J. Blank; P. Exner

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Remarks on Tensor Products and their Applications in Quantum Theory — II. Spectral Properties

JIŘÍ BLANK, PAVEL EXNER

Nuclear center, Faculty of Mathematics and Physics, Charles University, Prague*)

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The study of tensor product formalism is continued by examining the spectral properties of tensor product operators. The paper is completed by a discussion of several typical quantum — theoretical applications.

О тензорных произведениях и их применениях в квантовой теории. II. Спектральные свойства. — Работа является продолжением изучения формализма тензорных произведений. Рассмотрены спектральные свойства операторов тензорного произведения и обсуждены некоторые типичные применения развитого формализма в квантовой теории.

Poznámky k tensorovým součinům a jejich použití v kvantové teorii. II. Spektrální vlastnosti. — Zkoumání formalismu tensorových součinů pokračuje v této části práce rozborem spektrálních vlastností tensorových součinů operátorů. Práce je završena diskusí několika typických kvantově-teoretických aplikací.

Introduction

This paper is a direct continuation of the first part (Acta Univ. Carolinae 17 (1976), 75—98) referred hereafter as I, where the basic notions are defined and discussed. The notation introduced in I is used and the numeration of sections, theorems, references etc. is continued.

In Section 5 a fundamental theorem concerning the tensor product of self-adjoint operators is proved and basic spectral properties are investigated; attention is paid mainly to self-adjoint or essentially self-adjoint operators and to one-parameter groups of unitary operators.

The quantum-theoretical applications in Section 6 concern the description of observables, states and time evolution of a joint quantum system in terms of its subsystems. Some other applications (second quantization, symmetries etc.) are also briefly discussed.

*) Myslíkova 7, 110 00 Praha 1

5. Spectral properties of tensor product operators

In this section we shall discuss how spectral charateristics of a tensor product operator are connected to those of its constituent operators. We are interested mainly in self-adjoint operators. Theorem 4 states that the tensor product of self--adjoint operators is symmetric. We shall now prove that it is, moreover, essentially self-adjoint (e.s.a.), i.e. its closure is self-adjoint. For this purpose we shall need several auxialiary statements.

First of all we make two remarks concerning notation.

1. Let M_r be a subset of $\mathscr{H}_r$ $(r = 1,2)$ and let $\mathscr{H}$, φ be a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$; then

$$(\mathsf{M}_1 \circ \mathsf{M}_2)_\varphi \equiv \varphi(\mathsf{M}_1 \times \mathsf{M}_2)_\lambda$$
$$(\mathsf{M}_1 \otimes \mathsf{M}_2)_\varphi \equiv \overline{\varphi(\mathsf{M}_1 \times \mathsf{M}_2)_\lambda}.$$

Thus $(\mathsf{M}_1 \circ \mathsf{M}_2)_\varphi$ is a linear manifold in $\mathscr{H}$ and $(\mathsf{M}_1 \otimes \mathsf{M}_2)_\varphi$ a subspace of $\mathscr{H}$. The subscript φ will be omitted unless an ambiguity can arise.

2. By $\langle \mathsf{M}, \mu \rangle$ a measure space will be denoted, i.e. the symbol $\langle \mathsf{M}, \mu \rangle$ involves a space (set) M together with a σ – algebra $\mathfrak{M}$ of subsets of M and a mapping $\mu : \mathfrak{M} \to [0, +\infty)$ with the following properties:

$$\mu(\varnothing) = 0 \; ; \; \mu\Big(\bigcup_{i=1}^{\infty} \mathsf{A}_i\Big) = \sum_{i=1}^{\infty} \mu(\mathsf{A}_i)$$

for any system of mutually disjoint sets $\mathsf{A}_i \in \mathfrak{M}$. For definitions of such notions as μ – measurable functions, integral on $\langle \mathsf{M}, \mu \rangle$ etc. see refs. [3], [4].

Lemma 5.1: Let $\langle \mathsf{M}, \mu \rangle$ be a measure space, $\mu(\mathsf{M}) < \infty$. Further let f be a real-valued, μ – measurable function on M, which is finite almost everywhere with respect to μ. Define operator A_f on L^2 $(\mathsf{M}, d\mu)$ with domain
$\mathsf{D}(A_f) = \{x \mid (fx)(t) \in L^2(\mathsf{M}, d\mu)\}$ by $(A_f x)(t) = f(t)x(t)$. Then
(a) A_f is self-adjoint;
(b) a real number λ is in spectrum of A_f if and only if

$$\mu\{t \in \mathsf{M} \mid |f(t) - \lambda| < \varepsilon\} > 0 \qquad (5.1)$$

for any $\varepsilon > 0$.

Proof: (a) For $n = 1, 2, \ldots$ consider the sequence of sets $\mathsf{M}_n = \{t \in \mathsf{M} \mid |f(t)| \leq n\}$ and denote by χ_n the characteristic function of M_n. Then
$x_n(t) = \chi_n(t)\, x(t) \in \mathsf{D}(A_f)$ for any $x \in L^2(\mathsf{M}, d\mu)$. By means of the dominated convergence (Lebesgue) theorem one easily finds

$$\int_{\mathsf{M}} |x_n(t) - x(t)|^2 \, d\mu \to 0$$

i.e. $\overline{\mathsf{D}(A_f)} = L^2(\mathsf{M}, d\mu)$. Clearly $(A_f x,y) = (x,A_f y)$ for any $x,y \in \mathsf{D}(A_f)$, so that A_f is symmetric. Its self-adjointness can be proved in the same way as in the case of Q (operator of multiplication by t on $L^2(\mathbf{R})$), the only difference consisting in replacing the characteristic functions of intervals $[-n,n]$ by χ_n (see e.g. [2]).

(b) Denote by $\sigma(A_f)$ the spectrum of A_f and remind the Weyl's criterion (refs. [2], [3]), according to which $\lambda \in \sigma(A_f)$ if and only if there is a sequence of unit vectors $x_n \in \mathsf{D}(A_f)$ such that $\|(A - \lambda I)x_n\| \to 0$. Let $\lambda \in \sigma(A_f)$, i.e.

$$\int_{\mathsf{M}} |f(t) - \lambda|^2 \, |x_n(t)|^2 \, d\mu \to 0$$

for some sequence of unit vectors $x_n \in \mathsf{D}(A_f)$ and denote for any $\varepsilon > 0 : \mathsf{N}(\varepsilon,\lambda) = \{t \in \mathsf{M} \mid |f(t) - \lambda| < \varepsilon\}$.
If there were $\varepsilon_0 > 0$ such that $\mu(\mathsf{N}(\varepsilon_0, \lambda)) = 0$, then

$$\int_{\mathsf{M}} |f(t) - \lambda|^2 \, |x_n(t)|^2 \, d\mu = \int_{\mathsf{M}-\mathsf{N}(\varepsilon_0,\lambda)} |f(t) - \lambda|^2 \, |x_n(t)|^2 \, d\mu \geq \varepsilon_0^2$$

which contradicts to the assumption $\lambda \in \sigma(A_f)$. Thus (5.1) holds for each $\lambda \in \sigma(A_f)$. On the other hand, if (5.1) holds then $\mu\left(\mathsf{N}\left(\frac{1}{n}, \lambda\right)\right) \neq 0$ for each natural n. The unit vectors

$$x_n = \left[\mu\left(\mathsf{N}\left(\frac{1}{n}, \lambda\right)\right)\right]^{-1/2} \chi_{\mathsf{N}\left(\frac{1}{n},\lambda\right)}$$

are clearly in $\mathsf{D}(A_f)$. Now

$$\|(A_f - \lambda\, \mathrm{I})x_n\|^2 = \int_{\mathsf{M}} |f(t) - \lambda|^2 \, |x_n(t)|^2 \, d\mu < \frac{1}{n^2} \to 0,$$

so that $\lambda \in \sigma(A_f)$. ■

Remark: If $\mathsf{M} = \mathbf{R}$, μ is a Lebesgue-Stieltjes measure on $\mathbf{R}$ and $f(t) = t$, one usually denotes A_f by Q_μ; especially for $\mu = m$, where m is the Lebesgue measure on $\mathbf{R}$, one has $Q_\mu = Q$.

Let us remind that a linear manifold N is called *core* for a closed operator T if $\mathsf{N} \subset \mathsf{D}(T)$ and $\overline{T \restriction \mathsf{N}} = T$. Especially N is a core for a self-adjoint operator A if and only if $A \restriction \mathsf{N}$ is e.s.a.

Lemma 5.2: Suppose the assumptions of Lemma 5.1 to be fulfilled. Moreover, let $f \in L^p(\mathsf{M}, d\mu)$ for some $p > 2$ and let N be a dense set in $L^q(\mathsf{M}, d\mu)$, where $1/p + 1/q = 1/2$. Then N is a core for A_f.

Proof: Take any $x \in L^q$ *) and set $p' = p/2$, $q' = q/2$, so that $1/p' + 1/q' = 1$. The Hölder inequality gives

$$\|x\|_2 \leq (\|1\|_{p'} \|x^2\|_{q'})^{1/2} = \|1\|_p \|x\|_q < \infty,$$

and similarly $\|A_f x\|_2 \leq \|f\|_p \|x\|_q < \infty$, i.e. $L^q \subset \mathsf{D}(A_f)$.

*) We shall write briefly L^q instead of L^q $(\mathsf{M}, d\mu)$ and denote by $\|\,.\,\|_q$ the norm in L^q; especially $\|\,1\,\|_q = [\mu(\mathsf{M})]^{1/q} < \infty$ since μ is finite.

The A_f is self-adjoint and therefore closed; hence

(*) $$\overline{A_f \upharpoonright L^q} \in A_f$$

and also

(**) $$\overline{A_f \upharpoonright \mathsf{N}} \subset \overline{A_f \upharpoonright L^q}.$$

In order to prove that equality holds in (*) and (**) take any $x \in \mathsf{D}(A_f)$ and denote

$$x_n(t) = \begin{Bmatrix} x(t) & \text{if} & |x(t)| \leq n \\ 0 & \text{if} & |x(t)| > n \end{Bmatrix}.$$

Now $\|x_n\|_q \leq n\|1\|_q$, i.e. $x_n \in L^q$ and further, using the dominated convergence theorem, we find $\|x_n - x\|_2 \to 0$, $\|A_f x_n - A_f x\|_2 \to 0$. Thus $x \in \mathsf{D}(\overline{A_f \upharpoonright L^q})$ and $\overline{A_f \upharpoonright L^q}x = A_f x$, i.e. $A_f = \overline{A_f \upharpoonright L^q}$.

Further for any $y \in L^q$ there is a sequence $\{y_n\} \in \mathsf{N}$, such that $\|y_n - y\|_q \to 0$. Since $\|y_n - y\|_2 \leq \|y_n - y\|_q\|1\|_p$ and $\|A_f y_n - A_f\|_2 \leq \|f\|_p\|y_n - y\|_q$, one has $y \in \mathsf{D}(\overline{A_f \upharpoonright \mathsf{N}})$ and $A_f y = \overline{A_f \upharpoonright \mathsf{N}}y$. ■

Lemma 5.3 (spectral theorem): Let A be a self-adjoint operator on a separable $\mathscr{H}$ with domain $\mathsf{D}(A)$. Then there is a measure space $\langle \mathsf{M}, \mu \rangle$ with $\mu(\mathsf{M}) < \infty$, a unitary operator U: $\mathscr{H} \to L^2(\mathsf{M}, d\mu)$ and a function f on M obeying the conditions of Lemma 5.1, so that A is unitarily equivalent to A_f, i.e. $\mathsf{D}(A_f) = U\mathsf{D}(A)$ and $Ax = U^{-1}A_f Ux$ for each $x \in \mathsf{D}(A)$.

For a proof see [3], [5], [12].

Remark 1: A mapping which assigns to each Borel set M on $\mathbf{R}$ a projection $E(\mathsf{M})$ on $\mathscr{H}$ so that

(a) $E(\varnothing) = 0$, $E(\mathbf{R}) = I$

(b) $E(\mathsf{M}_1 \cap \mathsf{M}_2) = E(\mathsf{M}_1)E(\mathsf{M}_2)$

(c) $E(\bigcup_n \mathsf{M}_n) = \sum_n E(\mathsf{M}_n)$ for each at most countable system $\{\mathsf{M}_n\}$ of mutually disjoint Borel sets

is called *spectral* or *projection-valued measure* on $\mathscr{H}$. An equivalent formulation of the spectral theorem states that there is a one-to-one correspondence between self-adjoint operators and spectral measures on $\mathscr{H}$. By means of the spectral measure corresponding to a given self-adjoint A one can define operators $\varphi(A)$ for each Borel function φ; especially one has $\varphi(A) = A$ for $\varphi(t) = t$ and $\chi_{\mathsf{M}}(A) = E(\mathsf{M})$ for each projection belonging to the spectral measure corresponding to A. These topics are discussed in more detail in refs. [3], [5], [12]. We shall hereafter use the "functional expression" $\chi_{\mathsf{M}}(A)$ of the spectral measure corresponding to A.

Remark 2: The measure space $\langle \mathsf{M}, \mu \rangle$ which occurs in the first formulation of the spectral theorem is, in general, an abstract space with an abstract measure. However, it appears that for an important class of self-adjoint operators, for the so-called multiplicity-free operators, M is simply $\mathbf{R}$, μ is a Lebesgue-Stieltjes measure on $\mathbf{R}$

and moreover $f(t) = t$. The corresponding definition reads: a self-adjoint operator A on a separable $\mathscr{H}$ is *multiplicity free,* if there is a vector $y^{(A)}$ (generating vector for A) such that the linear envelope of $\{\chi_{\mathsf{I}}(A)y^{(A)} | \mathsf{I}$ – intervals on $\mathbf{R}\}$ is dense in $\mathscr{H}$. If $\dim \mathscr{H} < \infty$ then the class of multiplicity-free operators is identical with that of Hermitian operators having a simple spectrum (no repeated eigenvalues). The spectral theorem for multiplicity-free operators can be formulated as follows: Each multiplicity-free operator A is unitarily equivalent to Q_μ on $L^2(\mathbf{R}, d\mu)$, the measure μ being given by

$$\mu(\mathsf{M}) = (\chi_{\mathsf{M}}(A)y^{(A)}, y^{(A)}),$$

where M is any Borel set on $\mathbf{R}$.

The inverse of this statement is also true:
If a self-adjoint operator A on a separable $\mathscr{H}$ is unitarily equivalent to Q_μ, i.e. $A = U^{-1}Q_\mu U$, then A is multiplicity free; if $y^{(Q)}$ is a generating vector for Q_μ then $U^{-1}y^{(Q)}$ is a generating vector for A.

For details see [3], [5], [12].

The following statement makes use of the fact that the measure space in the spectral theorem is not uniquely determined by A, and shows that $\langle \mathsf{M}, \mu \rangle$ can always be chosen so that Lemma 5.2 is applicable.

Lemma 5.4: Let A be a self-adjoint operator on a separable $\mathscr{H}$. Then for each $p \geq 1$ a measure space $\langle \mathsf{M}, \mu \rangle$ can be found such that A is unitarily equivalent to A_f where $f \in L^p(\mathsf{M}, d\mu)$.

Proof: We shall consider only multiplicity-free operators A. According to the spectral theorem there is a measure μ_0 on $\mathbf{R}$ such that A is unitarily equivalent to $Q_{\mu_0} : A = U_0^{-1}\, Q_{\mu_0} U_0$ where U_0 is a unitary operator from $\mathscr{H}$ onto $L^2(\mathbf{R}, d\mu_0)$. By means of μ_0 and of the positive continuous function $\varphi(t) = e^{-t^2}$ one obtains the following function of intervals on $\mathbf{R}$:

$$\mu(\mathsf{I}) = \int_{\mathsf{I}} e^{-t^2}\, d\mu_0.$$

The Lebesgue extension of this function is a measure on $\mathbf{R}$ and it holds then for any Borel function g on $\mathbf{R}$

$$(*) \qquad \int_{\mathbf{R}} g\, d\mu = \int_{\mathbf{R}} g\varphi\, d\mu_0$$

(see e.g. [14], [12]).
Consider linear mapping $V\colon L^2(\mathbf{R}, d\mu_0) \to L^2(\mathbf{R}, d\mu)$ given by
$(Vx)(t) = \exp(t^2/2)x(t)$. From $(*)$ we find that $x \in L^2(\mathbf{R}, d\mu_0)$ implies
$|x|^2 \exp(t^2) \in L(\mathbf{R}, d\mu)$, i.e. $x(t) \exp(t^2/2) \in L^2(\mathbf{R}, d\mu)$ and thus V is defined for all $x \in L^2(\mathbf{R}, d\mu_0)$. One easily verifies that V preserves norm and is surjective; this means that V is a unitary operator. It is further not difficult to check that $\mathsf{D}(Q_\mu) = V\mathsf{D}(Q_{\mu_0})$ and $Q_\mu = VQ_{\mu_0}V^{-1}$.

Then $A = (VU_0)^{-1}Q_\mu(VU_0)$ and VU_0 is a unitary operator from $\mathscr{H}$ onto $L^2(\mathbf{R}, d\mu)$. Finally, since $t^p\exp(-t^2)$ is in $L(\mathbf{R}, d\mu_0)$ for any $p \geq 1$, one gets from $(*)$

$$t^p \in L(\mathbf{R}, d\mu), \text{ i.e. } t \in L^p(\mathbf{R}, d\mu).$$

Thus A is unitarily equivalent to $A_f = Q_\mu$ and $f(t) = t \in L^p(\mathbf{R}, d\mu)$. ■

The next auxiliary statement concerns one special dense set in $L^p(\mathbf{R}^n, d\mu)$. It could be formulated for $L^p(\mathsf{M}, d\mu)$, where $\langle \mathsf{M}, \mu \rangle$ is a general measure space, as well. However such a formulation would require some prerequisites from abstract measure theory which cannot be presented here (see e.g. [4]).

Consider the system of all intervals $\mathsf{I}_n \in \mathbf{R}^n$. Each linear combination of $\chi_{\mathsf{I}_n^{(j)}}$ for mutually disjoint $\mathsf{I}_n^{(j)}$ is called *step function* on $\mathbf{R}^n$. Thus each step function s can be written as

$$s = \sum_{j=1}^{k} \alpha_j \chi_{\mathsf{I}_n^{(j)}}$$

where $\alpha_j \in \mathbf{C}$, $\alpha_j \neq 0$ and $\mathsf{I}_n^{(j)} \cap \mathsf{I}_n^{(j')} = \varnothing$ if $j \neq j'$. Clearly $s \in L^p(\mathbf{R}^n, d\mu)$ for any $p \geq 1$ and any measure μ satisfying $\mu(\mathbf{R}^n) < \infty$. Making use of simple properties of intervals in $\mathbf{R}^n$ *) one easily verifies that the set S_n of step functions on $\mathbf{R}^n$ is a linear manifold in $L^p(\mathbf{R}^n, d\mu)$. Moreover it holds:

Lemma 5.5: Let μ be a measure on $\mathbf{R}^n$ such that $\mu(\mathbf{R}^n) < \infty$ and let $p \geq 1$. Then the set S_n of step functions on $\mathbf{R}^n$ is a dense linear manifold in $L^p(\mathbf{R}^n, d\mu)$.

For a proof see ref. [12].

The last of the auxiliary statements we shall need is closely related to Theorem 4.

Lemma 5.6: Let T_r $(r = 1, 2)$ be a closed, densely defined operator on $\mathscr{H}_r$. Then it holds for $\boldsymbol{T}_\Sigma = \boldsymbol{T}_1 + \boldsymbol{T}_2$ with domain $\mathsf{D}_\Sigma = \mathsf{D}(T_1) \circ \mathsf{D}(T_2)$
(a) $\boldsymbol{T}_\Sigma$ is densely defined and closable
(b) if the T_r's are symmetric so is $\boldsymbol{T}_\Sigma$.

Proof: (a) Clearly $\overline{\mathsf{D}_\Sigma} = \mathscr{H}$ (see Lemma 2.2) so that $\boldsymbol{T}_\Sigma^+$ exists. If we prove that $\boldsymbol{T}_\Sigma^+$ is densely defined then $\boldsymbol{T}_\Sigma$ is closable. Theorem 4 gives
$\boldsymbol{T}_1^+ \supset T_1^+ \otimes I_2$, i.e.

$$\mathsf{D}(\boldsymbol{T}_1^+) \supset \mathsf{D}(T_1^+ \otimes I_2) = \mathsf{D}(T_1^+) \circ \mathscr{H}_2 \supset \mathsf{D}(T_1^+) \circ \mathsf{D}(T_2^+)$$

and similarly $\mathsf{D}(\boldsymbol{T}_2^+) \supset \mathsf{D}(T_1^+) \circ \mathsf{D}(T_2^+)$. Since the T_r's are closed, $\overline{\mathsf{D}(T_r^+)} = \mathscr{H}_r$ and thus $\overline{\mathsf{D}(T_1^+) \circ \mathsf{D}(T_2^+)} = \mathscr{H}$. Now

$$\mathsf{D}(\boldsymbol{T}_\Sigma^+) \supset \mathsf{D}(\boldsymbol{T}_1^+ + \boldsymbol{T}_2^+) = \mathsf{D}(\boldsymbol{T}_1^+) \cap \mathsf{D}(\boldsymbol{T}_2^+) \supset \mathsf{D}(T_1^+) \circ \mathsf{D}(T_2^+)$$

so that $\boldsymbol{T}_\Sigma$ is closable.

*) In fact one needs these two statements (see [14]):
1. If $\mathsf{I}_n^{(1)}$, $\mathsf{I}_n^{(2)}$ are intervals in $\mathbf{R}^n$ so is their intersection;
2. If I_n and $\mathsf{J}_n^{(1)}, \mathsf{J}_n^{(2)}, \dots \mathsf{J}_n^{(k)}$ are intervals then the difference of I_n and of the union of $\mathsf{J}_n^{(j)}$ can be expressed as a finite disjoint union of intervals.

(b) If the T_r's are symmetric so are the $\boldsymbol{T}_r$'s and hence $\boldsymbol{T}_r^+ \supset \boldsymbol{T}_r$. Then $\boldsymbol{T}_\Sigma^+ \supset \boldsymbol{T}_1^+ + \boldsymbol{T}_2^+ \supset \boldsymbol{T}_1 + \boldsymbol{T}_2$, i.e. $\boldsymbol{T}_\Sigma$ is symmetric. ■

After these preliminaries we can pass to the following theorem which is of basic importance for studying spectral properties of tensor product of self-adjoint operators.

Theorem 7: Let A_r $(r = 1, 2)$ be a self-adjoint operator with domain D_r on a separable $\mathscr{H}_r$, and let $\mathscr{H}$, φ be a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then operators $A_1 \otimes A_2$ and $\boldsymbol{A}_1 + \boldsymbol{A}_2$, which are both defined on $\mathsf{D}_1 \circ \mathsf{D}_2$, are essentially self-adjoint.

Proof: We shall again restrict ourselves to the case when the A_r's are multiplicity free (a general proof is sketched in [3]). According to Lemma 5.3 there is a unitary operator V_r which maps $\mathscr{H}_r$ onto $L^2(\mathbf{R}, d\mu_r)$ in such a way that

$$\mathsf{D}(Q_{\mu_r}) = V_r\mathsf{D}_r, \quad A_r = V_r^{-1}Q_{\mu_r}V_r. \tag{5.2}$$

Let ψ be the multiplicative mapping introduced in Example 2.1; then

$$(L^2(\mathbf{R}, d\mu_1) \otimes L^2(\mathbf{R}, d\mu_2))_\psi = L^2(\mathbf{R}^2, d\mu_{12}),$$

where μ_{12} is the product measure on $\mathbf{R}^2$ that is obtained by the Lebesgue extension from the additive function of intervals I_2 in $\mathbf{R}^2$ defined by

$$\mu_{12}(\mathsf{I}_2) = \mu_{12}(\mathsf{I}_1 \times \mathsf{I}_1') = \mu_1(\mathsf{I}_1)\,\mu_2(\mathsf{I}_1').$$

Using unitarity of the V_r's one easily checks that the mapping $\boldsymbol{V}_{12}^{(0)}$ from $(\mathscr{H}_1 \circ \mathscr{H}_2)_\varphi$ to $(L^2(\mathbf{R}, d\mu_1) \circ L^2(\mathbf{R}, d\mu_2))_\psi$:

$$\boldsymbol{V}_{12}^{(0)}\, x = \sum_{i,j} c_{ij}\, \psi(V_1x_1^{(i)}, V_2x_2^{(j)}) \text{ if } x = \sum_{i,j} c_{ij}\, \varphi(x_1^{(i)}, x_2^{(j)})$$

is surjective, linear and preserves norm. Since

$$\overline{(\mathscr{H}_1 \circ \mathscr{H}_2)_\varphi} = \mathscr{H}, \; \overline{(L^2(\mathbf{R}, d\mu_1) \circ L^2(\mathbf{R}, d\mu_2))_\psi} = L^2(\mathbf{R}^2, d\mu_{12})$$

there is a unique extension of $\boldsymbol{V}_{12}^{(0)}$ to a unitary operator $\boldsymbol{V}_{12}$ from $\mathscr{H}$ onto $L^2(\mathbf{R}^2, d\mu_{12})$. For any $y \in \boldsymbol{V}_{12}(\mathsf{D}_1 \circ \mathsf{D}_2)_\varphi$,

$$y = \boldsymbol{V}_{12} \sum_{i,j} c_{ij}\, \varphi(x_1^{(i)}, x_2^{(j)}), \;\; x_1^{(i)} \in \mathsf{D}_1,\; x_2^{(j)} \in \mathsf{D}_2,$$

one gets with the help of (5.2)

$$\boldsymbol{V}_{12}(A_1 \otimes A_2)\, \boldsymbol{V}_{12}^{-1} y = \sum_{i,j} c_{ij}\, \psi(V_1A_1\, x_1^{(i)}, V_2A_2x_2^{(j)}) = \tag{5.3}$$

$$= \sum_{i,j} c_{ij}\, \psi(Q_{\mu_1}\, V_1x_1^{(i)}, Q_{\mu_2}\, V_2x_2^{(j)}) = \boldsymbol{A}_m y,$$

and similarly

$$\boldsymbol{V}_{12}(\boldsymbol{A}_1 + \boldsymbol{A}_2)\boldsymbol{V}_{12}^{-1} y = \boldsymbol{A}_s y.$$

Here $\boldsymbol{A}_m$ and $\boldsymbol{A}_s$ are the self-adjoint operators on $L^2(\mathbf{R}^2, d\mu_{12})$ that are obtained, according to Lemma 5.1, for

$$f_m(t_1, t)\,_2 = t_1t_2 \tag{5.4}$$

and

$$f_s(t_1, t_2) = t_1 + t_2 \tag{5.4'}$$

respectively. Now V_r and μ_r can be chosen in such a way that

$$f_r(t_r) = t_r \in L^4(\mathbf{R}, d\mu_r) \tag{5.5}$$

which implies

$$f_m(t_1, t_2) \in L^4(\mathbf{R}^2, d\mu_{12})\,, \quad f_s(t_1, t_2) \in L^4(\mathbf{R}^2, d\mu_{12}). \tag{5.5'}$$

From now on we shall consider only $A_1 \otimes A_2$; in view of Lemma 5.6 each step of what follows can immediately be applied for $\mathbf{A}_1 + \mathbf{A}_2$ as well.

Consider a restriction $\mathbf{A}_{12} = (A_1 \otimes A_2) \restriction \mathsf{D}_{12}$, where $\mathsf{D}_{12} \subset (\mathsf{D}_1 \circ \mathsf{D}_2)_\varphi$. In view of (5.3) it holds

$$\mathbf{V}_{12}\mathbf{A}_{12}\mathbf{V}_{12}^{-1} = \mathbf{A}_m \restriction \mathbf{V}_{12}\mathsf{D}_{12}.$$

Suppose that there be a restriction $\mathbf{A}_{12}$ with the following additional properties:

(i) $\mathbf{V}_{12}\mathsf{D}_{12} \subset L^4(\mathbf{R}^2, d\mu_{12})$

(ii) $\mathbf{V}_{12}\mathsf{D}_{12}$ is a core for $\mathbf{A}_m$, i.e. $\overline{\mathbf{A}_m \restriction \mathbf{V}_{12}\mathsf{D}_{12}} = \overline{\mathbf{V}_{12}\mathbf{A}_{12}\mathbf{V}_{12}^{-1}} = \mathbf{A}_m$.

Bearing in mind that $\mathbf{V}_{12}$ is unitary, $\mathbf{A}_{12}$ closable and $\mathbf{A}_m$ self-adjoint, we conclude that $\overline{\mathbf{A}_{12}} = \mathbf{V}_{12}^{-1}\mathbf{A}_m\mathbf{V}_{12}$ is self-adjoint*). Futher $A_1 \otimes A_2$ is symmetric so that $\overline{A_1 \otimes A_2}$ exists and is symmetric. Now $\overline{\mathbf{A}_{12}} \subset \overline{A_1 \otimes A_2}$, and as $\overline{\mathbf{A}_{12}}$ is self-adjoint, it must hold $\overline{\mathbf{A}_{12}} = \overline{A_1 \otimes A_2}$. Thus $\overline{A_1 \otimes A_2}$ is self-adjoint, i.e. $A_1 \otimes A_2$ is e.s.a.

Hence the proof will be finished if we find a restriction which has the above properties. To this purpose we use Lemma 5.5: the set S_1 of step functions on $\mathbf{R}$ is dense in $L^4(\mathbf{R}, d\mu_r)$ $(r = 1,2)$ and S_2 is dense in $L^4(\mathbf{R}^2, d\mu_{12})$. It follows from (5.5) that $L^4(\mathbf{R}, d\mu_r) \subset \mathsf{D}(Q\mu_r)$ (see proof of Lemma 5.2 for $p = q = 4$) and thus $\mathsf{S}_1 \subset \mathsf{D}\,(Q\mu_r)$. Then (5.2) yields $\mathsf{D}_r^{(0)} = V_r^{-1}\mathsf{S}_1 \subset \mathsf{D}_r$. Denote $\mathsf{D}_{12}^{(0)} = (\mathsf{D}_1^{(0)} \circ \mathsf{D}_2^{(0)})_\varphi$ and consider the restriction $\mathbf{A}_{12}^{(0)} = (A_1 \otimes A_2) \restriction \mathsf{D}_{12}^{(0)}$. Clearly

$$\mathbf{V}_{12}\mathsf{D}_{12}^{(0)} = (\mathsf{S}_1 \circ \mathsf{S}_1)_\psi \subset (L^4(\mathbf{R}, d\mu_1) \circ L^4(\mathbf{R}, d\mu_2))_\psi \subset L^4(\mathbf{R}^2, d\mu_{12}),$$

so that (i) is satisfied. Further one has for any interval $\mathsf{I}_2 \subset \mathbf{R}^2$, $\mathsf{I}_2 = \mathsf{I}_1 \times \mathsf{I}_1'$:

$$\chi_{\mathsf{I}_2}(t_1, t_2) = \chi_{\mathsf{I}_1}(t_1)\chi_{\mathsf{I}_1'}(t_2).$$

Consequently, each step function on $\mathbf{R}^2$ is in $(\mathsf{S}_1 \circ \mathsf{S}_1)_\psi$, i.e.

$$\mathsf{S}_2 \subset \mathbf{V}_{12}\mathsf{D}_{12}^{(0)} \subset L^4(\mathbf{R}^2, d\mu_{12}).$$

Then

$$\overline{\mathsf{S}_2} = \overline{\mathbf{V}_{12}\mathsf{D}_{12}^{(0)}} = L^4(\mathbf{R}^2, d\mu_{12});$$

taking into account (5.5′) and applying Lemma 5.2 for $p = q = 4$, we conclude that $\mathbf{V}_{12}\mathsf{D}_{12}^{(0)}$ is a core for $\mathbf{A}_m$. ■

*) Let us remind that for each unitary operator V the following holds:
(a) if A is self-adjoint so is VAV^{-1};
(b) if T is closable so is VTV^{-1} and $\overline{VTV^{-1}} = V\overline{T}V^{-1}$.

Corollary: Let $A_r^{(e)}$ $(r = 1,2)$ be an e.s.a. operator on an a separable $\mathscr{H}_r$ with domain $\mathsf{D}_r^{(e)}$. Then $A_1^{(e)} \otimes A_2^{(e)}$ and $\mathbf{A}_1^{(e)} + \mathbf{A}_2^{(e)}$, which are both defined on $\mathsf{D}_1^{(e)} \circ \mathsf{D}_2^{(e)}$, are e.s.a.

Proof: Let us denote

$$(*) \qquad A_r = \overline{A_r^{(e)}}\ , \quad \mathsf{D}_r = \mathsf{D}(A_r).$$

By Theorem 7 operator $A_1 \otimes A_2$ with domain $\mathsf{D}_1 \circ \mathsf{D}_2$ is e.s.a. Clearly $\overline{A_1^{(e)} \otimes A_2^{(e)}} \subset \overline{A_1 \otimes A_2}$. It is thus sufficient to prove that $A_1 \otimes A_2 \subset \overline{A_1^{(e)} \otimes A_2^{(e)}}$. Because of $(*)$ there is for each $x_r \in \mathsf{D}_r$ a sequence $\{x_r^{(n)}\} \subset \mathsf{D}_r^{(e)}$ such that $x_r^{(n)} \to x_r$ and $A_r^{(e)} x_r^{(n)} \to A_r x_r$. Now $\varphi(x_1, x_2) \in \mathsf{D}_1 \circ \mathsf{D}_2$, $\{\varphi(x_1^{(n)}, x_2^{(n)})\}$ is a sequence in $\mathsf{D}_1^{(e)} \circ \mathsf{D}_2^{(e)}$, and using the same reasoning as in the proof of Lemma 2.2, we get

$$\|\varphi(x_1, x_2) - \varphi(x_1^{(n)}, x_2^{(n)})\| \to 0$$
$$\|(A_1 \otimes A_2)\varphi(x_1, x_2) - (A_1^{(e)} \otimes A_2^{(e)})\, \varphi\, (x_1^{(n)}, x_2^{(n)})\| =$$
$$= \|\varphi(A_1 x_1, A_2 x_2) - \varphi(A_1^{(e)} x_1^{(n)}, A_1^{(e)} x_2^{(n)})\| \to 0.$$

Hence

$$\varphi(x_1, x_2) \in \mathsf{D}(\overline{A_1^{(e)} \otimes A_2^{(e)}})$$
$$(A_1 \otimes A_2)\, \varphi\, (x_1, x_2) = (\overline{A_1^{(e)} \otimes A_2^{(e)}})\, \varphi\, (x_1, x_2),$$

and, in view of linearity of $A_1 \otimes A_2$ and $\overline{A_1^{(e)} \otimes A_2^{(e)}}$, we have $A_1 \otimes A_2 \subset \overline{A_1^{(e)} \otimes A_2^{(e)}}$. The same procedure can be applied for proving that $\mathbf{A}_1^{(e)} + \mathbf{A}_2^{(e)}$ is e.s.a. ■

Remark 1: Theorem 7 can be generalized in an obvious way for general real operator polynomials formed from

$$\mathbf{A}_r = I_1 \otimes I_2 \otimes \ldots I_{r-1} \otimes A_r \otimes I_{r-1} \otimes \ldots \otimes I_n$$

$(r = 1, 2, .., n)$ (see [3]).

Remark 2: Notice that $L^2(\mathbf{R}^2, d\mu_{12})$ is a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$, the corresponding mapping being given by

$$\zeta(x_1, x_2) = \psi(V_1 x_1, V_2 x_2).$$

In this sense the choice of a suitable realization is essential in the above proof.

Let us now examine how the spectral properties of a tensor-product operator are related to those of its component operators. We shall see that, though the relations between spectra are not so simple as is usually supposed in textbooks on quantum theory, they nevertheless confirm the intuitive understanding of tensor-product operators. In addition to operators $T_1 \otimes T_2$, which will be shortly denoted by $\mathbf{T}_\pi$, we shall consider operators $\mathbf{T}_\Sigma = \mathbf{T}_1 + \mathbf{T}_2$ as well.

Let λ_r be an eigenvalue of a densely defined and closable operator T_r on $\mathscr{H}_r$ $(r = 1, 2)$; we denote by $\mathsf{N}_r(\lambda_r)$ the linear manifold spanned by all the eigenvectors of T_r which belong to λ_r ($\mathsf{N}_{\bar{r}}(\lambda_r)$ has the same meaning for $\overline{T_r}$), and by $\mathscr{D}(T_r)$ the set of all eigenvalues of T_r. Clearly $\mathscr{D}(T_r) \subset \mathscr{D}(\overline{T_r})$ and $\mathsf{N}_r(\lambda_r) \subset \mathsf{N}_{\bar{r}}(\lambda_r)$ for each $\lambda_r \in \mathscr{D}(T_r)$. Moreover $\mathsf{N}_{\bar{r}}(\lambda_r)$ is closed, i.e. a subspace in $\mathscr{H}_r$*) and thus $\overline{\mathsf{N}_r(\lambda_r)} \subset \mathsf{N}_{\bar{r}}(\lambda_r)$. For tensor-product operators $\boldsymbol{T}_\Pi$ and $\boldsymbol{T}_\Sigma$ we introduce analogously $\mathsf{N}_\Pi(\lambda)$, $\mathsf{N}_{\bar{\Pi}}(\lambda)$, $\mathsf{N}_\Sigma(\lambda)$, $\mathsf{N}_{\bar{\Sigma}}(\lambda)$ and again

$$\overline{\mathsf{N}_\Pi(\lambda)} \subset \mathsf{N}_{\bar{\Pi}}(\lambda),\ \overline{\mathsf{N}_\Sigma(\lambda)} \subset \mathsf{N}_{\bar{\Sigma}}(\lambda). \tag{5.6}$$

It is further obvious that $\lambda_r \in \mathscr{D}(T_r)$ implies $\lambda_1\lambda_2 \in \mathscr{D}(\boldsymbol{T}_\Pi)$ and $\lambda_1 + \lambda_2 \in \mathscr{D}(\boldsymbol{T}_\Sigma)$; for the corresponding sets $\mathsf{N}_\Pi(\lambda_1\lambda_2)$ and $\mathsf{N}_\Sigma(\lambda_1 + \lambda_2)$ we get

$$\left.\begin{matrix}\mathsf{N}_\Pi(\lambda_1\lambda_2)\\ \mathsf{N}_\Sigma(\lambda_1+\lambda_2)\end{matrix}\right\} \supset \mathsf{N}_1(\lambda_1) \circ \mathsf{N}_2(\lambda_2) \tag{5.7}$$

and, according to (5.6)

$$\left.\begin{matrix}\mathsf{N}_{\bar{\Pi}}(\lambda_1\lambda_2)\\ \mathsf{N}_{\bar{\Sigma}}(\lambda_1+\lambda_2)\end{matrix}\right\} \subset \mathsf{N}_1(\lambda_1) \otimes \mathsf{N}_2(\lambda_2). \tag{5.8}$$

If one considers Hermitian operators A_r ($r = 1, 2$) on finite-dimensional spaces $\mathscr{H}_r$, one easily verifies, with the help of orthonormal bases formed from the eigenvectors of the A_r's, that each eigenvalue λ of operators $\mathbf{A}_\Pi$ ($\mathbf{A}_\Sigma$) can be expressed as $\lambda = \lambda_1\lambda_2$ ($\lambda = \lambda_1 + \lambda_2$) where $\lambda_r \in \mathscr{D}(A_r)$. If we make use of functions f_m and f_s (see (5.4), (5.4')) we can write

$$\mathscr{D}(\mathbf{A}_\Pi) = f_m(\mathscr{D}(A_1) \times \mathscr{D}(A_2))$$
$$\mathscr{D}(\mathbf{A}_\Sigma) = f_s(\mathscr{D}(A_1) \times \mathscr{D}(A_2)).$$

We shall now examine how these statements must be modified when considering arbitrary self-adjoint operators A_r on infinite-dimensional separable $\mathscr{H}_r$ and the tensor-product operators $\mathbf{A}_\Pi$ and $\mathbf{A}_\Sigma$ formed from them. We shall start with self-adjoint operators A_r having *pure point spectra***) because of their importance in quantum theory and of the fact that their properties are very similar to those of Hermitian operators on a finite-dimensional $\mathscr{H}$.

Theorem 8: Let A_r ($r = 1, 2$) be a self-adjoint operator with a pure point spectrum on a separable $\mathscr{H}_r$: Then the self-adjoint operators $\overline{\mathbf{A}_\Pi} = \overline{A_1 \otimes A_2}$ and $\overline{\mathbf{A}_\Sigma} = \overline{\mathbf{A}_1 + \mathbf{A}_2}$ also have pure point spectra and it holds:

$$\mathscr{D}(\overline{\mathbf{A}_\Pi}) = \mathscr{D}(\mathbf{A}_\Pi) = f_m(\mathscr{D}(A_1) \times \mathscr{D}(A_2)), \tag{5.9a}$$

$$\mathscr{D}(\overline{\mathbf{A}_\Sigma}) = \mathscr{D}(\mathbf{A}_\Sigma) = f_s(\mathscr{D}(A_1) \times \mathscr{D}(A_2)). \tag{5.9b}$$

For each $\lambda \in \mathscr{D}(\mathbf{A}_\Pi)$

$$\mathsf{N}_{\bar{\Pi}}(\lambda) = \sum_{[\lambda_1, \lambda_2] \in \mathsf{P}(\lambda)} \dotplus\ \mathsf{N}_1(\lambda_1) \otimes \mathsf{N}_2(\lambda_2), \tag{5.10a}$$

where $\mathsf{P}(\lambda) = \{[\lambda_1, \lambda_2] \in \mathscr{D}(A_1) \times \mathscr{D}(A_2) \mid \lambda = \lambda_1\lambda_2\}$.

*) This subspace is often called eigenspace belonging to eigenvalue λ_r.

**) Let us remind that a self-adjoint operator A on a separable $\mathscr{H}$ is said to have pure point spectrum if the eigenvectors of A form an orthonormal basis in $\mathscr{H}$; it holds then for the spectrum of A: $\sigma(A) = \overline{\mathscr{D}(A)}$ (see [2], [15]).

For each $\lambda \in \mathscr{D}(\mathbf{A}_\Sigma)$

(5.10b) $$\mathsf{N}_{\bar\Sigma}(\lambda) = \sum_{[\lambda_1, \lambda_2] \in \mathsf{S}(\lambda)} \dotplus \mathsf{N}_1(\lambda_1) \otimes \mathsf{N}_2(\lambda_2),$$

where $\mathsf{S}(\lambda) = \{[\lambda_1, \lambda_2] \in \mathscr{D}(A_1) \times \mathscr{D}(A_2) \mid \lambda = \lambda_1 + \lambda_2\}$. Further

(5.11a) $$\sigma(\overline{\mathbf{A}_\Pi}) = \overline{f_m(\sigma(A_1) \times \sigma(A_2))} \quad {}^\star),$$

(5.11a) $$\sigma(\overline{\mathbf{A}_\Sigma}) = \overline{f_s(\sigma(A_1) \times \sigma(A_2))} \quad {}^\star).$$

Proof: Denote by $e_r^{(i)}$ $(i = 1, 2, \dots, \dim \mathscr{H}_r)$ the eigenvectors of A_r, so that $\mathscr{E}_r = \{e_r^{(i)}\}_{i=1}^{\dim \mathscr{H}_r}$ is an orthonormal basis in $\mathscr{H}_r$. Then each vector $e_{ij} = \varphi(e_1^{(i)}, e_2^{(j)})$ in the orthonormal basis $\varphi(\mathscr{E}_1 \times \mathscr{E}_2)$ in $(\mathscr{H}_1 \otimes \mathscr{H}_2)_{\mathscr{H},\varphi}$ is an eigenvector of both operators $\overline{\mathbf{A}_\Pi}$, $\overline{\mathbf{A}_\Sigma}$ and belongs to eigenvalues $\lambda_1^{(i)}\lambda_2^{(j)}$ and $\lambda_1^{(i)} + \lambda_2^{(j)}$, respectively. Thus the $\overline{\mathbf{A}_\Pi}$ and $\overline{\mathbf{A}_\Sigma}$ have pure point spectra. Take any $x \in \mathsf{D}(\overline{\mathbf{A}_\Pi})$ and denote $\xi_{ij} = (x, e_{ij})$. Clearly

$$(\overline{\mathbf{A}_\Pi} x, e_{ij}) = (x, \overline{\mathbf{A}_\Pi} e_{ij}) = \lambda_1^{(i)}\lambda_2^{(j)}\xi_{ij}$$

so that

(*) $$\overline{\mathbf{A}_\Pi} x = \sum_{i,j} \lambda_1^{(i)}\lambda_2^{(j)}\xi_{ij}e_{ij}.$$

Let $\lambda \in \mathscr{D}(\overline{\mathbf{A}_\Pi})$, i.e. $\overline{\mathbf{A}_\Pi} x = \lambda x$ for some $x \neq 0$. Then (*) implies $\xi_{ij} = 0$ if $\lambda_1^{(i)}\lambda_2^{(j)} \neq \lambda$; thus $\lambda \in f_m(\mathscr{D}(A_1) \times \mathscr{D}(A_2))$ and

(**) $$x = \sum_{\{i,j \mid [\lambda_1^{(i)}, \lambda_i^{(2)}] \in \mathsf{P}(\lambda)\}} \xi_{ij}e_{ij}.$$

Hence $\mathscr{D}(\overline{\mathbf{A}_\Pi}) \subset f_m(\mathscr{D}(A_1) \times \mathscr{D}(A_2))$ and, since the opposite inclusion

$$f_m(\mathscr{D}(A_1) \times \mathscr{D}(A_2)) \subset \mathscr{D}(\mathbf{A}_\Pi) \subset \mathscr{D}(\overline{\mathbf{A}_\Pi}),$$

is obvious, we get (5.9a). Further (**) implies $\mathsf{N}_{\bar\Pi}(\lambda) \subset \mathsf{N}(\mathsf{P}(\lambda))$**). On the other hand, (5.8) yields $\mathsf{N}_{\bar\Pi}(\lambda) \supset \mathsf{N}_1(\lambda_1) \otimes \mathsf{N}_2(\lambda_2)$ for each $[\lambda_1, \lambda_2] \in \mathsf{P}(\lambda)$, which further implies $\mathsf{N}_{\bar\Pi}(\lambda) \supset \mathsf{N}(\mathsf{P}(\lambda))$, so that (5.10a) is proved.
As $\sigma(\overline{\mathbf{A}_\Pi})$ is pure point, we get with the help of (5.9a)

$$\sigma(\overline{\mathbf{A}_\Pi}) = \overline{\mathscr{D}(\overline{\mathbf{A}_\Pi})} = \overline{f_m(\mathscr{D}(A_1) \times \mathscr{D}(A_2))} \subset \overline{f_m(\sigma(A_1) \times \sigma(A_2))}.$$

For proving the opposite inclusion we use the Weyl's criterion (see proof of Lemma 5.1). Let $\lambda_r \in \sigma(A_r)$ and let $\{x_r^{(n)}\}$ be a sequence of unit vectors satisfying the Weyl's condition for A_r, λ_r. Now $\{\varphi(x_1^{(n)}, x_2^{(n)})\} \subset \mathsf{D}(A_1) \circ \mathsf{D}(A_2) \subset \mathsf{D}(\overline{\mathbf{A}_\Pi})$, $\|\varphi(x_1^{(n)}, x_2^{(n)})\| = 1$ and

$$\|(\overline{\mathbf{A}_\Pi} - \lambda_1\lambda_2 I_1 \otimes I_2)\, \varphi(x_1^{(n)}, x_2^{(n)})\| = \|(\overline{A_1 \otimes A_2} - \lambda_2 A_1 \otimes I_2 + \lambda_2 A_1 \otimes I_2 - \lambda_1\lambda_2 I_1 \otimes I_2)\, \varphi(x_1^{(n)}, x_2^{(n)})\| \leq$$
$$\|A_1 x_1^{(n)}\|_1 \|(A_2 - \lambda_2 I_2)x_2^{(n)}\|_2 + |\lambda_2|\, \|(A_1 - \lambda_1 I_1)x_1^{(n)}\|_1.$$

*) This is true for arbitrary self-adjoint A_r's (see Theorem 9). Notice also that $f_m(\sigma(A_1) \times \sigma(A_2))$ and $f_s(\sigma(A_1) \times \sigma(A_2))$ need not be closed though the $\sigma(A_r)$'s are (Example 5.1).

**) We write briefly $\mathsf{N}(\mathsf{P}(\lambda))$ instead of the right-hand side of (5.10a).

Further $\|A_1 x_1^{(n)}\|_1 \to |\lambda_1|$ and hence

$$\|(\overline{\mathbf{A}_\Pi} - \lambda_1\lambda_2 I_1 \otimes I_2)\, \varphi(x_1^{(n)}, x_2^{(n)})\| \to 0$$

i.e. $f_m(\sigma(A_1) \times \sigma(A_2)) \subset \sigma(\overline{\mathbf{A}_\Pi})$. Now $\sigma(\overline{\mathbf{A}_\Pi})$ **is** closed and so we get (5.11a).

The same reasoning can be used for proving the statements concerning operator $\overline{\mathbf{A}_\Sigma}$. ■

Remark: Notice that we have not made use of the assumption that the A_r's have pure point spectra when proving $f_m(\sigma(A_1) \times \sigma(A_2)) \subset \sigma(\overline{\mathbf{A}_\Pi})$, so that this relation, together with the analogous relation for $\overline{\mathbf{A}_\Sigma}$, holds for each pair of self-adjoint A_r's.

Example 5.1: Consider the following two operators:
(a) $\mathscr{P}$ on $L^2(0,2\pi)$, $(\mathscr{P}x)(t) = i\,(dx/dt)$, whose domain consists of all the absolutely continuous functions on $(0,2\pi)$ which obey $x(0) = x(2\pi)$ and $dx/dt \in L^2(0,2\pi)$. Operator $\mathscr{P}$ is self-adjoint (see [2], [5]) and each vector of the trigonometric basis $\mathscr{E}_T = \{e_k\}_{k=-\infty}^{\infty}$ in $L^2(0,2\pi)$ is an eigenvector of $\mathscr{P}$: $\mathscr{P}e_k = ke_k$, so that $\mathscr{P}$ has a pure point spectrum.
(b) C on a separable $\mathscr{H}_0$ defined as follows: let $\mathscr{F} = \{f_n\}_{n=1}^{\infty}$ be an orthonormal basis in $\mathscr{H}_0$. Then for any $x \in \mathscr{H}_0$, $x = \xi_1 f_1 + \xi_2 f_2 + \dots$:

$$Cx = \sum_{n=1}^{\infty} (\xi_n/n) f_n.$$

It is clear that C is Hermitian and that each f_n is an eigenvector of C; thus the spectrum of C is pure point.

Consider now operator $\overline{\mathscr{P} \otimes C}$ on $L^2((0,2\pi); \mathscr{H}_0)$ which is a realization of $L^2(0,2\pi) \otimes \mathscr{H}_0$ (see (2.13)). According to Theorems 7 and 8, this operator is self-adjoint and has a pure point spectrum; we see that λ is in $f_m(\sigma(\mathscr{P}) \times \sigma(C))$ if and only if λ is a rational number. Then

$$\sigma(\overline{\mathscr{P} \otimes C}) = \overline{f_m(\sigma(\mathscr{P}) \times \sigma(C))} = \mathbf{R}.$$

This shows that in general $f_m(\sigma(A_1) \times \sigma(A_2)) \subset \sigma(\overline{\mathbf{A}_\Pi})$ while equality need not hold. A similar example can be constructed for $\mathbf{A}_\Sigma$.

Theorem 9: Let A_r be a self-adjoint operator on a separable $\mathscr{H}_r (r = 1,2)$. Then the spectra of operators

$$\overline{\mathbf{A}_\Pi} = \overline{A_1 \otimes A_2}\,,\ \overline{\mathbf{A}_\Sigma} = \overline{\mathbf{A}_1 + \mathbf{A}_2}$$

satisfy

$$\sigma(\overline{\mathbf{A}_\Pi}) = \overline{f_m(\sigma(A_1) \times \sigma(A_2))},$$
$$\sigma(\overline{\mathbf{A}_\Sigma}) = \overline{f_s(\sigma(A_1) \times \sigma(A_2))}\,.$$

Proof: We only have to prove that $\sigma(\overline{\mathbf{A}_\Pi}) \subset \overline{f_m(\sigma(A_1) \times \sigma(A_2))}$ (see Remark to Theorem 8) *). To this purpose we shall apply the spectral theorem (Lemma 5.3)

*) Again only operator $\overline{\mathbf{A}_\Pi}$ is considered; everything of what follows can be applied for $\overline{\mathbf{A}_\Sigma}$ as well.

and use the same notation and restrictions as in the proof of Theorem 7. Operator $\overline{\mathbf{A}_{II}}$ is unitarily equivalent to the self-adjoint operator $\mathbf{A}_m$ – multiplication by $f_m(t_1, t_2) = t_1 t_2$ on $L^2(\mathbf{R}^2, d\mu_{12})$ – and consequently $\sigma(\overline{\mathbf{A}_{II}}) = \sigma(\mathbf{A}_m)$ (see [5], [12]). Similarly

$$(*) \qquad \sigma(A_r) = \sigma(Q_{\mu_r}).$$

Let $\lambda \in \sigma(\overline{\mathbf{A}_{II}})$; then, according to Lemma 5.1, $\mu_{12}(\mathsf{N}_m(\varepsilon, \lambda)) \neq 0$ where

$$\mathsf{N}_m(\varepsilon, \lambda) = \{[t_1, t_2] \in \mathbf{R}^2 | \, |f_m(t_1, t_2) - \lambda| < \varepsilon\}$$

and ε is any positive number. Now, f_m is a continuous mapping from $\mathbf{R}^2$ to $\mathbf{R}$, so that $\mathsf{N}_m(\varepsilon, \lambda)$ is an open set in $\mathbf{R}^2$ and can be therefore expressed as a countable union of bounded open intervals in $\mathbf{R}^2$:

$$(**) \qquad \mathsf{N}_m(\varepsilon, \lambda) = \bigcup_{k=1}^{\infty} \mathsf{I}_2^{(k)}.$$

Each of these intervals is a Cartesian product of bounded open intervals $\mathsf{I}_1^{(k)}, \tilde{\mathsf{I}}_1^{(k)}$ in $\mathbf{R}$. Suppose

$$(***) \qquad \mathsf{N}_m(\varepsilon_0, \lambda) \cap (\sigma(A_1) \times \sigma(A_2)) = \emptyset,$$

i.e. $\mathsf{I}_1^{(k)} \cap \sigma(A_1) = \emptyset$ or $\tilde{\mathsf{I}}_1^{(k)} \cap \sigma(A_2) = \emptyset$ for $k = 1, 2, \ldots$.
Condition $\mathsf{I}_1 \cap \sigma(A_r) = \emptyset$ implies, together with Lemma 5.1 and $(*)$, that for each $t \in \mathsf{I}_1$ there is an open interval $\mathsf{U}_t = (t - \varepsilon_t, t + \varepsilon_t)$ such that $\mu_r(\mathsf{U}_t) = 0$. Now μ_r is a regular measure on $\mathbf{R}$, i.e. $\mu_r(\mathsf{I}_1) = \sup\{\mu_r(\mathsf{F}) \mid \mathsf{F} \subset \mathsf{I}_1, \mathsf{F} = \overline{\mathsf{F}}\}$ (see [3], [4]). The system $\{\mathsf{U}_t \mid t \in \mathsf{I}_1\}$ is a cover of each F and, since F is compact, a finite subset $\{\mathsf{U}_{t_1}, \mathsf{U}_{t_2}, \ldots, \mathsf{U}_{t_n}\}$ exists such that $\mathsf{F} \subset \bigcup_{k=1}^{n} \mathsf{U}_{t_k}$. Then $\mu_r(\mathsf{F}) \leq \sum_{k=1}^{n} \mu_r(\mathsf{U}_{t_k}) = 0$, which implies $\mu_r(\mathsf{I}_1) = 0$. Consequently, asumption $(***)$ together with $(**)$ yields $\mathsf{N}_m(\varepsilon_0, \lambda) = \emptyset$. Hence $\mathsf{N}_m(\varepsilon, \lambda) \cap (\sigma(A_1) \times \sigma(A_2)) \neq \emptyset$ for each $\varepsilon > 0$; since $f_m(\mathsf{N}_m(\varepsilon, \lambda)) = (\lambda - \varepsilon, \lambda + \varepsilon) = \mathsf{U}_\varepsilon(\lambda)$, we conclude that each neigborhood $\mathsf{U}_\varepsilon(\lambda)$ of λ satisfies $\mathsf{U}_\varepsilon(\lambda) \cap f_m(\sigma(A_1) \times \sigma(A_2)) \neq \emptyset$, i.e. $\lambda \in \overline{f_m(\sigma(A_1) \times \sigma(A_2))}$. ■

Remark: The generalization mentioned in Remark 1 to Theorem 7 refers to this theorem as well: if $P(t_1, t_2, .., t_n)$ is a real polynomial function of n variables, then the spectrum σ_P of polynomial tensor-product operator $\overline{P(\mathbf{A}_1, \mathbf{A}_2, .., \mathbf{A}_n)}$ is related to the spectra of the A_r's by

$$(5.11c) \qquad \sigma_P = \overline{P(\sigma(A_1) \times \sigma(A_2) \times .. \times \sigma(A_n))}.$$

Concluding this section, we show how the fundamental Theorem 7 can be applied when studying tensor products of *strongly continuous one-parameter unitary groups (SCOPUG)*. Let us remind that a SCOPUG on $\mathscr{H}$ is an operator-valued function $U : \mathbf{R} \rightarrow \mathfrak{L}(\mathscr{H})$*) satisfying the following conditions:

*) $\mathfrak{L}(\mathscr{H})$ is the Banach space of bounded operators on $\mathscr{H}$.

(a) Each $U(t)$ is a unitary operator, $U(t)U(s) = U(t+s)$, $U(0) = I$.
(b) $U(.)$ is a strongly continuous function, i.e.

$$\lim_{t \to t_0} \|(U(t) - U(t_0))x\| = 0 \text{ for each } x \in \mathscr{H} \text{ and } t_0 \in \mathbf{R}.$$

The well-known Stone's theorem (see [3], [12], [15]) states that for each SCOPUG $U(.)$ on $\mathscr{H}$ there is just one self-adjoint operator A on $\mathscr{H}$ (the generator of $U(.)$) such that

$$(5.12) \qquad U(t) = \exp(iAt).$$

for all $t \in \mathbf{R}$*),

$$(5.13a) \qquad \lim_{t \to 0} \left\| \frac{U(t) - I}{t} x - iAx \right\| = 0$$

for all $x \in \mathsf{D}(A)$, and conversely, if

$$y(t_n) = \frac{U(t_n) - I}{t_n} x$$

is a strongly convergent sequence for any $\{t_n\} \subset \mathbf{R}$, $t_n \to 0$, then

$$(5.13b) \qquad x \in \mathsf{D}(A)\,,\; y(t_n) \to iAx.$$

Lemma 5.7: Let $U_r(.)$ be a SCOPUG on $\mathscr{H}_r$ $(r = 1, 2)$ and let $\mathscr{H}$, φ be a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then $\boldsymbol{U}(.) = U_1(.) \otimes U_2(.)$ is a SCOPUG on $\mathscr{H}$.

Proof: Using Lemmas 4.1 and 4.4 one finds that $\boldsymbol{U}(.)$ satisfies condition (a). Further it is not difficult to verify

$$(*) \qquad \lim_{t \to t_0} \|(\boldsymbol{U}(t) - \boldsymbol{U}(t_0))x\| = 0 \quad \text{if} \quad x \in \mathscr{H}_1 \circ \mathscr{H}_2.$$

Now $\mathscr{H}_1 \circ \mathscr{H}_2$ is dense in $\mathscr{H}$, $\|\boldsymbol{U}(t)\|_{\mathfrak{L}(\mathscr{H})} = 1$, and hence $(*)$ holds for any $x \in \mathscr{H}$. ■

Let $\boldsymbol{A}$ be the generator of $U_1(.) \otimes U_2(.)$. Is there a relation between $\boldsymbol{A}$ and generators A_r of the constituent SCOPUG's $U_r(.)$?

Theorem 10: Let $U_r(.)$ be a SCOPUG on a separable $\mathscr{H}_r (r = 1, 2)$, let A_r be the generator of $U_r(.)$ and let $\mathscr{H}$, φ be a realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$. Then the generator $\boldsymbol{A}$ of the SCOPUG $U_1(.) \otimes U_2(.)$ on $\mathscr{H}$ satisfies $\boldsymbol{A} = \overline{\boldsymbol{A}_\Sigma} = \overline{\boldsymbol{A}_1 + \boldsymbol{A}_2}$, i.e. it holds

$$(5.14) \qquad \exp(iA_1 t) \otimes \exp(iA_2 t) = \exp(i\overline{(\boldsymbol{A}_1 + \boldsymbol{A}_2)}t)$$

for all $t \in \mathbf{R}$.

Proof: Take some $x = \varphi(x_1, x_2)$, $x_r \in \mathsf{D}(A_r)$ and let $\{t_n\} \subset \mathbf{R}$ be an arbitrary sequence converging to zero. Taking into account that $x_r^{(n)} \to x_r (r = 1,2)$ implies $\varphi(x_1^{(n)}, x_2^{(n)}) \to \varphi(x_1, x_2)$, we obtain

*) See Remark 1 to Lemma 5.3 for an explication of the symbol exp (iAt).

$$\frac{1}{t_n}[(U_1(t_n) \otimes U_2(t_n))\, x - x] =$$

$$\frac{1}{t_n}[((U_1(t_n) - I_1) \otimes U_2(t_n))\, x + (I_1 \otimes (U_2(t_n) - I_2))x] =$$

$$\varphi\left(\frac{1}{t_n}[U_1(t_n) - I_1]x_1, U_2(t_n)x_2\right) + \varphi\left(x_1, \frac{1}{t_n}[U_2(t_n) - I_2]x_2\right) \to$$

$$\varphi(iA_1x_1, x_2) + \varphi(x_1, iA_2x_2) = i\mathbf{A}_\Sigma x.$$

This further implies that

$$y(t_n) = \frac{U_1(t_n) \otimes U_2(t_n) - I_1 \otimes I_2}{t_n}\, x$$

is a strongly convergent sequence for any $x \in \mathsf{D}(A_\Sigma) = \mathsf{D}(A_1) \circ \mathsf{D}(A_2)$, its limit being $i\mathbf{A}_\Sigma x$. According to (5.13b) we conclude that $\mathbf{A}_\Sigma \subset \mathbf{A}$. Since $\mathbf{A}$ is self-adjoint, it is closed and thus $\overline{\mathbf{A}_\Sigma} \subset \mathbf{A}$. Now, $\overline{\mathbf{A}_\Sigma}$ is self-adjoint and therefore it has no symmetric extensions; hence $\overline{\mathbf{A}_\Sigma} = \mathbf{A}$. ■

6. Applications in quantum theory

In the beginning of this section we shall remind some important points concerning the general description of quantum systems in terms of Hilbert spaces. This description is based on the assumption that an „appropriate" separable Hilbert space is assigned to each quantum system S. This Hilbert space is called the *state Hilbert space of* S. The relation of S to its state Hilbert space $\mathscr{H}$ is established by several postulates (see e.g. [(16]), one of which asserts that each observable (measurable quantity) is represented by a self-adjoint operator on $\mathscr{H}$.

We shall restrict ourselves for simplicity to such systems for which every Hermitian operator represents an observable. Denote by $\mathscr{S}$ the set of all Hermitian operators on $\mathscr{H}$ and by $\mathscr{S}'$ the commutant of $\mathscr{S}$, i.e. $\mathscr{S}' = \{B \in \mathfrak{L}(\mathscr{H}) \mid [B, A] = 0$ for each $A \in \mathscr{S}\}$. Since $\mathscr{S}$ is an irreducible symmetric set in $\mathfrak{L}(\mathscr{H})$, the Schur's lemma implies that $\mathscr{S}'$ contains only multiples of the identity operator (see [12], [13]). In other words, the above restriction means that there are no superselection rules*) in the system; such a system (and also its state Hilbert space) is called *coherent***).

The definition of the commutant implies further $\mathscr{S}'' = \mathfrak{L}(\mathscr{H})$. The same reasoning can be repeated for each irreducible subset $\mathscr{C}$ of $\mathscr{S}$ so that again $\mathscr{C}'' =$

*) Let us remind that presence of a *superselection rule* in the system may be equivalently expressed as follows: there exists a bounded operator B on $\mathscr{H}$ which commutes with every Hermitian operator (representing an observable), but is not a multiple of the identity operator (cf. e.g. [13].

**) The state Hilbert space of any system can be expressed as an orthogonal sum of coherent subspaces [17] so that the restriction to coherent systems is not substantial.

$= \mathfrak{L}(\mathscr{H})$. Now $\mathscr{C}''$ is the von Neumann algebra generated by $\mathscr{C}$ and thus each bounded operator (i.e. also each operator representing an observable with a bounded spectrum) is the weak operator topology limit of a sequence of operators belonging to the minimal symmetric algebra generated by $\mathscr{C}$ ([12], [13], [18]).

One says that a *state* of S is given if a non-negative number $p(E)$ is assigned to each projection E on $\mathscr{H}$ so that $p(I) = 1$ and $p(\sum_k E_k) = \sum_k p(E_k)$ for every at most countable set $\{E_k\}$ of mutually orthogonal projections. In other words, states are positive σ-additive functionals on the set of all projections on $\mathscr{H}$. The fundamental Gleason's theorem (see e.g. [13]) asserts that for each such functional $p(.)$ there is just one statistical operator W on $\mathscr{H}$ satisfying

$$p(E) = \text{Tr } WE.$$

It is not difficult to check the converse statement: if W is a statistical operator then $p : E \mapsto \text{Tr } WE$ is a positive σ-additive functional. Hence there is a one-to-one correspondence between states and statistical operators.

We shall therefore in the following identify states with statistical operators; similarly each observable will be identified with the corresponding self-adjoint operator. The state is *pure* if W is a projection of rank one; a pure state can be represented by any unit vector from the one-dimensional space $W\mathscr{H}$. Using the fact that W is a positive operator with a pure point spectrum one easily proves that $W^2 \leq W$ and that W is pure if and only if $W^2 = W$. The states which are not pure are called *mixed*; thus W is mixed if and only if $W^2 \neq W$.

Let A be an observable and M a Borel set on **R**. One postulates that for each projection $\chi_{\mathsf{M}}(A)$ belonging to the spectral measure of A (see Remark to Lemma 5.3) the non-negative number

$$p(\chi_{\mathsf{M}}(A)) = \text{Tr } W\chi_{\mathsf{M}}(A)$$

is the probability that a measurement of A in the state W gives the value within M. The mapping $\mu_{(W,A)}(.)$ from the system of Borel sets on **R** to $[0, \infty)$ defined by

$$\mu_{(W,A)}(\mathsf{M}) = \text{Tr } W\chi_{\mathsf{M}}(A) \tag{6.1}$$

is a Lebesgue-Stieltjes measure on **R**. Expressing the trace on the r.h.s. of (6.1) in the orthonormal basis $\{e_i\}$ formed by the eigenvectors of W we get

$$\mu_{(W,A)}(\mathsf{M}) = \sum_i \mu_i(\mathsf{M})$$

where $\mu_i(\mathsf{M}) = w_i(\chi_{\mathsf{M}}(A)e_i, e_i)$ and $We_i = w_i e_i$.
If A is Hermitian so that its spectrum is bounded, it further holds (see [14])

$$\int_{\sigma(A)} t \, d\mu_{(W,A)} = \sum_i \int_{\sigma(A)} t \, d\mu_i = \sum_i w_i(Ae_i, e_i) = \text{Tr } WA. \tag{6.2}$$

Owing to the physical meaning of $\mu_{(W,A)}$ it is clear that Tr WA has to be interpreted as the expectation value of observable A in the state W.

Observables A, A' are *compatible* (simultaneously measurable) if A commutes with A'. Let $\{A^{(1)}, A^{(2)}, \ldots, A^{(N)}\}$ be a set of compatible observables with pure point spectra and let $E_{j_i}^{(i)}$ ve the projection on the eigenspace of $A^{(i)}$ belonging to the eigenvalue $\lambda_{j_i}^{(i)}$. The set $\{A^{(1)}, A^{(2)}, \ldots, A^{(N)}\}$ is called *complete set of commuting operators* (CSCO) if the rank of each projection

$$E_{j_1}^{(1)} E_{j_2}^{(2)} \ldots E_{j_N}^{(N)}$$

is unity or zero*). Then there is an orthonormal basis in $\mathscr{H}$ formed by the unit vectors $e_{j_1 \ldots j_N} \in E_{j_1}^{(1)} \ldots E_{j_N}^{(N)} \mathscr{H}$ which are common eigenvectors of operators $A^{(1)}, A^{(2)}, \ldots, A^{(N)}$ belonging to eigenvalues $\lambda_{j_1}^{(1)}, \ldots, \lambda_{j_N}^{(N)}$ (only such N-tuples $(j_1, \ldots, j_N)$ are considered for which the corresponding subspace is one-dimensional). Each state $E_{j_1}^{(1)} \ldots E_{j_N}^{(N)}$ is clearly pure and uniquely determined by $\lambda_{j_1}^{(1)}, \ldots, \lambda_{j_N}^{(N)}$. It follows from the properties of the spectral measure

$$\mathrm{Tr}(\chi_{\mathsf{M}}(A^{(i)}) E_{j_1}^{(1)} \ldots E_{j_N}^{(N)}) = \begin{cases} 1 & \text{if} \quad \lambda_{j_i}^{(i)} \in \mathsf{M} \\ 0 & \text{if} \quad \lambda_{j_i}^{(i)} \in \mathsf{M} \end{cases}$$

One further finds with the use of the basis $\{e_{j_1 \ldots j_N}\}$

$$\mathrm{Tr}(A^{(i)} E_{j_1}^{(1)} \ldots E_{j_N}^{(N)}) = \lambda_{j_i}^{(i)}.$$

Finishing this introduction we mention the quantum theoretical description of the time evolution of a quantum system S. Let H be the energy operator, i.e. the Hamiltonian of S. We shall consider only *conservative systems*, i.e. systems whose Hamiltonians are time-independent. The fundamental dynamical postulate of the quantum theory states that the time evolution of a given system S is described by the strongly continuous one-parameter unitary group (SCOPUG) $U(t)$ whose generater is $-1/\hbar\, H$:

$$U(t) = \exp\left(-\frac{i}{\hbar} Ht\right); \tag{6.3}$$

the time evolution of states is then given by

$$W(t) = U(t - t_0)\, W(t_0)\, U^{-1}(t - t_0). \tag{6.3a}$$

Especially if W is a pure state and ψ is a unit vector in $W\mathscr{H}$**) then

$$\psi(t) = U(t - t_0)\psi(t_0). \tag{6.3b}$$

After these preliminaries let us examine how the tensor-product formalism can

*) There is a more general definition of CSCO: a set $\mathscr{S}$ of mutually commuting self-adjoint operators is CSCO if any $B \in \mathscr{S}'$ is a function of operators from $\mathscr{S}$ ([12], [13]). For example the self-adjoint operator Q on $L^2(\mathbf{R})$, which has a pure continuous spectrum, forms itself a CSCO. Both the definitions are of course equivalent if all the operators in $\mathscr{S}$ have pure point spectra.

**) One could prove easily that if $W(t_0)$ is pure then $W(t)$ is pure for all t (see e.g. section 6.3).

be applied in quantum theory. Let S be a quantum system whose state Hilbert space serves as a realization space of some tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$*). Such a system is called *joint system.* Again only *coherent* joint systems will be considered. If A_1 is a Hermitian operator on $\mathscr{H}_1$ then

$$(*) \qquad \mathbf{A}_1 = A_1 \otimes I_2$$

is a Hermitian operator on $\mathscr{H}$ and therefore it is an observable of S. One can thus regard those observables of S which are of the form (*) (and more generally of the form $\overline{A_1 \otimes I_2}$ where A_1 is a self-adjoint operator on $\mathscr{H}_1$) as observables of a system S_1 whose state Hilbert space is $\mathscr{H}_1$. This system will be called *subsystem* of S. In the same way one obtains the subsystem S_2. The generalization for a joint system with the state Hilbert space $(\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \ldots \otimes \mathscr{H}_n)_{\mathscr{H},\varphi}$ is straightforward. It is further obvious that each subsystem of a coherent joint system is again coherent.

Notice that subsystems can, but need not correspond to systems really separable from the joint system. Consider e.g. the system S_e consisting of one electron. Its state Hilbert space can be expressed as $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$ (cf. Example 2.2). Thus S_e is a joint system and both its subsystems are „fictitious" — they describe the orbital and the spin part of degrees of freedom respectively. On the contrary, the system S_{2e}, consisting of two electrons, with the state Hilbert space $L^2(\mathbf{R}^3; \mathbf{C}^2) \otimes L^2(\mathbf{R}^3; \mathbf{C}^2)$ is a joint system composed from „real" subsystems S_e, S_e.

On the other hand, many pairs $\mathscr{H}_1$, $\mathscr{H}_2$ could generally exist to given $\mathscr{H}$, namely if $\mathscr{H}$ is infinitedimensional. In fact, we call S joint system and speak about its subsystems only if we have a reasonable physical interpretation for $\mathscr{H}_1$, $\mathscr{H}_2$. It is further clear from the discussion that the notions of subsystem and joint system are relative: the same system, which is a subsystem of a „greater" joint system, can simultaneously be a joint system with respect to some „smaller" subsystems.

6.1 OBSERVABLES

Let S be a joint system with subsystems S_1, S_2. We know that each observable A_r of S_r $(r = 1, 2)$ is an observable of S represented by $\mathbf{A}_r$, the operators A_r and $\mathbf{A}_r$ having the same spectra**). However, the observables of the form $\mathbf{A}_1$, $\mathbf{A}_2$ do not exhaust the set of all observables of S. If A_1, A_2 are Hermitian then $A_1 \otimes A_2$ is Hermitian and hence it is an observable of S. The same holds for $\mathbf{A}_1 + \mathbf{A}_2$ so that neither $A_1 \otimes A_2$ can be regarded as a general form of observables of S. Nevertheless, the set of observables of the form $\overline{A_1 \otimes A_2}$ is „large enough" in the following sense:

*) We are not interested in the trivial case: $\mathscr{H} \otimes \mathrm{U}_1$ and $\mathrm{U}_1 \otimes \mathscr{H}$, U_1 being the one-dimensional space, are always realized in $\mathscr{H}$. Also other tensor products $\mathscr{H}_1 \otimes \mathscr{H}_2$ realized in $\mathscr{H}$ could appear as physically non-interesting (see below).

**) Cf. Theorem 9. This fact is understood physically as follows: a quantity a referring to A_r and $\mathbf{A}_r$ is measured by the same apparatus on S_r and S, respectively.

Theorem 11: If $\mathscr{S}_r$ $(r = 1,2)$ is a set of self-adjoint operators on $\mathscr{H}_r$ generating the von Neumann algebra of all bounded operators on $\mathscr{H}_r$*) (so that $\mathscr{S}_r'' = \mathfrak{L}(\mathscr{H}_r)$), then the set $\mathscr{S} = \overline{\{A_1 \otimes A_2} \mid A_r \in \mathscr{S}_r\}$ generates the von Neumann algebra of all bounded operators on $\mathscr{H}$.
For a proof see [1].

The next theorem is of great importance for describing a joint system S in terms of its subsystems:

Theorem 12: Let $\mathscr{C}_1 = \{A_1^{(1)}, A_1^{(2)}, \ldots, A_1^{(m)}\}$ be a CSCO on $\mathscr{H}_1$, $\mathscr{C}_2 = \{A_2^{(1)}, A_2^{(2)}, \ldots, A_2^{(n)}\}$ a CSCO on $\mathscr{H}_2$. Then the set $\mathscr{C} = \{\mathbf{A}_1^{(1)}, \mathbf{A}_1^{(2)}, \ldots, \mathbf{A}_1^{(m)}, \mathbf{A}_2^{(1)}, \mathbf{A}_2^{(2)}, \ldots, \mathbf{A}_2^{(n)}\}$ is a CSCO on the realization space $\mathscr{H}$ of $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Proof: We shall consider only the case when every $A_r^{(i)}$ has a pure point spectrum**). According to Theorem 8 the $\mathbf{A}_r^{(i)}$'s have also pure point spectra, $\mathscr{D}(\mathbf{A}_r^{(i)}) = \mathscr{D}(A_r^{(i)})$, and the eigenspace of $\mathbf{A}_1^{(i)}(\mathbf{A}_2^{(j)})$ belonging to an eigenvalue $\lambda_k^{(i)}$ $(\mu_l^{(j)})$ is

$$(*) \qquad \mathsf{N}_1(\lambda_k^{(i)}) \otimes \mathscr{H}_2 \quad (\mathscr{H}_1 \otimes \mathsf{N}_2(\mu_l^{(j)})).$$

Denote by $P_k^{(i)}$ and $Q_l^{(j)}$ the projections on eigenspaces $\mathsf{N}_1(\lambda_k^{(i)})$ and $\mathsf{N}_2(\mu_l^{(j)})$, respectively. It follows from Lemma 4.4 that $\mathbf{P}_k^{(i)} = P_k^{(i)} \otimes I_2$ and $\mathbf{Q}_l^{(j)} = I_1 \otimes Q_l^{(j)}$ are projections on eigenspaces $(*)$. Now each $\mathbf{P}$ commutes with each $\mathbf{Q}$ and since $\mathscr{C}_r$ is a CSCO, the projections $P_k^{(i)}, P_{k'}^{(i')}$ $(Q_l^{(j)}, Q_{l'}^{(j')})$ commute with each other. Hence for each $(k_1, k_2, \ldots, k_m)$, $(l_1, l_2, \ldots, l_n)$ the set $\{\mathbf{P}_{k_1}^{(1)}, \mathbf{P}_{k_2}^{(2)}, \ldots, \mathbf{P}_{k_m}^{(m)}, \mathbf{Q}_{l_1}^{(1)}, \mathbf{Q}_{l_2}^{(2)}, \ldots, \mathbf{Q}_{l_n}^{(n)}\}$ is a set of mutually commuting projections and thus $\mathscr{C}$ is a set of mutually commuting self-adjoint operators***). Consider projections $E(k_1, k_2, \ldots, k_m) = P_{k_1}^{(1)} P_{k_2}^{(2)} \ldots P_{k_m}^{(m)}$ and $F(l_1, l_2, \ldots, l_n) = Q_{l_1}^{(1)} Q_{l_2}^{(2)} \ldots Q_{l_n}^{(n)}$. Due to the starting assumption it holds $\dim E(k_1, k_2, \ldots, k_m) \leq 1$, $\dim F(l_1, l_2, \ldots, l_n) \leq 1$. Using the relation

$$(**) \qquad \mathbf{P}_{k_1}^{(1)} \ldots \mathbf{P}_{k_m}^{(m)} \mathbf{Q}_{l_1}^{(1)} \ldots \mathbf{Q}_{l_n}^{(n)} = E(k_1, \ldots, k_m) \otimes F(l_1, \ldots, l_n)$$

and (4.10), we conclude that the rank of each projection $(**)$ is less or equal to unity. ■

Example 6.1: Consider firstly a one-electron system S_e. Its state Hilbert space $L^2(\mathbf{R}^3; \mathbf{C}^2)$ serves for a realization of $L^2(\mathbf{R}) \otimes L^2(\mathbf{R}) \otimes L^2(\mathbf{R}) \otimes \mathbf{C}^2$. A CSCO on $L^2(\mathbf{R})$ is formed e.g. by the operator Q; further the operator s representing one component of spin has a simple pure point spectrum and thus s is a CSCO on $\mathbf{C}^2$.

*) The commutant of a set $\mathscr{S}$ which may contain unbounded self-adjoint operators and the von Neumann algebra generated by $\mathscr{S}$ are defined in [12].

**) Hint for a proof in the general case is given in [1].

***) Two self-adjoint operators A, A' commute if and only if $[\chi_{(-\infty,t)}(A), \chi_{(-\infty,t')}(A')] = 0$ for any $t, t' \in \mathbf{R}$. If A has a pure point spectrum, then $\chi_{(-\infty,t)}(A) = \sum_{\{k \mid \lambda_k < t\}} P_k$ where P_k is the projection on the eigenspace belonging to eigenvalue λ_k of A. Then the necessary and sufficient condition that A and A' with pure point spectra commute becomes: $[P_k, P_{k'}] = 0$ for all k, k' (see [12]).

Let I_L and $I_{\mathbf{C}}$ be the identity operators on $L^2(\mathbf{R})$ and $\mathbf{C}^2$, respectively, and denote

$$\mathbf{Q}_1 = Q \otimes I_L \otimes I_L \otimes I_{\mathbf{C}} \quad \text{(similarly } \mathbf{Q}_2, \mathbf{Q}_3 \text{ — see (4.1a))}$$
$$\mathbf{s} = I_L \otimes I_L \otimes I_L \otimes s$$

Then $\{\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{s}\}$ is a CSCO on $L^2(\mathbf{R}^3; \mathbf{C}^2)$. Passing to a two-electron system S_{2e}, we denote by I the identity operator on $L^2(\mathbf{R}^3; \mathbf{C}^2)$ and introduce $\mathbf{A}_j^{(1)} = A_j \otimes I$, $\mathbf{A}_j^{(2)} = I \otimes \mathbf{A}_j$ for each operator A_j on $L^2(\mathbf{R}^3; \mathbf{C}^2)$. Then $\{\mathbf{Q}_1^{(r)}, \mathbf{Q}_2^{(r)}, \mathbf{Q}_3^{(r)}, \mathbf{s}^{(r)} \mid r = 1, 2\}$ is a CSCO on the state Hilbert space $L^2(\mathbf{R}^3; \mathbf{C}^2) \otimes L^2(\mathbf{R}^3; \mathbf{C}^2)$ of S_{2e}.

One can obtain the von Neumann algebra $\mathfrak{L}(L^2(\mathbf{R}^3; \mathbf{C}^2))$ by means of operators of the form $\overline{A_1 \otimes A_2}$ where A_1 and A_2 belong to irreducible sets of self-adjoint operators on $L^2(\mathbf{R}^3)$ and $\mathbf{C}^2$, respectively. Let us consider for simplicity only the „one--dimensional" case, i.e. let the state Hilbert space of S_e be $L^2(\mathbf{R}) \otimes \mathbf{C}^2$. The operators Q and P form an irreducible set $\mathscr{S}_L$ on $L^2(\mathbf{R})$ (see [12], [13]); it follows further from the commutation relations of the spin components that any two of them, say s and s', form an irreducible set $\mathscr{S}_{\mathbf{C}}$ on $\mathbf{C}^2$. Then, by Theorem 11, the set $\mathscr{S} = \{\overline{A_i^{(L)} \otimes A_j^{(\mathbf{C})}} \mid A_i^{(L)} \in \mathscr{S}_L, A_j^{(\mathbf{C})} \in \mathscr{S}_{\mathbf{C}}\}$ generates the von Neumann algebra $\mathfrak{L}\,(L^2(\mathbf{R};\ \mathbf{C}^2))$*). Generalizations for the three-dimensional case and for two-electron systems are obvious.

6.2 STATES

Let S be a joint system with a state Hilbert space $\mathscr{H}$ and let $\mathscr{H}_r$ be state Hilbert spaces of subsystems S_r ($r = 1, 2$) of S such that $\mathscr{H}_1 \otimes \mathscr{H}_2$ is realized in $\mathscr{H}$. Let further $\mathbf{W}$ be a state of S. Take any projection E_r on $\mathscr{H}_r$ and consider the functional $p : p(E_r) = \operatorname{Tr} \mathbf{W}\,\mathbf{E}_r$. If one expresses the trace by means of the orthonormal basis formed by the eigenvectors of $\mathbf{W}$ one easily verifies that p satisfies the conditions of the Gleason theorem; hence there is a unique statistical operator $\mathscr{W}_r(\mathbf{W})$ on $\mathscr{H}_r$ such that $p(E_r) = \operatorname{Tr} \mathscr{W}_r(\mathbf{W})\, E_r$, i.e.

$$\operatorname{Tr} \mathbf{W}\,\mathbf{E}_r = \operatorname{Tr} \mathscr{W}_r(\mathbf{W})\, E_r \tag{6.4}$$

for every projection E_r on $\mathscr{H}_r$. We have thus obtained a mapping $\mathscr{W}_r(.)$ from the set of all states of S to the set of states of S_r. It is quite natural to interpret $\mathscr{W}_r(\mathbf{W})$ as follows**): if $\mathbf{W}$ is a state of S then the subsystem S_r is in the state $\mathscr{W}_r(\mathbf{W})$. The $\mathscr{W}_r(\mathbf{W})$'s are called *reduced (component) states*. Thus the reduced states $\mathscr{W}_r(\mathbf{W})$ are uniquely determined by $\mathbf{W}$.

Let A_r be a Hermitian operator on $\mathscr{H}_r$. Taking into account that the spectral measure $\chi_{\mathsf{M}}(A_r)$ of the Hermitian operator A_r satisfies for each Borel set M:

$$\chi_{\mathsf{M}}(\mathbf{A}_1) = \chi_{\mathsf{M}}(A_1) \otimes I_2, \quad \chi_{\mathsf{M}}(\mathbf{A}_2) = I_1 \otimes \chi_{\mathsf{M}}(A_2)$$

*) In practice one usually expresses electron observables with the use of a „greater" irreducible set on $\mathbf{C}^2$, especially that consisting of all three spin components.

**) The physical relevance of this interpretation is expressed in the most illustrative way by the formula (6.5).

where $\chi_M(A_r)$ is the spectral measure of A_r (see [12]), and using (6.1), (6.4), we obtain

$$\mu_{(\mathbf{W},\mathbf{A}_r)}(M) = \mathrm{Tr}\, \mathbf{W}\chi_M(\mathbf{A}_r) = \mathrm{Tr}\, \mathscr{W}_r(\mathbf{W})\chi_M(A_r) = \mu_{(\mathscr{W}_r(\mathbf{W}),A_r)}(M).$$

Further $\sigma(A_r) = \sigma(\mathbf{A}_r) = \sigma$ (see Theorem 9) and then (6.2) implies

$$\mathrm{Tr}\, \mathbf{W}\mathbf{A}_r = \int_\sigma t\, d\mu_{(\mathbf{W},\mathbf{A}_r)} = \int_\sigma t\, d\mu_{(\mathscr{W}_r(\mathbf{W}),A_r)} = \mathrm{Tr}\, \mathscr{W}_r(\mathbf{W})A_r. \tag{6.4a}$$

Thus the component states $W_r = \mathscr{W}_r(\mathbf{W})$ of S_r corresponding to the state $\mathbf{W}$ of S have to obey the relations

$$\mathrm{Tr}\, W_r A_r = \mathrm{Tr}\, \mathbf{W}\mathbf{A}_r\,, \quad r = 1, 2, \tag{6.5}$$

for any Hermitian operator A_r on $\mathscr{H}_r$. Validity of the relations (6.5) represents itself a natural physical requirement. On the other hand, the relations (6.5) in the particular case give (6.4); it shows that the above choice of the functional p was physically reasonable.

Let $\mathscr{W}_r(\mathbf{W}) = \mathscr{W}_r(\mathbf{W}')$, $r = 1, 2$; do these relations imply $\mathbf{W} = \mathbf{W}'$? In view of relations (6.5) we can formulate the problem as follows: is there for a given pair W_1, W_2 only one $\mathbf{W}$ which „solves" the relations (6.5)? It is clear that $\mathbf{W} = W_1 \otimes W_2$ is always a solution. The next example shows that in general this solution is not unique.

Example 6.2: Consider a joint system S whose state Hilbert space $\mathscr{H}$ is a realization of $\mathbf{C}^2 \otimes \mathbf{C}^2 : \mathscr{H} = (\mathbf{C}^2 \otimes \mathbf{C}^2)_\varphi$; e.g. let S represent the spin degrees of freedom of a two-electron system. Let $\mathscr{E} = \{e_+, e_-\}$ be the orthonormal basis in $\mathbf{C}^2$ in which the spin components s_i of an electron are represented by $\frac{1}{2}\sigma_i$ (σ_i are the Pauli matrices). Introduce projections

$$E_\pm = \frac{1}{2}(I \pm \sigma_3), \tag{*}$$

so that $E_\pm \mathbf{C}^2$ are one-dimensional subspaces spanned by $e_\pm$. The „coupled" orthonormal basis $\{f_{ij}\}$ in $\mathscr{H}$ is obtained from the „uncoupled" basis $\varphi(\mathscr{E} \times \mathscr{E})$ by standard formulae

$$\begin{aligned} f_{10} &= \frac{1}{\sqrt{2}}\,[\varphi(e_+, e_-) + \varphi(e_-, e_+)] \\ f_{11} &= \varphi(e_+, e_+) \\ f_{1,-1} &= \varphi(e_-, e_-) \\ f_{00} &= \frac{1}{\sqrt{2}}\,[\varphi(e_+, e_-) - \varphi(e_-, e_+)]\,. \end{aligned} \tag{6.6}$$

The first three of these vectors span the *triplet subspace* $\mathscr{H}_t$, the last one spans the *singlet subspace* $\mathscr{H}_s$ in $\mathscr{H}$. Further $\mathscr{H}_t$ and $\mathscr{H}_s$ are eigenspaces of $\sum_{i=1}^{3} \sigma_i \otimes \sigma_i$ (this operator will be briefly written as $\vec{\sigma} \otimes \vec{\sigma}$) belonging to eigenvalues 1 and —3, respectively, and the corresponding projections are

$$(6.7a) \qquad \mathbf{E}_t = \frac{3\mathbf{I} + \vec{\sigma} \otimes \vec{\sigma}}{4}, \qquad \mathbf{E}_s = \frac{\mathbf{I} - \vec{\sigma} \otimes \vec{\sigma}}{4}.$$

Using (∗) one obtains for the projection $\mathbf{E}^{(0)}$ on the eigenspace of $s_3^{(1)} \otimes I + I \otimes s_3^{(2)}$ belonging to eigenvalue zero:

$$(6.7b) \qquad \mathbf{E}^{(0)} = I - E_+ \otimes E_+ - E_- \otimes E_- = \frac{1}{2}(\mathbf{I} - \sigma_3 \otimes \sigma_3).$$

It can be easily checked that $\mathbf{E}_t\mathbf{E}^{(0)} = \mathbf{E}^{(0)}\mathbf{E}_t$ and $\mathbf{E}_s\mathbf{E}^{(0)} = \mathbf{E}^{(0)}\mathbf{E}_s = \mathbf{E}_s$, and consequently projections on the one-dimensional subspaces spanned by f_{ij} are

$$(6.8) \qquad \mathbf{E}_{10} = \mathbf{E}_t\mathbf{E}^{(0)},\ \mathbf{E}_{11} = E_+ \otimes E_+,\ \mathbf{E}_{1,-1} = E_- \otimes E_-,\ \mathbf{E}_{00} = \mathbf{E}_s{}^{\star}).$$

Consider the one-electron spin state $W = \frac{1}{2} I$; clearly $\mathrm{Tr}(Ws_i) = 0$ for $i = 1, 2, 3$. Such a density matrix could describe e.g. a totally unpolarized electron. For each Hermitian operator A on $\mathbf{C}^2$ one has

$$\mathrm{Tr}\, WA = \frac{1}{2}[(Ae_+, e_+) + (Ae_-, e_-)] = \frac{1}{2}\mathrm{Tr} A.$$

Calculation of $\mathrm{Tr}\, \mathbf{E}_t\mathbf{A}_r$, $\mathrm{Tr}\, \mathbf{E}_s\mathbf{A}_r$ in the basis (6.6) yields

$$\mathrm{Tr}\, \mathbf{E}_t\mathbf{A}_r = \frac{3}{2}[(Ae_+, e_+) + Ae_-, e_-)] = 3\mathrm{Tr}\, WA,$$
$$\mathrm{Tr}\, \mathbf{E}_s\mathbf{A}_r = \mathrm{Tr}\, WA.$$

Hence for any non-negative α, β, obeying $3\alpha + \beta = 1$, the statistical operator

$$\mathbf{W}_{(\alpha,\beta)} = \alpha\mathbf{E}_t + \beta\mathbf{E}_s$$

satisfies relations (6.5) if $W_1 = W_2 = 1/2I$. Especially

$$\mathbf{W}_{(1/4,\,1/4)} = \frac{1}{4}\mathbf{I} = W_1 \otimes W_2.$$

The knowledge of states of both subsystems S_1, S_2 is therefore in general not sufficient for determining uniquely a state of the joint system S. This result is closely connected with the existence of observables of S which cannot be expressed as $\mathbf{A}_r$. However, the following statement holds:

(i) If at least one of the states W_1, W_2 is pure then relations (6.5) are satisfied only by $\mathbf{W} = W_1 \otimes W_2$ (for a proof see Appendix).

⋆) Of course $\mathbf{E}_{10} + \mathbf{E}_{11} + \mathbf{E}_{1,-1} = \mathbf{E}_t$, what can be checked by a simple calculation.

Consequently, if both W_1 and W_2 are pure (i.e. projections of rank one) then the state of S determined (uniquely) by them is pure. Conversely, consider a pure state $\mathbf{W}$ of S which has the form $W_1 \otimes W_2$. Taking into account that the W_r's are positive, one obtains from the condition $\mathbf{W}^2 = \mathbf{W}$ and from Lemma 4.1 that $W_r^2 = W_r$, i.e. that the W_r's are pure. According to (4.17) the W_r's obey relations (6.5) and therefore $\mathscr{W}_r(W_1 \otimes W_2) = W_r$. We have thus obtained the following results:

(ii) Any pair of pure states W_1, W_2 of subsystems S_1, S_2 uniquely determines a pure state $\mathbf{W} = W_1 \otimes W_2$ of S such that the corresponding reduced states $\mathscr{W}_r(\mathbf{W})$ are just W_r, $r = 1, 2$.

(iii) If $\mathbf{W}$ is a pure state of S having the form $W_1 \otimes W_2$ then both reduced states W_r are pure.

Let $\mathbf{W}$ be a mixed state. According to (ii), at least one of the reduced states $\mathscr{W}_r(\mathbf{W})$ must be mixed. In order to study the structure of $\mathscr{W}_r(\mathbf{W})$ in more detail we express firstly $\mathbf{W}$ by means of the projections $\mathbf{E}^{(i)}$ on the one-dimensional eigenspaces of $\mathbf{W}$ corresponding to eigenvalues w_i:

(6.9a) $$\mathbf{W} = \sum_{i=1}^{\infty} w_i \mathbf{E}^{(i)}$$

and

(*) $$\sum_{i=1}^{\infty} w_i = 1 \;{}^{\star)}.$$

Expressing the trace in the basis formed by the eigenvectors $e^{(i)}$ of $\mathbf{W}$ one has

$$\mathrm{Tr}\, \mathbf{W}\mathbf{A}_r = \sum_{i=1}^{\infty} w_i(\mathbf{A}_r e^{(i)}, e^{(i)}) = \sum_{i=1}^{\infty} w_i \,\mathrm{Tr}\,(\mathbf{E}^{(i)}\mathbf{A}_r).$$

Denote $W_r^{(i)} = \mathscr{W}_r(\mathbf{E}^{(i)})$; then (6.4a) gives

(**) $$\mathrm{Tr}\, \mathbf{W}\mathbf{A}_r = \sum_{i=1}^{\infty} w_i \,\mathrm{Tr}\,(W_r^{(i)} A_r).$$

The operators $S_r^{(n)} = \sum_{i=1}^{n} w_i W_r^{(i)}$ are positive and $S_r^{(n+1)} \geq S_r^{(n)}$. Using (*) and the inequality $W_r^{(i)} \leq I_r$ one finds that $S_r^{(n)} \leq I_r$, and hence there is a positive operator $W_r \leq I_r$ such that $S_r^{(n)} x \to W_r x$ for each $x \in \mathscr{H}_r$. One easily checks that W_r is a statistical operator and

$$\mathrm{Tr}\, W_r B_r = \sum_{i=1}^{\infty} w_i \,\mathbf{Tr}(W_r^{(i)} B_r)$$

for each $B_r \in \mathfrak{L}(\mathscr{H}_r)$. Then (**) can be rewritten

$$\mathrm{Tr}\, \mathbf{W}\mathbf{A}_r = \mathrm{Tr}\, W_r A_r,$$

⋆) We assume dim $\mathbf{W}\mathscr{H} = \infty$; otherwise the following considerations are trivial.

which implies

$$(6.9b) \qquad \mathscr{W}_r(\mathbf{W}) = W_r = \sum_{i=1}^{\infty} w_i W_r^{(i)} = \sum_{i=1}^{\infty} w_i \mathscr{W}_r(\mathbf{E}^{(i)}).$$

Comparing (6.9a) to (6.9b) we see that it is sufficient to study the dependence of the reduced states $\mathscr{W}_r(\mathbf{W})$ on $\mathbf{W}$ for pure states $\mathbf{W}$ only.

In view of (iii) we shall suppose that $\mathbf{W}$ is a pure state but not of the form $W_1 \otimes W_2$. Thus $\mathbf{W}$ is a projection and $\dim \mathbf{W}\mathscr{H} = 1$. Let $\psi \in \mathbf{W}\mathscr{H}$, take any orthonormal bases $\mathscr{E} = \{e_i\}_{i=1}^{\dim \mathscr{H}_1}$ in $\mathscr{H}_1$ and $\mathscr{F} = \{f_j\}_{j=1}^{\dim \mathscr{H}_2}$ in $\mathscr{H}_2$ and express ψ by means of the basis $\varphi(\mathscr{E} \times \mathscr{F})$:

$$\psi = \sum_{i,j} \alpha_{ij} \varphi(e_i, f_j), \quad \sum_{i,j} |\alpha_{ij}|^2 = 1$$

We denote $W_r = \mathscr{W}_r(\mathbf{W})$ and use (6.4a) for expressing the action of W_1 and W_2 on the vectors of bases $\mathscr{E}$ and $\mathscr{F}$ by means of the α_{ij}'s:

$$\mathrm{Tr}\ \mathbf{W}\mathbf{A}_1 = (\mathbf{A}_1\psi, \psi) = \sum_{i,j} \sum_{k,l} \alpha_{ij}\bar{\alpha}_{kl}(A_1 e_i, e_k)_1 \delta_{jl} =$$
$$= \sum_k \sum_{i,j} \alpha_{ij}\bar{\alpha}_{kj}(A_1 e_i, e_k)_1.^{\star)}$$

On the other hand,

$$\mathrm{Tr}\ \mathbf{W}\mathbf{A}_1 = \mathrm{Tr}\ W_1 A_1 = \sum_i (A_1 e_i, W_1 e_i)_1.$$

Introduce $\qquad c_{ki} = \sum_j \alpha_{kj}\bar{\alpha}_{ij}$ for $i, k = 1,2, \ldots, \dim \mathscr{H}_1$. Since

$$\sum_{i,k} |c_{ki}|^2 = \sum_{i,k} \Big| \sum_j \alpha_{kj}\bar{\alpha}_{ij} \Big|^2 \leq \sum_{i,k} \sum_j |\alpha_{kj}|^2 \sum_j |\alpha_{ij}|^2 = 1$$

we can define a Hilbert-Schmidt operator $\tilde{W}_1$ by

$$\tilde{W}_1 e_i = \sum_k c_{ki} e_k.$$

For any $x \in \mathscr{H}_1$, $x = \sum_i \xi_i e_i$, one has

$$(\tilde{W}_1 x, x)_1 = \sum_k \sum_i c_{ki}\xi_i\bar{\xi}_k = \sum_{k,i,j} \alpha_{kj}\bar{\xi}_k\bar{\alpha}_{ij}\xi_i = \sum_j \Big| \sum_i \bar{\alpha}_{ij}\xi_i \Big|^2 \geq 0,$$

i.e. $\tilde{W}_1$ is positive. Further

$$\sum_i (\tilde{W}_1 e_i, e_i)_1 = \sum_i c_{ii} = \sum_{i,j} |\alpha_{ij}|^2 = 1$$

so that $\tilde{W}_1$ is a statistical operator. Finally

$$\mathrm{Tr}\ \tilde{W}_1 A_1 = \sum_i (A_1 e_i, \tilde{W}_1 e_i)_1 = \sum_{i,k} \bar{c}_{ki}(A_1 e_i, e_k)_1 = \sum_{i,k,j} \alpha_{ij}\bar{\alpha}_{kj}(A_1 e_i, e_k)_1 = \mathrm{Tr}\ W_1 A_1$$

and hence $W_1 = \tilde{W}_1$. Thus we get

$$(6.10a) \qquad W_1 e_i = \sum_{k=1}^{\dim \mathscr{H}_1} \sum_{j=1}^{\dim \mathscr{H}_2} \alpha_{kj}\bar{\alpha}_{ij} e_k$$

*) Owing to the normalization condition, this series is absolutely convergent so that the summation may be performed in any order.

and similarly

(6.10b) $$W_2 f_i = \sum_{k=1}^{\dim \mathscr{H}_2} \sum_{j=1}^{\dim \mathscr{H}_1} \alpha_{jk} \bar{\alpha}_{ji} f_k \; ;$$

these are so-called *reduction formulae.* Another form of them can be found in [1] where a special realization of $\mathscr{H}_1 \otimes \mathscr{H}_2$ is used. The above derivation is realization--independent; (6.10a, b) hold for arbitrary bases $\mathscr{E}$ and $\mathscr{F}$.

Let us now choose as $\mathscr{E}$ the basis $\tilde{\mathscr{E}}$ formed by the eigenvectors $\tilde{e}_i$ of W_1; the corresponding eigenvalues will be denoted as $w_i^{(1)}$ and ψ becomes

$$\psi = \sum_{i,j} \beta_{ij} \varphi(\tilde{e}_i, f_j).$$

Then (6.10a) yields

$$\sum_{=1}^{\dim \mathscr{H}_2} \beta_{kj} \bar{\beta}_{ij} = \delta_{ik} w_i^{(1)}$$

and, since the basis $\mathscr{F}$ is arbitrary, we see that the left-hand-side expression does not depend on $\mathscr{F}$.

Suppose $w_i^{(1)} > 0$ (for given i) so that at least one of the β_{ij}'s ($j = 1, 2, \ldots, \dim \mathscr{H}_2$) is non-zero. Then the unit vector

(6.10c) $$\tilde{f}_i = \frac{1}{\sqrt{w_i^{(1)}}} \sum_j \beta_{ij} f_j$$

satisfies

$$W_2 \tilde{f}_i = \frac{1}{\sqrt{w_i^{(1)}}} \sum_j \beta_{ij} W_2 f_j = \frac{1}{\sqrt{w_i^{(1)}}} \sum_{j,k,l} \beta_{ij} \beta_{lk} \bar{\beta}_{lj} f_k = w_i^{(1)} \tilde{f}_i .$$

Thus each non-zero eigenvalue of W_1 is simultaneously an eigenvalue of W_2, and consequently the number of non-zero eigenvalues of W_1 does not exceed $\dim \mathscr{H}_2$. If we interchange the roles of W_1 and W_2 we conclude that $\mathscr{D}(W_1) = \mathscr{D}(W_2)$ (by $\mathscr{D}(W)$ the set of eigenvalues of W is denoted). These common eigenvalues will be denoted as w_i. Further the number n of mutually orthogonal eigenvectors belonging to non-zero w_i's obeys

(6.11a) $$n \leq \min (\dim \mathscr{H}_1, \dim \mathscr{H}_2)$$

and it holds

(6.11b) $$\sum_{i=1}^{n} w_i = 1.$$

The orthonormal set $\{\tilde{f}_i\}_{i=1}^n$ can be completed to a basis $\tilde{\mathscr{F}}$ in $\mathscr{H}_2$. Denoting by E_i and F_i ($i = 1, 2, \ldots, n$) the projections on the one-dimensional subspaces spanned by the vectors $\tilde{e}_i$ and $\tilde{f}_i$, respectively, we can write

(6.11c) $$W_1 = \sum_{i=1}^{n} w_i E_i \, , \quad W_2 = \sum_{i=1}^{n} w_i F_i \, .$$

Let us finally express the vector ψ in the basis $\varphi(\tilde{\mathscr{E}} \times \tilde{\mathscr{F}})$:

$$(\psi, \varphi(\tilde{e}_k, \tilde{f}_l)) = \sum_{i,j} \beta_{ij}\delta_{ik}(f_j, \tilde{f}_l)_2 = \delta_{kl} \sqrt{w_l}\,,$$

which gives

$$\psi = \sum_{i=1}^{n} \sqrt{w_i}\, \varphi(\tilde{e}_i, \tilde{f}_i)\,. \tag{6.11d}$$

Notice that n is greater than unity; otherwise $w_1 = 1, w_i = 0$ for $i > 1$ and $\psi = \varphi(\tilde{e}_1, \tilde{f}_1)$, i.e. $\mathbf{W} = E_1 \otimes F_1$, which contradicts to the starting assumption about $\mathbf{W}$. Thus $n \geq 2$, and consequently both the reduced states are mixed. Notice further that the states (6.11c) provide another illustration of the possibility $\mathscr{W}_r(\mathbf{W}) = \mathscr{W}_r(\mathbf{W}')$ for $\mathbf{W} \neq \mathbf{W}'$.

The above discussion can be summarized as follows:

(iv) If $\mathbf{W}$ is a pure state of $\mathbf{S}$ which cannot be written in the form $W_1 \otimes W_2$, then there is a set of positive numbers $\{w_i\}_{i=1}^{n}$, where $2 \leq n \leq \infty$, orthonormal bases $\tilde{\mathscr{E}} = \{\tilde{e}_i\}_{i=1}^{\dim \mathscr{H}_1}, \tilde{\mathscr{F}} = \{\tilde{f}_i\}_{j=1}^{\dim \mathscr{H}_2}$ and corresponding sets of projections $\{E_i\}_{i=1}^{\dim \mathscr{H}_1}$, $\{\mathscr{F}_j\}_{j=1}^{\dim \mathscr{H}_2}$ so that formulae (6.11a) — (6.11d), representing the *normal form* of reduction, hold. Both the reduced states (6.11c) are mixed.

Example 6.3: Consider again the joint system from Example 6.2. The pure states $\mathbf{E}_{1,\pm 1}$ are of the form $W_1 \otimes W_2$ and by (iii) they reduce to $W_r = E_\pm$. On the other hand, the singlet state $\mathbf{E}_s$ provides an example of a pure state which is not of the form $W_1 \otimes W_2$. This state can be represented by the vector f_{00} whose components with respect to the „uncoupled" basis are $\alpha_{++} = \alpha_{--} = 0, \alpha_{+-} = -\alpha_{-+} = 1/\sqrt{2}$. Using (6.10a), we get for $W_r^{(s)} = \mathscr{W}_r(\mathbf{E}_s)$:

$$W_r^{(s)} e_\pm = \frac{1}{2}\, e_\pm\,,$$

i.e. $e_\pm$ are eigenvectors of $W_1^{(s)}$ and $W_1^{(s)} = 1/2\, I$. Thus $\alpha_{ij} = \beta_{ij}\,, w_1 = w_2 = 1/2$ and (6.10c) yields

$$\tilde{f}_1 = e_+\,, \qquad \tilde{f}_2 = -e_-\,.$$

Again $F_1 = E_+\,,\ F_2 = E_-$ and $W_2^{(s)} = 1/2\, I$. Finally

$$f_{00} = \frac{1}{\sqrt{2}}\, [\varphi(e_+, \tilde{f}_1) + \varphi(e_-, \tilde{f}_2)]\,.$$

For $\mathbf{E}_{10}$ the same reduced states are obtained. Reduction of the mixed state $\mathbf{W}_t = 1/3\, \mathbf{E}_t$ can be then performed by means of (6.9a, b):

$$\mathscr{W}_r(\mathbf{W}_t) = \frac{1}{6}\, I + \frac{1}{3}\, (E_+ + E_-) = \frac{1}{2}\, I\,.$$

A summary of (i)—(iv) is given in the following table:

1) W_1, W_2 are given and $\mathbf{W}$ is searched which satisfies $\mathscr{W}_r(\mathbf{W}) = W_r$, $r = 1,2$.	
a) W_1, W_2 are both mixed	$\mathbf{W}$ is not unique and can be either mixed or pure
b) one of the W_r's is pure	$\mathbf{W}$ is unique (equal to $W_1 \otimes W_2$) and mixed
c) W_1, W_2 are both pure	$\mathbf{W}$ is unique (equal to $W_1 \otimes W_2$) and pure
2) $\mathbf{W}$ is given (the reduced states $\mathscr{W}_r$ ($\mathbf{W}$) are denoted as W_r, $r = 1,2$).	
a) $\mathbf{W}$ is mixed	at least one of W_r's is mixed
b) $\mathbf{W}$ is pure, $\mathbf{W} = \widetilde{W}_1 \otimes \widetilde{W}_2$	W_1, W_2 are both pure, $W_r = \widetilde{W}_r$
c) $\mathbf{W}$ is pure but not of the form $W_1 \otimes W_2$	W_1, W_2 are both mixed; normal form of reduction — formulae (6.11a-d)

6.3 TIME EVOLUTION

In the previous considerations no observable was of special importance. Now we shall be interested in *energy operators* (*Hamiltonians*) of the considered systems; they will be denoted as H_r, $r = 1, 2$, and $\mathbf{H}$. The subsystems S_1, S_2 of S are *non-interacting* if $\overline{\mathbf{H}_1 + \mathbf{H}_2} = \mathbf{H}$ and *interacting* in the opposite case. However, usually the Hamiltonian of a joint system is expressed as

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_{int} .$$

If $\mathbf{H}_{int} = 0$ then $\mathbf{H}$ is e.s.a. (see Theorem 7) and therefore has a unique self-afjoint extension. However, the presence of non-zero $\mathbf{H}_{int}$ can cause serious troubles, because often we neither know whether a common dense domain for $\mathbf{H}_1 + \mathbf{H}_2$ and $\mathbf{H}_{int}$ exists. We can verify that $\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_{int}$ is e.s.a. in special cases only. If we are not able to perform such a verification we can assume that both H_r and $\mathbf{H}$ are bounded, i.e. that an energy cut-off exists.

Theorem 10 states that the evolution operator $\mathbf{U}(t)$ of the joint system S is of the form

$$\mathbf{U}(t) = (U_1 \otimes U_2)(t) \tag{6.12}$$

(see Lemma 5.7) if and only if the subsystems S_1, S_2 are non-interacting. The relation between time evolution of a system and time evolution of its subsystems is then simple. On the other hand, in a lot of physically interesting cases the subsystems interact. The general relation between the time evolution operators is then complicated; however, we are not always interested in it*). One can, of course, to any

*) Suppose as an example that the systems S_1, S_2 are two interacting particles. If we shall study *the bound states* of the joint system S, then the time evolution $\mathbf{U}(t)$ of S is simple (as for stationary states) and we are usually not interested in $U_r(t)$. On the other hand, studying *scattering* of S_1 on S_2 we are ussually interested in both $\mathbf{U}(t)$ and $\mathbf{U}_r(t)$, however, in different time regions.

state $\mathbf{W}(t) = \mathbf{U}(t)\, \mathbf{W}(0)\, \mathbf{U}^{+}(t)$ determine the reduced states $W_r(t) = \mathscr{W}_r(\mathbf{W}(t))$, but even if we find an operator $\tilde{U}_r(t)$ such that $W_r(t) = \tilde{U}_r(t) W_r(0)\, \tilde{U}_r^{-1}(t)$ for all t, it may happen that

a) $\tilde{U}_r(t)$ depends on the state $\mathbf{W}$,

b) $\tilde{U}_r(t)$ has not the group property,

c) $\tilde{U}_r(t)$ is not unitary.

Let a system, whose time evolution is described by $U(t)$, be at $t = t_0$ in a mixed state $W(t_0)$, so that $W^2(t_0) \neq W(t_0)$. It follows from (6.3a) that the same holds for $W(t)$, i.e. that the mixed state of a system remains mixed during the time evolution*). This statement *is not valid* if one considers the time evolution of a reduced state W_r; if the subsystems interact then the time evolution of such a reduced state is determined by the whole joint system and not only by the corresponding subsystem itself. The reduced mixed state can evolve (see c) above) into a pure state and vice versa. Notice that in the case when the joint system is in a pure state then both reduced states are mixed or pure simultaneously (see the table); if both the reduced states become pure at $t = \bar{t}$ then $\mathbf{W}(\bar{t}) = W_1(\bar{t}) \otimes W_2(\bar{t})$.

Example 6.4: Consider the joint system of Examples 6.2, 6.3. Let $\mathbf{H}_1 = \mathbf{H}_2 = 0$ and $\mathbf{H} = \varepsilon\,\Sigma$, $\Sigma = 1/2\,(\sigma_2 \otimes \sigma_1 - \sigma_1 \otimes \sigma_2)$.**) We shall firstly prove that

$$\mathbf{U}(t) = \exp(-i\mathbf{H}t) = \mathbf{E}^{(1)} + \mathbf{E}^{(0)} \cos \varepsilon t - i\,\Sigma \sin \varepsilon t\,,$$

where $\mathbf{E}^{(1)} = \mathbf{I} - \mathbf{E}^{(0)} = 1/2\,(\mathbf{I} + \sigma_3 \otimes \sigma_3)$. One easily verifies that the relations

$$\mathbf{E}^{(1)}\mathbf{E}^{(0)} = 0\,, \quad \mathbf{E}^{(0)}\,\Sigma = \Sigma\,\mathbf{E}^{(0)} = \Sigma\,;$$

(*)

$$\mathbf{E}^{(1)}\,\Sigma = \Sigma\,\mathbf{E}^{(1)} = 0\,, \quad \Sigma^2 = \mathbf{E}^{(0)}$$

hold. Clearly $\mathbf{U}(0) = \mathbf{I}$, $\mathbf{U}^{+}(t) = \mathbf{U}(-t)$ for any $t \in \mathbf{R}$, and further the relations (*) imply

$$\mathbf{U}(t)\,\mathbf{U}(s) = \mathbf{E}^{(1)} + \mathbf{E}^{(0)}(\cos \varepsilon t \cos \varepsilon s - \sin \varepsilon t \sin \varepsilon s) - \\ - i\,\Sigma\,(\sin \varepsilon t \cos \varepsilon s + \cos \varepsilon t \sin \varepsilon s) = \mathbf{U}(t+s)$$

for any $t, s \in \mathbf{R}$ and

$$\mathbf{U}^{+}(t)\,\mathbf{U}(t) = \mathbf{U}(-t)\,\mathbf{U}(t) = \mathbf{U}(0) = \mathbf{I}\,.$$

Continuity of $\mathbf{U}(t)$ is obvious. Thus $\mathbf{U}(t)$ is a SCOPUG on $\mathscr{H}$; one obtains with the use of (5.13) that its generator equals $-\varepsilon\,\Sigma = -\mathbf{H}$.

Consider now the states $\mathbf{W}^{(\pm)}(0) = 1/2\,\mathbf{E}^{(0)} \pm (\mathbf{E}_s - 1/2\,\mathbf{E}^{(0)})$ of S , i.e.

$$\mathbf{W}^{(+)}(0) = \mathbf{E}_s\,, \quad \mathbf{W}^{(-)}(0) = \mathbf{E}^{(0)} - \mathbf{E}_s = \mathbf{E}_{10}\,.$$

*) Also the quantity Tr $W^2(t)$ is conserved, which characterizes „how much the state $W(t)$ is mixed".

**) Such a Hamiltonian could be obtained e.g. from $\mathbf{H}_{int} = \text{const}\ \vec{H} \cdot [\vec{s}^{(1)} \times \vec{s}^{(2)}]$, where $\vec{s}^{(r)}$ are spin operators of the r-th electron and $\vec{H}$ is intensity of an external magnetic field, if this field is homogenous and its intensity is paralel to the third axis. This interaction Hamiltonian is not realistic, however, it is convenient for illustrating of the above statesments.

We shall use the following relations

$$\mathbf{E}^{(1)}\mathbf{E}_s = \mathbf{E}_s\mathbf{E}^{(1)} = 0\,, \quad \mathbf{E}^{(1)}\mathbf{E}_{10} = \mathbf{E}_{10}\mathbf{E}^{(1)} = 0\,,$$

$$[\,\Sigma, \mathbf{E}_s] = \frac{i}{2}(\sigma_3 \otimes I - I \otimes \sigma_3) = [\mathbf{E}_{10}, \Sigma]\,,$$

$$\Sigma\, \mathbf{E}_s\, \Sigma = \mathbf{E}_{10}\,, \quad \Sigma\, \mathbf{E}_{10}\, \Sigma = \mathbf{E}_s\,,$$

which can be checked easily with the help of (6.7), (6.8), (*) and properties of the Pauli matrices. Using these relations we obtain

$$\mathbf{W}^{(+)}(t) = \mathbf{U}(t)\, \mathbf{E}_s \mathbf{U}^+(t) =$$

$$= \mathbf{E}_s \cos^2 \varepsilon t + \mathbf{E}_{10} \sin^2 \varepsilon t + \frac{1}{4}(\sigma_3 \otimes I - I \otimes \sigma_3) \sin 2\varepsilon t$$

and similarly for $\mathbf{W}^{(-)}(t)$, which gives

$$\mathbf{W}^{(\pm)}(t) = \frac{1}{2}\mathbf{E}^{(0)} \pm \frac{1}{2}(\mathbf{E}_s - \mathbf{E}_{10}) \cos 2\varepsilon\, t \pm \frac{1}{4}(\sigma_3 \otimes I - I \otimes \sigma_3) \sin 2\varepsilon t\,.$$

It holds

$$(**) \qquad \mathbf{W}^{(-)}(t) = \mathbf{W}^{(+)}\left(t + \frac{\pi}{2\varepsilon}(2k + 1)\right)$$

for every integer k and any $t \in \mathbf{R}$. Hence it is sufficient to find the reduced states e.g. for $\mathbf{W}^{(+)}(t)$. This state is pure and can be represented by the vector

$$f(t) = \mathbf{U}(t) f_{00} = f_{00} \cos \varepsilon t + f_{10} \sin \varepsilon t\,,$$

whose components with respect to the „uncoupled" basis are $\alpha_{++}(t) = \alpha_{--}(t) = 0$, $\alpha_{+-}(t) = (1/\sqrt{2})(\cos \varepsilon t + \sin \varepsilon t)$ and $\alpha_{+-}(t) = (1\sqrt{2})(-\cos \varepsilon t + \sin \varepsilon t)$. The formulae (6.10a, b) then give

$$W_1^{(+)}(t) = \frac{1}{2}(1 + \sin 2\varepsilon t)\, E_+ + \frac{1}{2}(1 - \sin 2\varepsilon t)\, E_-$$

and

$$W_2^{(+)}(t) = \frac{1}{2}(1 - \sin 2\varepsilon\, t)\, E_+ + \frac{1}{2}(1 + \sin 2\varepsilon\, t)\, E_-.$$

With the help of (*) we obtain an analogous result for $\mathbf{W}^{(-)}(t)$ so that we can write

$$W_1^{(\pm)}(t) = \frac{1}{2}(1 \pm \sin 2\varepsilon t)\, E_+ + \frac{1}{2}(1 \mp \sin 2\varepsilon t)\, E_- = W_2^{(\pm)}(-t)\,.$$

We see that though the states $\mathbf{W}_{(\pm)}(t)$ of $\mathbf{S}$ are pure, the „purity" of the reduced states $W_r^{(\pm)}(t)$ changes during the time evolution: they are simultaneously pure at $t = (\pi/4\varepsilon)(2k + 1)$, k integer, otherwise they are mixed. For the „rate of purity" we obtain

$$\mathrm{Tr}\, [W_r^{(\pm)}(t)]^2 = \frac{1}{4}(3 - \cos 4\varepsilon t)\,;$$

at $t = \pi/2\varepsilon\, k$, k integer, both W_r's are „maximally mixed". Moreover, no matrix $\tilde{U}_r(t)$ could be found such that $W_r^{(\pm)}(t) = \tilde{U}r(t)\, W_r^{(\pm)}(0)\, \tilde{U}_r^{-1}(t)$ — remember that $W_r^{(\pm)}(0) = 1/2\, I_r$.

6.4 SOME OTHER APPLICATIONS

Second quantization: To a system of n identical particles, each of them having a state Hilbert space $\mathscr{H}$, we ascribe a realization space $\mathscr{H}^n$ of the n-fold tensor product $\mathscr{H} \otimes \mathscr{H} \otimes \ldots \otimes \mathscr{H}$. In fact, if these particles are bosons (fermions), their states belong to $S_n\mathscr{H}^n\,(A_n\mathscr{H}^n)$ — see Example 2.3. Let A be an observable of a single particle, then *the one-particle observable* $\mathbf{A}^{(n)}$ *is defined as*

$$\mathbf{A}^{(n)} = \mathbf{A}_1 + \mathbf{A}_2 + \ldots + \mathbf{A}_n$$

(see 4.1a). The operator $\mathbf{A}^{(n)}$ is e.s.a. (see Remark 1 to Theorem 7) and so are the restrictions $\mathbf{A}^{(n)} \restriction S_n\mathscr{H}^n$, $\mathbf{A}^{(n)} \restriction A_n\mathscr{H}^n$, since $\mathbf{A}^{(n)}$ commutes with S_n, A^n. Analogously one can define two-particle observables etc.

If we consider further a system of non-interacting particles, where the number of particles is not conserved, we obtain *a free quantum field.* The state Hilbert space of a boson (fermion) field is then the Fock space $\mathscr{F}_s(\mathscr{H})\,(\mathscr{F}_a(\mathscr{H}))$. One can construct again *the one-particle observables*

$$\mathbf{A}_f = \sum_{n=1}^{\infty} \mathbf{A}^{(n)}$$

with the domain consisting of all vectors such that:

(a) their „components" in $\mathscr{H}^i$ belong to $\mathsf{D}(\mathbf{A}^{(i)})$, $i = 1, 2, \ldots, N$,

(b) their „components" in $\mathscr{H}^{N+1}, \mathscr{H}^{N+2}, \ldots$ are zero for some integer N.

It can be proved (see [3]) that such operator $\mathbf{A}_f$ is e.s.a., and that the same holds for its restrictions to $\mathscr{F}_s(\mathscr{H}), \mathscr{F}_a(\mathscr{H})$. If for example H (on $\mathscr{H}$) is a Hamiltonian of a free boson, then $\mathbf{H}_f \restriction \mathscr{F}_s(\mathscr{H})$ represents the Hamiltonian of the corresponding free boson field.

One usually uses the occupation number representation of Fock-space vectors and expresses the field operators by means of the creation and annihilation operators. This second quantization method is commonly known; for its detailed discussion see e.g. [1], [19].

Symmetries: Let G be a symmetry group of both systems S_r, i.e. let a unitary representation $U_r(.)$ of G be realized on $\mathscr{H}_r$. Then G is a symmetry group for the joint system S as well: $(U_1 \otimes U_2)(.)$ forms the representation of G on $\mathscr{H}$. This representation is in general reducible, even if the U_r's are irreducible. Reduction of this representation is usually of special interest; let us remind the well-known coupling of angular momenta or baryonic multiplets in the quark model (see e.g. [20]).

Separation of variables: One usually tries to simplify solving of the Schrödinger equation (and other equations as well) by separating the variables. This procedure can be described briefly as follows: we have the equation $(\boldsymbol{H} - \lambda \boldsymbol{I})\psi = 0$ on $\mathscr{H}$ and we look for appropriate Hilbert spaces $\mathscr{H}_r$, $r = 1, 2$, and operators H_r on $\mathscr{H}_r$ so that $\mathscr{H}_1 \otimes \mathscr{H}_2$ is realized in $\mathscr{H}$ and $\boldsymbol{H} = \overline{\boldsymbol{H}_1 + \boldsymbol{H}_2}$. Solving of the equations $(H_r - \lambda_r I_r)\psi_r = 0$ is often easier than that of the original one; then the spectrum, eigenvectors and other characteristics of the operator $\boldsymbol{H}$ are simply obtained with the help of the results of Section 5.

Appendix

We shall give here a proof of statement (i) of Section 6.2. Let W_r $(r = 1, 2)$ be states of subsystems S_r of a joint system S, whose state Hilbert space serves as a realization space for $\mathscr{H}_1 \otimes \mathscr{H}_2$, and let at least one of these states, say W_2, be pure. Further, let $\boldsymbol{W}$ be a state of S such that the corresponding reduced states $\mathscr{W}_r(\boldsymbol{W})$ are just W_r. Denote by $\mathscr{E} = \{e_i\}_{i=1}^{\dim \mathscr{H}_1}$ and $\mathscr{F} = \{f_j\}_{j=1}^{\dim \mathscr{H}_2}$ orthonormal bases formed by the eigenvectors of W_1 and W_2, respectively, by $\{E^{(i)}\}_{i=1}^{\dim \mathscr{H}_1}$ and $\{F^{(j)}\}_{j=1}^{\dim \mathscr{H}_2}$ the corresponding sequences of rank-one projections and let $W_2 f_j = \delta_{j1} f_1$ so that $W_2 = F^{(1)}$. Using (6.5) for $\boldsymbol{A}_2 = \boldsymbol{F}_2^{(j)} = I_1 \otimes F^{(j)}$ we find

(A.1a)
$$\delta_{j1} = \operatorname{Tr} W_2 F^{(j)} = \operatorname{Tr} \boldsymbol{W} \boldsymbol{F}_2^{(j)} = \\ = \sum_{i,l} (\boldsymbol{W}\varphi(e_i, F^{(j)} f_l), \varphi(e_i, f_l)) = \sum_{i} \boldsymbol{W}_{ij,\, ij}$$

where

$$\boldsymbol{W}_{ij,\, kl} = (\boldsymbol{W}\varphi(e_k, f_l), \varphi(e_i, f_j)) .$$

Since $\boldsymbol{W}$ is positive, (A.1a) implies

(A.2) $\quad \boldsymbol{W}_{ij,\, ij} = 0$ for $i = 1, 2, \ldots, \dim \mathscr{H}_1$, $j = 2, 3, \ldots, \dim \mathscr{H}_2$

Similarly we obtain for $\boldsymbol{A}_1 = \boldsymbol{E}_1^{(i)} = E^{(i)} \otimes I_2$:

(A.1b)
$$\sum_j \boldsymbol{W}_{ij,\, ij} = \boldsymbol{W}_{i1,\, i1} = \operatorname{Tr} W_1 E^{(i)} = w_i ,$$

where the w_i's are eigenvalues of W_1, and hence

$$\boldsymbol{W}_{ij,\, ij} = (W_1 \otimes W_2)_{ij,\, ij} .$$

Let λ_s and x_s, $s = 1, 2, \ldots, \dim \mathscr{H}$, be eigenvalues and corresponding eigenvectors of $\boldsymbol{W}$ so that $\{x_s\}_{s=1}^{\dim \mathscr{H}}$ is an orthonormal basis in $\mathscr{H}$. Denoting

$$(\varphi(e_i, f_j), x_s) = \alpha_{ij}^{(s)} ,$$

and taking into account that $\boldsymbol{W}$ is continuous, we get

$$\boldsymbol{W}\varphi(e_k, f_l) = \sum_s \alpha_{kl}^{(s)} \lambda_s x_s ,$$

i.e.

(A.3)
$$\boldsymbol{W}_{ij,kl} = \sum_s \alpha_{kl}^{(s)} \overline{\alpha_{ij}^{(s)}} \lambda_s .$$

Now $\lambda_s \geq 0$; then (A.2) implies

$$\alpha_{ij}^{(s)} = 0 \quad \text{if} \quad j \neq 1 \quad \text{and} \quad \lambda_s > 0 .$$

Substituting into (A.3) we have

(A.4) $\qquad \mathbf{W}_{ij,kl} = 0 \qquad$ if at least one of j, l differs from unity.

With the help of (A.1b) and (A.4) we can write (6.5) for any A_1 as follows

(A.5)
$$\sum_{\substack{i,k \\ i \neq k}} (A_1)_{ki} \overline{\mathbf{W}_{k1,i1}} = 0 .$$

Since

$$\sum_{k,i} |\mathbf{W}_{k1,i1}|^2 = \sum_{k,i,j,l} |\mathbf{W}_{kl,ij}|^2 \doteq \mathrm{N}^2(\mathbf{W}) \leq 1 ,$$

it follows that the matrix $(A'_{ki}) = (\mathbf{W}_{k1,i1})$ together with $\mathscr{E}$ defines a bounded operator A' on $\mathscr{H}_1$ such stat

$$A'x = \sum_k (\sum_i A'_{ki} \xi_i) e_k$$

for every $x \in \mathscr{H}_1$, $x = \sum_i \xi_i e_i$; the norm of A' satisfies $\|A'\| \leq \mathrm{N}(\mathbf{W})$. Further A' is Hermitian, because $\mathbf{W}$ is so. Therefore we can set $A_1 = A'$ in (A.5), which yields

$$\mathbf{W}_{k1,,i1} = 0 \quad \text{for all} \quad i, k = 1, 2, \ldots, \dim \mathscr{H}_1 \, , \ i \neq k \, .$$

These conditions, together with (A.1b) and (A.4), imply $\mathbf{W} = W_1 \otimes W_2$ and therefore only this $\mathbf{W}$ satisfies conditions (6.5).

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