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Unipotent Quasigroups

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This paper deals with unipotent quasigroups, i.e. quasigroups satisfying the identity $xx = yy$. This class of quasigroups has a nice property with respect to isotopy, namely every quasigroup is isotopic to some unipotent quasigroup. In the first section, several basic results concerning unipotent quasigroups are presented, e.g. the quasigroups having the isotopy-isomorphism property in this class are completely described. The following two sections are devoted to unipotent IP-quasigroups and unipotent medial quasigroups. The results of the last section determine the structure of some quasivariaties of quasigroups.

В статье изучены унипотентные квазигруппы, те квазигруппы выполнюящие равенство $xx = yy$. В первой части получены основные результаты, доказывается например, что всякая квазигруппа изотопна некоторой унипотентной квазигруппе и описываются все квазигруппы обладающие изотопно-изоморфным свойством относительно класса унипотентных квазигрупп. Вторая и третья части посвящены унипотентным IP-квазигруппам и унипотентным медиальным квазигруппам. В последней части исследованы некоторые квазипримитивные классы квазигрупп.

Článek se zabývá studiem unipotentních kvazigrup, tj. kvazigrup splňujících identitu $xx = yy$. V úvodní části jsou dokázány některé základní výsledky o unipotentních kvazigrupách. Kromě jiného je dokázáno, že každá kvazigrupa je izotopní některé unipotentní kvazigrupě. Další dvě části jsou věnovány unipotentním IP-kvazigrupám a unipotentním mediálním kvazigrupám. V závěrečné části jsou popsány některé kvazivariety kvazigrup.

As it is well-known, every quasigroup is isotopic to some loop. However, there are some classes of quasigroups other than loops which also have this remarkable property. One of them is the class of unipotent quasigroups, i.e. quasigroups satisfying the identity $xx = yy$, and namely this class is studied in the present paper. In the first section, several basic results concerning unipotent quasigroups are presented, e.g. the quasigroups having the isotopy-isomorphism property in this class are completely described. The following two sections are devoted to unipotent IP-quasigroups and unipotent medial quasigroups, respectively. The results of the last section determine the structure of the quasivarieties of quasigroups defined by the quasiidentities of the form

$$x_1x_2 = x_3x_4 \Rightarrow x_{\pi(1)}x_{\pi(2)} = x_{\pi(3)}x_{\pi(4)} ,$$

where π is some transformation of the set $\{1, 2, 3, 4\}$.

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1. Basic Properties

The terminology and notation is standard (see e.g. [1]). In particular, if Q is a quasigroup then the parastrophs of Q are denoted by Q^{-1}, ${}^{-1}Q$, $\overline{Q}$, ${}^{-1}(Q^{-1})$ and $({}^{-1}Q)^{-1}$, $L_x(y) = R_y(x) = xy$ and $xe(x) = f(x)x = x$ for all $x, y \in Q$, $C(Q)$ is the centre of Q and S_Q is the set of all permutations of the set Q. If Q is a group then its unit will always be denoted by 0.

A quasigroup Q is called unipotent if $xx = yy$ for all $x, y \in Q$. Obviously, every unipotent quasigroup has exactly one idempotent element. Immediately from the definition of parastrophs we have the following simple result:

1.1 Proposition. Consider the following conditions for a quasigroup Q:

Q	is	a left loop	a right loop	unipotent
Q^{-1}	is	a left loop	unipotent	a right loop
${}^{-1}Q$	is	unipotent	a right loop	a left loop
$\overline{Q}$	is	a right loop	a left loop	unipotent
${}^{-1}(Q^{-1})$	is	unipotent	a left loop	a right loop
$({}^{-1}Q)^{-1}$	is	a right loop	unipotent	a left loop..

Then the conditions in each column are equivalent.

1.2 Corollary. The following are equivalent for a quasigroup Q:

(i) Q is a unipotent loop.
(ii) All parastrophs of Q are unipotent loops.
(iii) At least one of the parastrophs of Q is a unipotent loop.
(iv) At least two of $\{Q, \overline{Q}\}$, $\{Q^{-1}, ({}^{-1}Q)^{-1}\}$, $\{{}^{-1}Q, {}^{-1}(Q^{-1})\}$ contain a loop.
(v) Each of $\{Q, \overline{Q}\}$, $\{Q^{-1}, ({}^{-1}Q)^{-1}\}$, $\{{}^{-1}Q, {}^{-1}(Q^{-1})\}$ contains a unipotent quasigroup.
(vi) All parastrophs of Q are loops.
(vii) All parastrophs of Q are unipotent.

1.3 Proposition. Every quasigroup Q is isotopic to a unipotent left (right) loop.

Proof. Let $a, b \in Q$ and $g(x) = L_b^{-1}(x)$ for each $x \in Q$. Define $x * y = R^{-1}_{g(x)}(a)\, L_b^{-1}(y)$. The mapping $x \mapsto R^{-1}_{g(x)}(a)$ clearly is a permutation, so $Q(*)$ is a quasigroup. Further, $x * x = a = y * y$ and $a * x = bL_b^{-1}(y) = y$ for all $x, y \in Q$. Similarly the other case.

1.4 Proposition. Every commutative unipotent quasigroup Q is isotopic to a commutative unipotent loop.

Proof. It suffices to take $a \in Q$ and put $x * y = L_a^{-1}(x)\, L_a^{-1}(y)$.

As it is easy to see, the three-element cyclic group is the only loop with three elements, however this loop is not unipotent. Hence there exist unipotent quasigroup which are not isotopic to unipotent loops.

It is an easy exercise to construct unipotent quasigroups of arbitrary finite order, unipotent loops of every finite order $n \geq 4$ and commutative unipotent quasigroups of every even order (see e.g. [2]). On the other hand, every non-trivial finite commutative unipotent quasigroup has even order, as follows from the following well-known (and obvious) result:

1.5 Proposition. A finite commutative quasigroup has odd order iff the mapping $x \mapsto xx$ is a permutation.

Now we turn our attention to the problem of determining all quasigroups with the isotopy-isomorphism property in the variety of unipotent quasigroups.

1.6 Proposition. Let Q be a unipotent quasigroup such that every unipotent quasigroup which is isotopic to Q is isomorphic to Q. Then Q has at most two elements.

Proof. According to 1.3, Q is a loop with unit 0. Further, let $a, b \in Q$ be arbitrary, $g(x) = L_b^{-1}(x)$ and $x * y = R_{g(x)}^{-1}(a)\, L_b^{-1}(y)$ for all $x, y \in Q$. Then $Q(*)$ is a loop with unit a (cf. the proof of 1.3) and, for $b = 0$ and $y = a$, we get $R_x^{-1}(a)\, a = x$. Similarly, $aL_x^{-1}(a) = x$ for all $x, a \in Q$, and so Q is a TS-loop. Consequently, for $x = bz$ and $y = a$, we have $bz = ((b \,.\, bz)a)\,(ba) = za \,.\, ba$, so that Q is a Moufang loop. Now, if $x, y, z \in Q$ are arbitry then, combining the preceding results, we have $x \,.\, yz = (y \,.\, xy)\,(yz) = xy \,.\, z$ and we conclude that Q is an Abelian group. Finally, let $h \in \mathrm{Aut}\, Q$ and $x \circ y = h(x)h(y)$. $Q\,(\circ)$ is clearly unipotent, thus being a loop with unit j. Now, for every $x \in Q$, $j = x \circ x = h(xx) = (h(0) = 0$, so $x = x \circ 0 = h(x)$ and we are through.

We conclude this section with a problem. We shall say that a variety $\mathscr{V}$ of quasigroups has property (U) if every quasigroup is isotopic to some quasigroup from $\mathscr{V}$. Denote by $\mathscr{Q}$, $\mathscr{Q}\mathscr{U}$, $\mathscr{R}\mathscr{U}$ the varieties of loops, unipotent left loops and unipotent right loops, respectively. We have shown that all these varieties have property (U). Moreover, considering the free loop on countably many free generators, it is not difficult to prove that $\mathscr{Q}$, $\mathscr{Q}\mathscr{U}$ and $\mathscr{R}\mathscr{U}$ are minimal varieties with property (U). Hence a question arises: Does every variety of quasigroups which has property (U) contain one of $\mathscr{Q}$, $\mathscr{Q}\mathscr{U}$, $\mathscr{R}\mathscr{U}$?

2. Unipotent IP-Quasigroups

A quasigroup Q is called an RIP-quasigroup (LIP-quasigroup) if there is a mapping $\delta\,(\tau)$ of Q into Q such that $yx \,.\, \delta(x) = y$ $(\tau(x) \,.\, xy = y)$ for all $x, y \in Q$. IP-quasigroup is a quasigroup which is simultaneously LIP and RIP. The following assertions are easy and well-known.

2.1 Lemna. Let Q be an RIP-quasigroup (LIP-quasigroup). Then

(i) $\delta^2 = 1$ and δ is a permutation ($\tau^2 = 1$ and τ is a permutation),

(ii) $y\, \delta(x) \,.\, x = y$ $(x \,.\, \tau(x)y = y)$ for all $x, y \in Q$,

(iii) $x\delta(x) = f(x)$ and $\delta(x)x = f(\delta(x))$ $(\tau(x)x = e(x)$ and $x\tau(x) = e(\tau(x)))$,

(iv) $e(x) = \delta(e(x))$ $(f(x) = \tau(f(x)))$.

2.2 Lemma. Let Q be an IP-quasigroup. Then $\delta(xy) = \tau(y)\tau(x)$ and $\tau(xy) = \delta(y)\delta(x)$ for all $x, y \in Q$.

2.3 Lemma. The following conditions are equivalent for a quasigroup Q:

(i) Q is a commutative IP-quasigroup.

(ii) There are a TS-quasigroup $Q(*)$ and $\delta \in \operatorname{Aut} Q(*)$ such that $\delta^2 = 1$ and $xy = \delta(x * y)$ for all $x, y \in Q$.

2.4 Proposition. The following conditions are equivalent for a quasigroup Q:

(i) Q is a TS-loop.

(ii) Q is a unipotent TS-quasigroup.

(iii) Q is a unipotent IP-loop.

Proof. Obvious.

2.5 Corollary. The following are equivalent for a quasigroup Q:

(i) Q is a commutative unipotent IP-quasigroup.

(ii) There are a TS-loop $Q(*)$ and its automorphism δ such that $\delta^2 = 1$ and $xy = \delta(x * y)$ for all $x, y \in Q$.

Let Q be a quasigroup. Then $\mathscr{Q}(Q)$ will be the set of all ordered pairs (λ, ϱ) such that λ, ϱ are mappings of Q into Q and $\lambda(xy) = \varrho(x)y$ for all $x, y \in Q$. Similarly, $\mathscr{R}(Q) = \{(\lambda, \varrho) \mid \lambda(xy) = x\varrho(y)\}$ and $\mathscr{M}(Q) = \{(\lambda, \varrho) \mid \lambda(x)y = x\varrho(y)\}$. Further, $L_l(Q)$ will be the set of all mappings λ such that $(\lambda, \varrho) \in \mathscr{Q}(Q)$ for some ϱ. Similarly we define $L_r(Q)$, $\mathscr{R}_l(Q)$, $R_r(Q)$, $M_l(Q)$, $M_r(Q)$.

2.6 Lemma. Let Q be a quasigroup. Then

(i) $\mathscr{Q}(Q)$ and $\mathscr{R}(Q)$ are subgroups of $S_Q \times S_Q$,

(ii) $\mathscr{M}(Q)$ is a subgroup of $S_Q \times \overline{S}_Q$,

(iii) $L_l(Q)$, $L_r(Q)$, $R_l(Q)$, $R_r(Q)$, $M_l(Q)$, $M_r(Q)$ are subgroups of S_Q,

(iv) $L_l(Q) \cong L_r(Q)$, $R_l(Q) \cong R_r(Q)$ and $M_l(Q) \cong M_r(Q)$.

Proof. Easy.

2.7 Lemma. Let Q be a quasigroup. Then

(i) $\lambda\varphi = \varphi\lambda$ for all $\lambda \in L_l(Q)$ and $\varphi \in R_l(Q)$,

(ii) $\varrho\varphi = \varphi\varrho$ for all $\varrho \in L_r(Q)$ and $\varphi \in M_l(Q)$,

(iii) $\lambda\psi = \psi\lambda$ for all $\lambda \in R_r(Q)$ and $\psi \in M_r(Q)$.

Proof. Easy.

2.8 Lemma. The following are equivalent for a quasigroup Q:

(i) Q is isotopic to a group.

(ii) At least one of the groups $L_l(Q)$, $L_r(Q)$, $R_l(Q)$, $R_r(Q)$, $M_l(Q)$, $M_r(Q)$ operates transitively on Q.

(iii) Each of the six groups operates transitively on Q.

Proof. See e.g. [4].

2.9 Lemma. Let Q be a commutative quasigroup. Then

(i) $\mathscr{L}(Q) = \mathscr{R}(Q)$,

(ii) $(\varrho, \varrho) \in \mathscr{M}(Q)$ for all $\varrho \in L_r(Q)$,

(iii) $L_1(Q) = R_1(Q)$ and $L_r(Q) = R_r(Q) \subseteq M_1(Q) \cap M_r(Q)$.

Proof. Obvious.

2.10 Lemma. Let Q be a commutative quasigroup. Then $L_1(Q)$ and $L_r(Q)$ are commutative groups.

Proof. Apply 2.6, 2.7 and 2.9.

2.11 Lemma. Let Q be an RIP-quasigroup. Then

(i) $(\lambda, \varrho) \in \mathscr{L}(Q)$ iff $(\varrho, \lambda) \in \mathscr{L}(Q)$,

(ii) $L_1(Q) = L_r(Q)$,

(iii) $(\lambda, \varrho) \in \mathscr{R}(Q)$ iff $(\lambda, \delta\varrho^{-1}\delta) \in \mathscr{M}(Q)$,

(iv) $R_1(Q) = M_1(Q)$,

(v) $R_r(Q) \cong M_r(Q)$.

Proof. (i) We have $\varrho(xy) = \varrho(xy)\delta(y) \,.\, y = \lambda(xy \,.\, \delta(y))y = \lambda(x)y$ for all $x, y \in Q$.

(ii) follows immeditely from (i).

(iii) Let $(\lambda, \varrho) \in \mathscr{R}(Q)$. Then $\lambda(x)y = \lambda(x\delta\varrho^{-1}\delta(y) \,.\, \varrho^{-1}\delta(y))y =$
$= (x\delta\varrho^{-1}\delta(y) \,.\, \delta(y))y = x\delta\varrho^{-1}\delta(y)$. Similarly the converse.

(iv) follows from (iii) and (v) follows from (iv) and 2.6(iv).

2.12 Lemma. Let Q be an LIP-quasigroup. Then

(i) $(\lambda, \varrho) \in \mathscr{R}(Q)$ iff $(\varrho, \lambda) \in \mathscr{R}(Q)$,

(ii) $R_r(Q) = R_1(Q)$,

(iii) $(\lambda, \varrho) \in \mathscr{L}(Q)$ iff $(\tau\varrho^{-1}\tau, \lambda) \in \mathscr{M}(Q)$,

(iv) $L_1(Q) = M_r(Q)$,

(v) $L_r(Q) \cong M_1(Q)$.

Proof. Dual to that of 2.11.

2.13 Lemma. Let Q be an IP-quasigroup. Then

(i) $L_1(Q) = L_r(Q) = M_r(Q) \cong R_1(Q) = R_r(Q) = M_1(Q)$,

(ii) $(\lambda, \varrho) \in \mathscr{L}(Q)$ iff $(\delta\lambda\delta, \tau\varrho\tau) \in \mathscr{R}(Q)$,

(iii) $\mathscr{L}(Q) \cong \mathscr{R}(Q) \cong \mathscr{M}(Q)$.

Proof. Apply 2.11 and 2.12.

2.14 Lemma. Let Q(be a commutative IP-quasigroup. Then

(i) $L_1(Q) = L_r(Q) = M_r(Q) = M_1(Q) = R_1(Q)$ is a commutative group,

(ii) $\lambda = \varrho$, whenever $(\lambda, \varrho) \in \mathscr{M}(Q)$.

Proof. (i) Apply 2.9 and 2.13.

(ii) We have $\lambda(x) = \delta(x) \,.\, \lambda(x)x = \delta(x) \,.\, x\,\varrho(x) = \varrho(x)$.

2.15 Lemma. Let Q be a unipotent quasigroup. Then
(i) $\varrho = \lambda^{-1}$ whenever $(\lambda, \varrho) \in \mathscr{M}(Q)$,
(ii) $M_l(Q) = M_r(Q)$.

Proof. (i) We have $xx = \lambda(x)\lambda(x) = x\varrho\lambda(x)$, hence $x = \varrho\lambda(x)$.
(ii) follows from (i).

2.16 Proposition. Let Q be a unipotent IP-quasigroup. Then $L_l(Q) = L_r(Q) = M_r(Q) = R_l(Q) = R_r(Q) = M_l(Q)$ is a commutative group. Moreover, if Q is commutative then every element of $L_l(Q)$ has order 2.

Proof. Use 2.13, 2.14 and 2.15.

2.17 Lemma. Let Q be a commutative quasigroup, $\alpha, \beta \in S_Q$ and $x * y = \alpha(x)\beta(y)$ for all $x, y \in Q$. Then $Q(*)$ is commutative iff $(\alpha\beta^{-1}, \alpha\beta^{-1}) \in \mathscr{M}(Q)$.

Proof. Trivial.

2.18 Proposition. Let Q be a commutative unipotent IP-quasigroup. Then every commutative quasigroup isotopic to Q is unipotent.

Proof. Let $a(x)\beta(y) = \alpha(y)\beta(x)$ for all $x, y \in Q$. Then $\alpha\beta^{-1} \in L_r(Q)$ by 2.17 and 2.16, hence $a\beta^{-1}(x)x = \alpha\beta^{-1}(y)y$, and so $\alpha(x)\beta(x) = \alpha(y)\beta(y)$ for all $x, y \in Q$.

3. Unipotent Medial Quasigroups

A quasigroup Q is called medial if $ab \,.\, cd = ac \,.\, bd$ for all $a, b, c, d \in Q$.

3.1 Proposition. The following are equivalent for a unipotent quasigroup Q:
(i) Q is medial.
(ii) There are an Abelian group $Q(+)$, $\varphi \in \operatorname{Aut} Q(+)$ and $e \in Q$ such that $xy = \varphi(x - y) + e$ for all $x, y \in Q$.
(iii) There is $x \in Q$ such that, for all $a, b, c \in Q$, $xa \,.\, bc = xb \,.\, ac$.
(iv) There is $x \in Q$ such that, for all, $a, b, c \in Q$, $ab \,.\, cx = ac \,.\, bx$.
(v) There is $x \in Q$ such that, for all $a, b, c \in Q$, $ax \,.\, bc = ab \,.\, xc$.
(vi) For all $a, b, c \in Q$, $aa \,.\, bc = ab \,.\, ac$ and $bc \,.\, aa = ba \,.\, ca$.

Proof. The equivalence of (i) and (ii) is obvious from Toyoda's theorem and the equivalence of (i), (iii), (iv) and (v) follows immediately from Theorems 1.3 and 2.3 of [5]. Finally, if (vi) holds then Q is isotopic to a group by 2.8, since $L_aL_b^{-1} \in M_l(Q)$ for all $a, b \in Q$, and an application of Proposition 4.3 from [5] finishes the proof.

A quasigroup Q is an F-quasigroup if $a \,.\, bc = ab \,.\, e(a)c$ and $bc.a = bf(a) \,.\, ca$ for all $a, b, c \in Q$. It is an open question (at least for the authors) whether every unipotent F-quasigroup is medial. The answer is positive in the commutative case.

In [6] was proved that the lattice $\mathscr{S}$ of all varieties of unipotent medial quasigroups is countable and modular, but not distributive. However, a complete description of $\mathscr{S}$ was obtained only in the commutative case. On the other hand, the variety of commutative unipotent medial groupoids contains uncountably many subvarieties.

4. Unipotent Quasigroups Isotopic to a Group

4.1 Proposition. The following conditions are equivalent for a quasigroup Q:
(i) Q is unipotent and isotopic to a group.
(ii) There are a group $Q(+)$ and $\alpha \in S_Q$ such that $\alpha(0) = 0$ and $xy = \alpha(x) - \alpha(y)$ for all $x, y \in Q$.

Proof. (i) implies (ii). Denote $0 = xx$, $x \in Q$. Now it suffices to define $x + y = R_0^{-1}(x) L_0^{-1}(y)$.
(ii) implies (i) trivially.

Denote by E the set of all transformations of the set $A = \{1, 2, 3, 4\}$. If $S \subseteq E$ then $\mathscr{K}_S$ will denote the class of all quasigroups satisfying the quasi-identities

$$x_1 x_2 = x_3 x_4 \Rightarrow x_{\pi(1)} x_{\pi(2)} = x_{\pi(3)} x_{\pi(4)}$$

for all $\pi \in S$. Instead of $\mathscr{K}_{\{\pi\}}$ we shall simply write $\mathscr{K}_\pi$.

4.2 Lemma. If $\alpha, \beta \in E$ then $\mathscr{K}_\alpha \cap \mathscr{K}_\beta \subseteq \mathscr{K}_{\alpha\beta} \cap \mathscr{K}_{\beta\alpha}$.

Proof. Obvious.

Now introduce the following notation for the elements of S_A: $1 = 1_A$, $2 = (12)(34)$, $3 = (12)$, $4 = (13)$, $5 = (23)$, $6 = (1234)$, $7 = (234)$, $8 = (123)$, $9 = (143)$, $10 = (134)$, $11 = (124)$, $12 = (243)$, $13 = (132)$, $14 = (142)$, $15 = (14)$, $16 = (1243)$, $17 = (1342)$, $18 = (1432)$, $19 = (24)$, $20 = (1423)$, $21 = (34)$, $22 = (1324)$, $23 = (14)(23)$ and $24 = (13)(24)$.

4.3 Proposition. (i) $\mathscr{K}_1 = \mathscr{K}_{24}$ is the class of all quasigroups.
(ii) $\mathscr{K}_2 = \mathscr{K}_{23}$.
(iii) $\mathscr{K}_3 = \mathscr{K}_{22} = \mathscr{K}_{21} = \mathscr{K}_{20}$ is the class of all commutative quasigroups.
(iv) $\mathscr{K}_4 = \mathscr{K}_{19}$.
(v) $\mathscr{K}_5 = \mathscr{K}_{17} = \mathscr{K}_{16} = \mathscr{K}_{15}$.
(vi) $\mathscr{K}_6 = \mathscr{K}_{18}$.
(vii) $\mathscr{K}_7 = \mathscr{K}_{14} = \mathscr{K}_{13} = \mathscr{K}_{12} = \mathscr{K}_{11} = \mathscr{K}_{10} = \mathscr{K}_9 = \mathscr{K}_8$.

Proof. Use 4.2.

4.4 Proposition. The following are equivalent for a quasigroup Q:
(i) $Q \in \mathscr{K}_2$.
(ii) $xy = e(yx) \,.\, yx$ for all $x, y \in Q$.
(iii) $xy = yx \,.\, f(yx)$ for all $x, y \in Q$.

Proof. If $Q \in \mathscr{K}_2$ then $xy = e(yx) \,.\, yx$, since $yx = yx \,.\, e(yx)$. Conversely, if (ii) holds and $ab = cd$ then $ba = e(ab) \,.\, ab = e(cd) \,.\, cd = dc$. The rest is similar.

4.5 Proposition. The following conditions are equivalent for a quasigroup Q:
(i) $Q \in \mathscr{K}_4$.
(ii) Q is isotopic to an Abelian group every nonzero element of which has order 2.
(iii) There are an Abelian group $Q(+)$, $\alpha, \beta \in S_Q$ and $e \in Q$ such that $\alpha(0) = \beta(0) = 0$, $x + x = 0$ and $xy = \alpha(x) + \beta(y) + e$ for all $x, y \in Q$.

Proof. Clearly, (ii) is equivalent to (iii) and implies (i). (i) implies (ii). Take $a \in Q$ and define $Q(+)$ via $xa + ay = xy$. Then $Q(+)$ is a loop which clearly belongs to $\mathscr{K}_4$, thus being unipotent and commutative. Finally, if $x, y, z \in Q$ are arbitrary then $(x + y) + 0 = y + x$, so that $(x + y) + x = y$, and $y = (x + y) + x = (y + z) + z$ implies $(x + y) + z = (y + z) + x = x + (y + z)$.

4.6. Proposition. The following conditions are equivalent for a quasigroup Q:
(i) $Q \in \mathscr{K}_5$.
(ii) Q is unipotent and isotopic to an Abelian group.
(iii) There are an Abelian group $Q(+)$ and $\alpha \in S_Q$ such that $\alpha(0) = 0$ and $xy = \alpha(x) - \alpha(y)$ for all $x, y \in Q$.

Proof. (i) implies (ii). For all $x, y, z \in Q$ let $x\varphi_z(x) = z = \psi_z(y)y$. Since $Q \in \mathscr{K}_5$, $x\psi_z(y) = \varphi_z(x)y$ and $xx = yy$, so that $\varphi_z \in M_1(Q)$. Hence $M_1(Q)$ operates transitively on Q and Q is isotopic to a group by 2.8. With respect to 4.1, there are a group $Q(+)$ and $\alpha \in S_Q$ such that $\alpha(0) = 0$ and $xy = \alpha(x) - a(y)$. Finally, $-x = y - (x + y)$, hence $\alpha^{-1}(0)\,\alpha^{-1}(x) = \alpha^{-1}(y)\,\alpha^{-1}(x + y)$, so $\alpha^{-1}(0)\,a^{-1}(y) = \alpha^{-1}(x)\,\alpha^{-1}(x + y)$, which means $-y = x - (x + y)$ and $x + y = y + x$.
(ii) implies (iii) by 4.1 and (iii) implies (i) trivially.

4.7 Proposition. The following conditions are equivalent for a quasigroup Q:
(i) $Q \in \mathscr{K}_6$.
(ii) There are an Abelian group $Q(+)$, $\varphi \in \operatorname{Aut} Q(+)$ and $\alpha \in S_Q$ such that $\varphi^2(x) = -x$ and $xy = \alpha(x) + \varphi\alpha(y)$ for all $x, y \in Q$.
(iii) There are an Abelian group $Q(+)$, $\varphi \in \operatorname{Aut} Q(+)$, $\alpha \in S_Q$ and $e \in Q$ such that $\alpha(0) = 0$, $\varphi^2(x) = -x$ and $xy = \alpha(x) + \varphi\alpha(y) + e$ for all $x, y \in Q$.

Proof. (i) implies (ii). For all $x, y, z \in Q$ and $x\psi_z(x) = z$ we have $x\psi_z(x) = y\psi_z(y)$, and so $\psi_z(x)y = \psi_z(y)x$. Using standard methods (similar to those showing that every transitive quasigroup is isotopic to a group), we can easily prove that Q is isotopic to an Abelian group. Now let $a \in Q$ be arbitrary. Then $Q(+)$ defined via $xa + ay = xy$ is an Abelian group and $aa = 0$. If $x + y = u + v$ then. since $Q \in \mathscr{K}_6$, $R_aL_a^{-1}(v) + L_aR_a^{-1}(x) = R_aL_a^{-1}(y) + L_aR_a^{-1}(u)$. However $R_aL_a^{-1}(0) = 0$, so that, in particular, $L_aR_a^{-1}(x + y) = L_aR_a^{-1}(x) - R_aL_a^{-1}(y)$ and $L_aR_a^{-1}(y) = -R_aL_a^{-1}(y)$. Thus $\alpha = R_a$, $\varphi = L_aR_a^{-1}$ have the desired properties.
The remaining implications are very easy.

4.8 Proposition. The following conditions are equivalent for a quasigroup Q:

(i) $Q \in \mathcal{K}_7$.

(ii) Q is commutative, unipotent and isotopic to a group.

(iii) There are an Abelian group $Q(+)$ and $\alpha \in S_Q$ such that $\alpha(0) = 0$, $x + x = 0$ and $xy = \alpha(x) + \alpha(y)$ for all $x, y \in Q$.

Proof. (i) implies (ii). For all $x, y, z \in Q$, let $x\varphi_z(x) = z = \psi_z(y)y$. Since $Q \in \mathcal{K}_7$ and $xy = xy$, we have $xx = yy$, $xy = yx$ and $x\psi_z(y) = y\varphi_z(x) = \varphi_z(x)y$. Hence Q is isotopic to a group by 2.8.

(ii) implies (iii) by 4.1 and 4.6 and (iii) implies (i) trivially.

4.9 Propositin. Let Q be a quasigroup with $C(Q) = \emptyset$. Then

(i) if $Q \in \mathcal{K}_2$ then Q is commutative,

(ii) if $Q \in \mathcal{K}i$, $i = 4, 5, 6$, then $Q \in \mathcal{K}$- .

Proof. An easy work.

4.10 Proposition. Let $\pi \in E$. Then

(i) if $|\mathrm{Im}\,\pi| = 3$ then $\mathcal{K}_\pi$ is the class of all one-element quasigroups,

(ii) if $|\mathrm{Im}\,\pi| = 2$ and $\pi(1) \neq \pi(3)$, $\pi(2) = \pi(4)$ (or $\pi(1) = \pi(3)$, $\pi(2) \neq \pi(4)$) then $\mathcal{K}_\pi$ is the class of all one-element quasigroups,

(iii) if $|\mathrm{Im}\,\pi| = 2$ and $\pi(1) = \pi(2)$, $\pi(3) = \pi(4)$ then $\mathcal{K}_\pi$ is the class of all unipotent quasigroups,

(iv) if $|\mathrm{Im}\,\pi| = 2$ and $\pi(1) = \pi(4)$, $\pi(2) = \pi(3)$ then $\mathcal{K}_\pi$ is the class of all commutative quasigroups,

(v) if $|\mathrm{Im}\,\pi| = 2$ and $\pi(1) = \pi(3)$, $\pi(2) = \pi(4)$ then $\mathcal{K}_\pi$ is the class of all quasigroups,

(vi) if $|\mathrm{Im}\,\pi| = 1$ then $\mathcal{K}_\pi$ is the class of all quasigroups.

Proof. Easy.

Finally, consider the following twelve classes of quasigroups: $\mathcal{K}_i$, $i = 1, \dots, 7$, $\mathcal{U}$ – the class of all unipotent quasigroups, $\mathcal{X}$ – the class of all quasigroups Q such that there are an Abelian group $Q(+)$, $\varphi \in \mathrm{Aut}\, Q(+)$ and $\alpha \in S_Q$ such that $\varphi^2 = 1$, $x + x = 0$ and $xy = \alpha(x) + \varphi\alpha(y)$ for all $x, y \in Q$, $\mathcal{Y}$ – the class of all unipotent quasigroups satisfying the identity $xy = yx \,.\, f(yx)$, $\mathcal{C}$ – the class of all commutative unipotent quasigroups and $\mathcal{O}$ – the class of all one-element quasigroups.

4.11 Proposition. All classes $\mathcal{K}_S$, $S \subseteq E$, are varieties of quasigroups and their intersection semilattice is as follows: $\mathcal{O} \subseteq \mathcal{K}_7 \subseteq \mathcal{C} \subseteq \mathcal{K}_3 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_1$, $\mathcal{K}_6 \subseteq \mathcal{K}_5 \subseteq \mathcal{Y} \subseteq \mathcal{U} \subseteq \mathcal{K}_1$, $\mathcal{C} \subseteq \mathcal{Y} \subseteq \mathcal{K}_2$, $\mathcal{K}_7 \subseteq \mathcal{X} \subseteq \mathcal{K}_4 \subseteq \mathcal{K}_1$, $\mathcal{X} \subseteq \mathcal{K}_6 \subseteq \mathcal{K}_1$, $\mathcal{X} \subseteq \mathcal{K}_2$.

Proof. In 4.3 – 4.10, we have described all classes $\mathcal{K}_\pi$, $\pi \in E$. Now we can use the results of [3] to conclude that all $\mathcal{K}_\pi$ are varieties of quasigroups. For the rest, it is an easy exercise to verify that each $\mathcal{K}_S$ coincides with one of the twelve classes.

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