteq L_0 \subseteq L$, and 2 card (K_0) is divisible by card (L_0).

Proof: Easy calculation employing 2.1. and 2.2.

2.4. Lemma: Let every φ-component of the algebra $(X, \varphi, \psi) \in (Set^{M_P})^\circ$ have a non-empty cyclic part. If φ is not 1—1, then (X, φ, ψ) has an endomorphism with the same property.

Proof: The mapping obtained by the obvious winding up of every φ-component on its cyclic part is evidently a φ-homomorphism, which is not 1—1. Its being a ψ-homomorphism follows from an easy calculation according to 2.3.b).

2.5. Notation: Denote by k the subcategory of Set with two objects $a' = \{0\}$, $a = \{1, 2, 3\}$ and all 1—1 mappings between them. Denote by $\bar{f} : a' \to a$, $\bar{g}, \bar{h} : a \to a$ the mappings such that $\bar{f}(0) = 1$, $\bar{g}(1) = \bar{h}(1) = 2$, $\bar{g}(2) = \bar{h}(3) = 1$, $\bar{g}(3) = \bar{h}(2) = 3$. Then $\bar{g}, \bar{h}$ is a set of generators of k(a, a) and we have

$$\bar{h}^3 = \bar{g}^2 = id_a, \qquad \bar{f} \neq \bar{h}\bar{f} = \bar{g}\bar{f}.$$

2.6. Proposition: The monoids a) $M = M_{P, \varphi^q = 1}$ and
b) $M = M_{P, \varphi^q \psi = \psi}$ are poor for all positive integers q.

Proof: If not otherwise stated, both cases a) and b) are treated simultaneously in 2.6. — 2.12.

A. It will be sufficient to prove that the algebraic category k from 2.5. cannot be fully embedded into the category $C = Set^M$. Thus, let us suppose that there exists a full embedding

$$\Phi : k \to C$$

Denote $\Phi(a') = A' = (X', \varphi', \psi')$, $\Phi(a) = A = (X, \varphi, \psi)$, $\Phi(\bar{f}) = f$, $\Phi(\bar{g}) = g$, $\Phi(\bar{h}) = h$. Then

$$(*) \qquad h^3 = id_X = g^2,$$

so each endomorphism of A is an isomorphism.
Denote by W the set of all $x \in X$ such that $h(x) = g(x)$ but $x, h(x)$, $h^2(x)$ are mutually distinct.

2.7. Lemma: $W \neq \varnothing$.

Proof: Since $\bar{f} \neq \bar{h}\bar{f}$, we have $f \neq hf$. Thus there exists $y \in X'$ such that $hf(y) \neq f(y)$. Put $x = f(y)$. We prove $x \in \mathrm{W}$. Since $hf = gf$, we have $g(x) = = h(x)$. If either $h^2(x) = h(x)$ or $h^2(x) = x$, then necessarily $h^3(x) = h(x)$. This is a contradiction because $h^3(x) = x$, while we have $h(x) \neq x$.

B. A finite sequence $p = \langle x_0, x_1, \dots, x_m \rangle, m \geq 1$, of points of X is said to be a *path* in A from x_0 to x_m if for any $i \in \{0, 1, \dots, m-1\}$ one of the following cases takes place:

$$\alpha)\ \varphi(x_i) = x_{i+1}, \qquad \gamma)\ \psi(x_i) = x_{i+1},$$
$$\beta)\ \varphi(x_{i+1}) = x_i, \qquad \delta\ \psi(x_{i+1}) = x_i.$$

An m-tuple of symbols $\alpha, \beta, \gamma, \delta$ is said to be a *type* of the above path if its i-th place denotes the case, which happens for x_{i-1}, x_i in p.

We say that p is a *φ-path* or *ψ-path* or *right path* or *left path* if its type contains only the symbols α, β (or γ, δ or α, γ or β, δ, respectively). A path that is neither a φ-path nor a ψ-path is called a *mixed path. A subpath* of a path $\langle x_0, \dots,$ $x_m \rangle$ is an arbitrary path $\langle x_i, x_{i+1}, \dots, x_j \rangle$ with $0 \leq i < j \leq m$.

A path is called reduced if it does not contain a subpath of the type $\langle \beta, \alpha \rangle$ or $\langle \delta, \gamma \rangle$. Clearly, for each path from x_0 to x_m we can construct a reduced path from x_0 to x_m.

C. Define a relation $\sim$ on the set X as follows:
$z \sim z' \leftrightarrow$ there exists a path in A from z to z'
$\sim$ is an equivalence; denote by $L(z)$ the set of all $z' \in X$ such that $z' \sim z$ and by $z \nsim z'$ the fact that $z' \notin L(z)$.

Let us suppose $x \nsim h(x)$ for some $x \in \mathrm{W}$. Since h is an isomorphism, then $h(x) \nsim h^2(x) \nsim x$. Then $L(x), L(h(x)), L(h^2(x))$ are disjoint. Thus, the mapping : $X \to X$ such that

$$F(z) = h(z) \qquad \text{whenever } z \in L(x),$$
$$F(z) = z \qquad \text{otherwise}$$

is an endomorphism of the algebra A, which is not 1—1. This contadicts to $(*)$.

D. Consequently, for any point $x \in \mathrm{W}$ there exists a path, say, $p = = \langle x = x_0, \dots, x_m = h(x) \rangle$ from x to $h(x)$, we may suppose p to be reduced. Then, $h(p) = \langle h(x_0), \dots, h(x_m) \rangle$, $h^2(p)$, $g(p)$ are reduced paths from $h(x)$ to $h^2(x)$ from $h^2(x)$ to x, from $g(x) = h(x)$ to x, respectively. Since g, h are isomorphisms, all the paths $p, h(p), h^2(p), g(p)$ are of the same type. In the following discussion we show that no type is posible for them. Thus, our assumption that k can be fully embedded into C leads to a contradiction.

In the sequel, for given $x \in \mathrm{W}$ p is always a reduced path from x to $h(x)$, U denotes the set $\{x, h(x), h^2(x)\}$, q is a positive integer.

2.8. Lemma: For any $x \in \mathrm{W}$ p is neither right nor left.

Proof: Let us suppose that p is a right path. Let π be a polynomial in the operations φ, ψ determined by the type of p. Then $h(x) = \pi(x)$, $h^2(x) =$

$= gh(x) = g^2(x) = x$, which is a contradiction. If p is a left path, the proof is analogous.

2.9. Lemma: For $M_{P,\varphi^q=1}$ and any $x \in W$, no path p is a φ-path.

Proof: A φ-path p would have to be of the type $\langle \underbrace{\alpha, \dots, \alpha}_{r\text{-times}}, \underbrace{\beta, \dots, \beta}_{s\text{-times}} \rangle$ with $r, s \geq 1$, which contradicts to φ's being 1—1.

2.10. Lemma: For $M_{P,\varphi^q\psi=\psi}$ and any $x \in W$, no path p is a φ-path.

Proof: Again, the type of such p would have to consist of r α's followed by s β's, with $r, s \geq 1$.

a) *Let* $r = s = 1$,

i.e. $p = \langle x, x_0, h(x) \rangle$, where $x_0 = \varphi h(x)$. For $u \in U$ denote

$$K(u) = \bigcup_{k=0}^{\infty} (\varphi^k)^{-1}(\{u\}) .$$

The sets $K(u)$, $u \in U$ are mutually disjoint, for otherwise there would exist a right or left path from x to $h(x)$.

If $\psi(w) \in K(u)$ for some $w \in X$, $u \in U$, i.e. $\varphi^k \psi(w) = u$ for some integer $k \geq 0$, then, using the identity $\varphi^q \psi = \psi$ $(k + 1)$ times, we obtain

$u = \varphi^k \psi(w) = \varphi^k \varphi^{(k+1)q} \psi(w) = \varphi^{kq+q-1} \varphi^{k+1} \psi(w) = \varphi^{kq+q-1}(x_0)$

(evidently $kq + q - 1 \geq 0$), and since

$$h(x_0) = h\varphi(x) = \varphi h(x) = x_0 ,$$

we have

$$h(u) = h\varphi^{kq+q-1}(x_0) = \varphi^{kq+q-1} h(x_0) = u ,$$

which is a contradiction. Consequently, $\psi^{-1}(K(u)) = \varnothing$ for all $u \in U$, the sets $K(u)$ are closed under proimages of both φ and ψ and under images of φ and ψ and under images of φ with the exception of the point u, for which we have $\varphi(u) = x_0$. The mapping $F : X \to X$ defined by

$$F(z) = h(z) \quad \text{whenever} \quad z \in K(x) ,$$
$$F(z) = z \quad \text{otherwise,}$$

is evidently a φ-homomorphism of the algebra A, which is not 1—1. To verify F's being also a ψ-homomorphism it is sufficient to prove that for $z \in K(x)$ we have $F\psi(z) = \psi F(z)$, i.e., since $\psi(z) \notin K(x)$, $\psi(z) = \psi h(z)$. Let $z \in K(x)$, $\varphi^k(z) = x$, $k \geq 0$. Then

$$\varphi^k h(z) = h\varphi^k(z) = h(x) ,$$
$$\psi\varphi^{k+1}(z) = \psi(x_0) = \psi\varphi^{k+1} h(z) ,$$
$$\varphi^{2k+2}\psi(z) = \psi(x_0) = \varphi^{2k+2}\psi h(z) .$$

But the identity $\varphi^q \psi = \psi$ implies that both $\psi(z)$ and $\psi h(z)$ are elements of φ-cycles, hence they coincide. Thus, the mapping F is an endomorphism of A, which is not 1—1 – a contradiction.

b) *Let* $r = s > 1$.

Put $y = \varphi^{r-1}(x)$. Sinde $h(p)$ are of the same type, we obtain $h(y) = \varphi^{r-1}h(x) = g(y)$ and $\varphi(y) = \varphi h(y)$. Since p is reduced, $y \neq h(y)$. Thus $y \in \mathrm{W}$ and there is a path of the type $\langle \alpha, \beta \rangle$ from y to $h(y)$, which contradicts to a).

c) *Let* $r > s \geq 1$.

Put $y = \varphi^s(x)$. Since $h(p)$ and $g(p)$ are of the same type as p, we obtain $h(y) = \varphi^s \mathrm{h}(x) = g(y)$. If $y = h(y)$, then $\langle \underbrace{\alpha, \ldots, \alpha}_{s-times}, \underbrace{\beta, \ldots, \beta}_{s-times} \rangle$ is the type of a path from x to $h(x)$, which has already been excluded. Thus, $y \neq h(y)$. But then $y \in \mathrm{W}$ and there is a right φ-path from y to $h(y)$, which contradicts to 2.8.

The remaining case $s > r \geq 1$ is excluded analogically as c).

2.11. Lemma: For any $x \in \mathrm{W}$, no path p is a ψ-path.

Proof: The type of a ψ-path p would have to consist of r γ's followed by s δ's, with $r, s \geq 1$. It will be sufficient to exclude the case $r = s = 1$, the rest being quite analogous to 2.10.b), c), d). Thus, suppose that p is of the type $\langle \gamma, \delta \rangle$, i.e. $p = \langle x, x_0, h(x) \rangle$, where $x_0 = \psi(x) = \psi h(x)$. Clearly $h(x_0) = x_0$. For $u \in U \cup \{x_0\}$ let $K(u)$ denote the φ-component of the point u, $K'(u) = \bigcup_{j=1}^{\infty} (\psi^j)^{-1}(K(u))$. The sets $K(u), u \in U \cup \{x_0\}$ are mutually disjoint for otherwise there would exist a φ-path from x to $h(x)$, which contradicts the previous two lemmas.

1) We prove that $K'(u_1) \cap K'(u_2) = \varnothing$ for $u_1, u_2 \in U, u_1 \neq u_2$. Let us suppose that there exists a point $z \in K'(u_1) \cap K'(u_2)$, $u_1, u_2 \in U$, $u_1 \neq u_2$, i.e. $\psi^m(z) = \varphi^k(u_1)$ and $\psi^n(z) = \varphi^l(u_2)$ for some $m, n \geq 1$ and $k, l \geq 0$. The assumption $m = n$ leads to a contradiction with the disjointness of $K(u_1)$ and $K(u_2)$, so we may suppose $m < n$. Thus, by lemma 2.1. we have

$$\varphi^l(u_2) = \psi^{n-m}\varphi^k(u_1) = \psi^{n-m-1}\varphi^{2k}\psi(u_1) = \psi^{n-m-1}\varphi^{2k}(x_0),$$

denote the point by x_1. $h(y_0) = x_0$ implies $h(x_1) = x_1$, so $\varphi^l h(u_2) = h\varphi^l(u_2) = h(x_1) = x_1$. But the resulting formula $\varphi^l(u_2) = \varphi^l h(u_2)$ means that there exists a φ-path from u_2 to $h(u_2)$ and consequently also from x to $h(x)$ — a contradiction.

2) For $u \in U$ put $K''(u) = K'(u) \cup K(u)$. By 2.3.a) we have $\varphi^{-1}(K''(u)) \subseteq K''(u)$, $\varphi(K''(u)) \subseteq K''(u)$, $\psi^{-1}(K''(u)) = K'(u)$, so $\psi(K'(u)) \subseteq K''(u)$, $\psi(K(u)) \subseteq K(x_0)$. Thus, the set $K''(u)$ is closed under the forming of images and proimages in both φ and ψ with the exception of $K(u) \subseteq K''(u)$, which is mapped by ψ into $K(x_0)$. The mapping $F : X \to X$ defined by

$$F(z) = h(z) \quad \text{whenever} \quad z \in K''(x),$$
$$F(z) = z \quad \text{otherwise}$$

is clearly a φ-homomorphism of A, which is not 1—1, for $K''(u), u \in U$ are disjoint and isomorphic as quasialgebras with a complete unary operation φ and a partial unary operation ψ. To prove F's being also a ψ-homomorphism it is sufficient to prove that $F\psi(z) = \psi F(z)$ for each $z \in K(x)$. Let $\varphi^l(x) = \varphi^k(z)$ for some $k, l \geq 0$. Then we have

(i) $\quad \varphi^{2k}F\psi(z) = \psi^{2k}\psi(z) = \psi\varphi^k(z) = \psi\varphi^l(x) = \psi^{2l}(x_0)$,

$\quad \varphi^k F(z) = \varphi^k h(z) = h\varphi^k(z) = h\varphi^l(x) = \varphi^l h(x)$, so that

(ii) $\quad \varphi^{2k}\psi F(z) = \psi\varphi^k F(z) = \psi\varphi^l h(x) = \varphi^{2l}(x_0)$.

But both the points $F\psi(z)$ and $F(z)$ are elements of φ-cycles, so $i)$ and (ii) imply their coincidence. Thus, F is an endomorphism of A, which is not 1—1 — a contradiction.

2.12. Lemma: For any $x \in W$, no path p is a mixed path.

Proof: 1) Maximal (as to their length) left (or right) subpaths of an arbitrary path p' are called left (or right, respectively) *blocks*. A left (right) block is said to be *ordered*, if it does not contain a subpath of the type $\langle \alpha, \gamma \rangle$ ($\langle \delta, \beta \rangle$, respectively). By 2.2. we may replace every block by the ordered block with the same ending points. The decomposition of a path p' on left (λ) and right (ϱ) blocks determines a finite sequence of symbols λ, ϱ called the *block type* of p'. The notions of a *subtype* and of a *length* of a block type are defined in an obvious manner. Every subtype of the block type of p' determines obviously a subpath p'' of p'.

2) Now, let p be a mixed path from x to $h(x)$, $n \in N$ be a length of its block type, all blocks of p are ordered. If the block type of p has a subtype $\langle \lambda, \varrho \rangle$, denote by p_0 the subpath determined by this subtype. Then p_0 must be of the type

a) $\langle \delta, \ldots, \delta, \alpha, \ldots, \alpha \rangle$ or b) $\langle \beta, \ldots, \beta, \delta, \ldots, \delta, \alpha, \ldots, \alpha \rangle$ or

c) $\langle \beta, \ldots, \beta, \gamma, \ldots, \gamma \rangle$ or d) $\langle \beta, \ldots, \beta, \gamma, \ldots, \gamma, \delta, \ldots, \delta \rangle$.

If a) is the case, i.e. $p_0 = \langle \psi^r(a), \ldots, \psi(a), a, \varphi(a), \ldots, \varphi^s(a) \rangle$ for some $r, s > 1$, $a \in X$, then we replace p_0 by

$p_1 = \langle \psi^r(a), \varphi\psi^r(a), \ldots, \varphi^{s2^r}\psi^r(a) = \psi^r\varphi^s(a), \ldots, \psi\varphi^s(a), \varphi^s(a) \rangle$

of the type $\langle \alpha, \ldots, \alpha, \delta, \ldots, \delta \rangle$ (see 2.1).

If b) is the case, i.e. $p_0 = \langle \varphi^t\psi^r(a), \ldots, a, \ldots, \varphi^s(a) \rangle$ for some $r, s, t \geq 1$, $a \in X$, then we replace p_0 by

$p_1 = \langle \varphi^t\psi^r(a), \ldots, \psi^r(a), \ldots, \varphi^{s2^r}\psi^r(a) = \psi^r\varphi^s(a), \ldots, \varphi^s(a) \rangle$

of the type $\langle \beta, \ldots, \beta, \alpha, \ldots, \alpha, \delta, \ldots, \delta \rangle$.

For c) and d) being the case we proceed symmetrically.

After forming a reduced path with ordered blocks from the path obtained we obtain a path p_2 from x to $h(x)$, whose block type is $\langle \varrho, \lambda \rangle$ if $n = 2$, or has the length less then n if $n \geq 3$. After a finite number of such procedures we obtain a path p_3 from x to $h(x)$, which is left or right — a contradiction with 2.8., or is of the block type $\langle \varrho, \lambda \rangle$.

3) Thus, let us suppose that

$p_3 = \langle x, \psi(x), \ldots, \psi^m(x), \ldots, \varphi^k\psi^m(x) = \varphi^l\psi^n h(x), \ldots, \psi^n h(x), \ldots, h(x) \rangle$

for some $k, l, m, n \geq 0$, $(k + m)(l + n) \geq 1$; we may suppose $m \geq n$. The assumption $\psi^n(x) = \psi^n h(x)$ leads to a contradiction with the previous lemma.

Consequently, $\psi^n(x) \in W$, so that we may suppose $n = 0$. Now, the assumption $m = 0$ would lead to a contradiction with 2.9. and 2.10., so we have

$$p_3 = \langle x, \dots, \psi^m(x), \dots, \varphi^k \psi^m(x) = \varphi^l h(x), \dots, h(x) \rangle \text{ for some } m, l \geq 1,$$
$$0 \leq k < q. \text{ Then } p_4 = \langle x, \dots, \psi^m(x) = \varphi^q \psi^m(x) = \varphi^{l+q-k} h(x), \dots, h(x) \rangle$$

is a path from x to $h(x)$ of the type $\langle \gamma, \dots, \gamma, \beta, \dots, \beta \rangle$.
Put $t = l + q - k$. Then $h(p_4) = \langle h(x), \dots, \psi^m h(x) = \varphi^t h^2(x), \dots, h^2(x) \rangle$, $g(p_4) = \langle h(x), \dots, \psi^m h(x) = \varphi^t(x), \dots, x \rangle$. Thus, there exists a φ-path $p_5 = \langle x, \dots, \varphi^t(x) = \varphi^t h^2(x), \dots, h^2(x) \rangle$ from x to $h^2(x)$, so that $h(p_5)$ is a φ-path from x to $h(x)$, which is a contradiction with 2.9. and 2.10.

This concludes the proof of 2.6.

2.13. Proposition: The monoid $M_{P, \varphi^{n+q} = \varphi^n}$ is poor for all integers $n \geq 0$, $q \geq 1$.

Proof: The small category k from 2.5. cannot be fully embedded into the category $SetM_{P, \varphi^{n+q} = \varphi^n}$, for by 2.4. each such embedding must factor through $SetM_{P, \varphi^q = 1}$, which contradicts to the proof of 2.6.a).

2.14 Proposition: The monoid $M = M_{P, \varphi^{n+q}\psi = \varphi^n \psi}$ is poor for all integers $q \geq 1$, $n \geq 0$.

Proof. For $n = 0$ the statement concurs with that of 2.6.b). Suppose that M were rich for some $n > 0$. Let k be the small category defined in 2.5., let $\Phi : k \to Set^M$ be a full embedding, $\Phi(a') = (X', \varphi', \psi')$, $\Phi(a) = (X, \varphi, \psi)$. If both φ and φ' are 1—1, then $\Phi(k) \subseteq SetM_{P, \varphi^q \psi = \psi}$, which contradicts to 2.6.

Hence, let us suppose that e.g. φ is not 1—1. Denote by $\bar{X}$ the set of all φ-components of (X, φ, ψ). By 2.3.a) ψ preserves φ-connectedness, so that the formula

$$\bar{\psi}(K) \supseteq \psi(K), \quad K \in \bar{X}$$

defines the mapping $\bar{\psi} : \bar{X} \to \bar{X}$.

Let $K \in \bar{\psi}(\bar{X})$. Then the identity $\varphi^{n+q}\psi = \varphi^n \psi$ implies that K has a non-empty cyclic part. Denote by $f_K : K \to K \subseteq X$ the obvious winding up of K on its cyclic part. For $K \in \bar{X} \setminus \bar{\psi}(\bar{X})$ the mapping $f_K : K \to X$ is defined by $f_K = \varphi^{nq} \mid K$. The mapping $F : X \to X$, $F = \bigcup_{K \in \bar{X}} f_K$ is an endomorphism of the algebra (X, φ, ψ), which is not 1—1. This contradicts the supposition of the full embedding of k, whose every endomorphism is an isomorphism.

2.15. Lemma: The monoid $M_1 = M_{P, \varphi^k \psi^m = \psi^l \psi^n}$ is a factormonoid of the monoid $M_2 = M_{P, \varphi^{l+2n}\psi^n = \varphi^{l+2m}\psi^n}$ for all integers $k, l, m, n \geq 0$.

Proof: Using the identities defining M_1 we have
$\varphi^{l+2n} \psi^n = \varphi^l \psi^n \varphi = \varphi^k \psi^m \varphi = \varphi^{k+2m} \psi^m = \varphi^{l+2m} \psi^n$.

2.16. Proposition: The monoid $M_{P, \varphi^k \psi^m = \varphi^l}$ is poor for all integers $k, l \geq 0$, $m \geq 1$.

Proof: follows from 2.13. and 2.15.

2.17. Proposition: The monoid $M_{P,\varphi^k\psi^m=\varphi^l\psi}$ is poor for all integers $k, l \geq 0$, $m \geq 2$.

Proof: 2.14 and 2.15.

2.18. Proposition: The monoid $M_{P,\varphi\psi^2=\varphi^4\psi^2}$ is poor.

Proof: Suppose that there exists a full embedding
$\Phi : k \to Set M_{P,\varphi\psi^2=\varphi^4\psi^2}$, $\Phi(a) = (X, \varphi, \psi)$, $\Phi(a') = (X', \varphi', \psi')$.
Then either $\psi^2 : X \to X$ is an endomorphism of the algebra (X, φ, ψ), which is not 1—1, or $\psi'^2 : X' \to X'$ has the same property, or $\Phi(k) \subseteq Set M_{P,\varphi=\varphi^4}$ — a contradiction in any case.

2.19. Proposition: The monoid $M_{P,\psi^2=\varphi\psi^2}$ is poor.

Proof: It is a factor monoid of the monoid from 2.18.

2.10. Theorem: The monoid $M = M_{P,\psi^3=\psi^2=\varphi^2\psi^2}$ is rich and each of its proper factomonoids is poor.

Proof: 1. Richness see 1.1

2. The monoid M has just the following distinct elements:

$$\begin{array}{l} 1, \varphi, \varphi^2, \varphi^3, \ldots \\ \psi, \varphi\psi, \varphi^2\psi, \varphi^3\psi, \ldots \\ \psi^2, \varphi\psi^2 . \end{array}$$

Consequently, the poorness statement follows from 2.13., 2.14., 2.16., 2.17. and 2.19.

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