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A Phenomenological Non-relativistic Two-nucleon Potential

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The *JINR* and Livermore two-nucleon phase shift data have been simultaneously fitted for all orbital states $j \leq 5$ by a phenomenological two-nucleon potential, the radial part of which is represented as a sum of Yukawa type terms. The potential contains central, spin-orbit and tensor components and is dependent on the total angular momentum. The χ^2 test gives χ^2 per number of degrees of freedom less than 1.63. The corresponding scattering lengths and the deuteron main characteristics are reproduced.

1. Introduction

The history of the two-nucleon phenomenological potential is well known and has been described in references [1], [2]. There have been several relatively successful attempts to derive the realistic two-nucleon potential, a non-relativistic quantity, which reproduces the experimental two-nucleon data.

The first class of realistic potentials was obtained under the assumption that the two-nucleon interaction contained a hard core — a repulsive and unpenetrable domain. The Hamada-Johnston[3] and Yale [4] potentials and their modifications belong to this type and are still widely used.

Some difficulties in explaining the properties of nuclear matter lead to the conclusion, that the hard core was too strong and the idea of a soft core two nucleon interaction was accepted as another possibility. The Reid potential [5] is a member of this family.

It had already been shown in 1966 [6] that most experimental two-nucleon phase shifts could be described on the basis of a simple assumption, according to which the radial part of the two-nucleon potential was represented as a sum of Yukawa terms. The repulsive soft core type interaction in the $^1S_0$ state and in some other states was then obtained automatically in the fitting process. This idea was then followed in a series of papers where it was shown that the Yukawa type interaction, which is interesting also from the theoretical point of view, is a suitable representation of the two-nucleon forces.

In the early studies of the two-nucleon problem attempts were made to obtain the potential in the frame of the strong interaction theory. Until now, the exact form of it has not been derived. The resulting potential is dependent on the starting principle, on approximations used during the calculation and on a relatively large number of not very well known coupling constants and masses of intermediary particles. They should be obtained by fitting the experimental two-nucleon data. Therefore this type of the potential does not lie very far from the phenomenological. Usually it is highly singular [7] and corresponding cuts are necessary if it is used for nuclear calculations.

The last group consists of the non-local or velocity dependent potentials, in which the radial non-locality is essential. In some cases they can be considered akin to the hard core potentials [8]. Although there are some theoretical indications, that the two-nucleon potential should be velocity dependent [9] this type is not commonly chosen for nuclear calculations because of real or expected mathematical complications. There is one exception — the separable Tabakin potential [10], which on the contrary was specifically constructed in a form suitable for nuclear computations. Its parameters can be determined fairly well and it can be considered as a realistic potential.

When the phenomenological approach is accepted, the question of the uniqueness of the potential naturally arises. It was proved by Agranovich and Marchenko [11] that the nuclear potential $V(x)$ is uniquely determined if the integral

$$\int_0^\infty x \mid V(x) \mid dx \tag{1.1}$$

converges, the unitary scattering matrix $S(k)$ is known for all real momenta k for given (fixed) orbital momentum or total angular momentum and the bound state energy and the so called normalization constants are given. On the other hand it was shown by Newton and by Sabatier [12] that the two-nucleon potential is unique if the S matrix is known for all real orbital momenta at a given (fixed) energy and if the potential decreases more rapidly than $x^{-3/2}$ for large x.

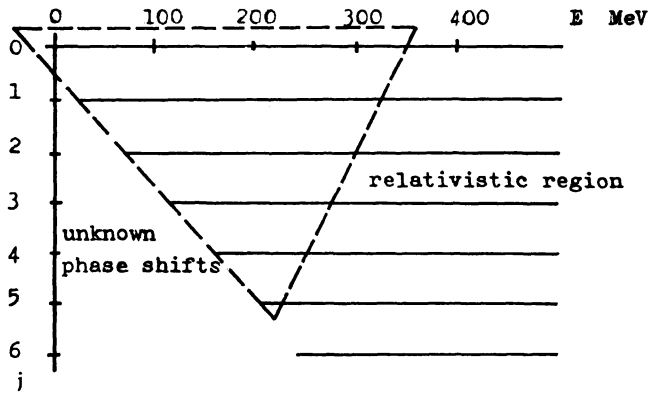

Fig. 1. The phase shifts for $j \geq 6$ are described by the one pion exchange interaction

In both cases it is assumed that the potential is a non relativistic quantity. This is certainly neither true for the energies higher than 300 MeV nor for large orbital momenta, where the notion of a potential is at least problematic. Hence, all these theorems and proofs have limited validity. The situation is made worse by the experimental fact, that the phase shifts for high orbital momenta and low energies are not known and cannot be obtained because of the small size. Schematically the basis on which the potential should be obtained is shown in Fig. 1. In the total angular momentum ($\mathcal{J}$)-energy (E) plane the full lines inside the dashed triangle represent the scattering data which can be seriously used for the derivation of a non-relativistic phenomenological two-nucleon potential.

So far realistic two-nucleon potentials have been established individually for given orbital states or for the few lowest orbital states. The fact, that the potential, if it exists, must lead to strong correlations among the phase shifts for different orbital momenta, has not been used in practice, although the idea of such correlations has already been successfully exploited in the Regge-Watson method of complex angular momenta.

The aim of this paper is to derive a two nucleon potential by the simultaneous use of known phase shifts inside the triangle — i.e. in the whole non-relativistic region together with the deuteron data and through this procedure eliminate at least partially ambiguities in the potential.

The computation has been based on three main assumptions:

A1. the potential is independent on "radial/velocity",
A2. the potential is linear in $\mathcal{J}^2$, where $\mathcal{J}$ is the total angular momentum,
A3. the potential has a Yukawa type structure.

II. The Representation of the Two-nucleon Potential

The theoretical form of the nonrelativistic two-nucleon potential is very well known [13]. Here, a slightly different representation is given.

The general scalar, isoscalar, time invariant and Hermitian potential operator satisfying A1 has the form

$$V = V_o + V_1 a + V_2 b + V_3 c + V_4 a^2 \tag{2.1}$$

where

$$V_i = U_i(r, \mathcal{J}^2) + \tau\, W_i(r, \mathcal{J}^2)$$
$$U_i = U_i^+ \,, \quad W_i = W_i^+$$

$$a \equiv (\vec{s} \,.\, \vec{L}) + c \,, \qquad b \equiv 2\,\frac{(\vec{r} \,.\, \vec{s})^2}{r^2} - c \,, \qquad c \equiv \frac{s^2}{2}$$

$$\tau \equiv I^2 \tag{2.2}$$

where $r = |\vec{r}|$ etc., r is the actual internucleon distance, $\vec{\mathcal{J}} = \vec{L} + \vec{s}$, $\vec{\mathcal{J}}$ is the total angular momentum, $\vec{L}$ the angular momentum, $\vec{s}$ the total spin defined by

$$\vec{s} = \frac{1}{2}\,(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) \tag{2.3}$$

$\vec{\sigma}^{(i)}$ are the spin matrices of the individual nucleons. The operator a contains the spon-orbit term, the operator b is connected with the S_{12} term known from the expression for the tensor force

$$S_{12} = 3b - c = 3(\vec{\sigma}^{(1)} \cdot \vec{r})(\vec{\sigma}^{(2)} \cdot \vec{r}) \frac{1}{r^2} - (\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) \tag{2.4}$$

The operator $\vec{I}$ is the isospin.

The *l-s-j* representations of the operators a, b and c are:

$$\text{singlet state, } j = l = 0, 1, 2, \ldots, \text{ (one row)}$$

$$a = b = c = 0$$

$$\text{triplet uncoupled state, } j = l = 1, 2, 3, \ldots \text{ (one-row)}$$

$$a = 0, b = c = 1 \tag{2.5}$$

$$\text{triplet coupled state } j = l \pm 1, j = 1, 2, 3, \ldots \text{ (two-row representation)}$$

$$a = \begin{pmatrix} j & 0 \\ 0 & -j-1 \end{pmatrix}, \quad b = \frac{1}{2j+1} \begin{pmatrix} 1 & 2\xi \\ 2\xi & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\xi = \sqrt{j(j+1)}$$

$$\text{triplet state } {}^3P_0 \quad j = 0, \quad l = 1$$

$$a = b = -1 \qquad c = 1$$

Because the operator $c = s^2/2$ generates two mutually orthogonal idempotent elements, the potential can be uniquely split into two terms. The first term represents the potential acting in the singlet state of the two-nucleon system and the second in the triplet state. The same procedure can be applied to the decomposition of the potential in isospin space. Because such a projection can be done with every term in (2.1), the non-relativistic two nucleon potential is in fact composed of four completely independent potential terms. Hence, one has to deal with the potential in the singlet iso-triplet or singlet even state, singlet isosinglet or singlet odd state, triplet isotriplet or triplet odd state and triplet isosinglet or triplet even state separately.

Now writing the potential in the form

$$V = V_0(1 - c) + V_1 a + V_2 b + (V_0 + V_3)\, c + V_4 a^2 \tag{2.6}$$

we see that in the singlet state

$$V = V_0 \tag{2.7}$$

and in the triplet state

$$V = V_1 a + V_2 b + V_3 + V_4 a^2 \tag{2.8}$$

where $V_0 + V_3 \to V_3$ can be considered as a new independent central term.

Using the definitions (2.2) and the relation (2.4) one can easily rearrange (2.6) to give

$$V = V_c + V_{SL}(\vec{S} \cdot \vec{L}) + V_T S_{12} + V_{(SL)^2} (\vec{S}\vec{L})^2 \tag{2.9}$$

where the central, spin-orbit, tensor and quadratic spin-orbit terms are linear combinations of the initial V_i:

$$\begin{aligned} V_c &= V_0(1-c) + \left(V_1 + \frac{1}{3} V_2 + V_3 + V_4\right) c \\ V_{SL} &= (V_1 + 2V_4)\, c \\ V_T &= \frac{1}{3} V_2\, c \\ V_{(SL)^2} &= V_4 c \end{aligned} \tag{2.10}$$

Both representations (2.6) and (2.9) of the potential will be used in this paper.

When the representation of the operators a, b and c for fixed j in the coupled triplet state is taken, one can also define the channel potentials:

$$\begin{aligned} V_{j,j-1} &= j\, V_1 + \frac{V_2}{2j+1} + V_3 + j^2 V_4 \\ &= V_c + (j-1)\, V_{SL} - \frac{2(j-1)}{2j+1} V_T + (j-1)^2\, V_{(SL)^2} \qquad j \neq 0 \\ T_j &= \frac{2\sqrt{j(j+1)}}{2j+1} V_2 = \frac{6\sqrt{j(j+1)}}{2j+1} V_T \qquad j \neq 0 \\ V_{j,j+1} &= -(j+1)\, V_1 - \frac{V_2}{2j+1} + V_3 + (j+1)^2 V_4 \\ &= V_c - (j+2)\, V_{SL} - \frac{2(j+2)}{2j+1} V_T + (j+2)^2\, V_{(SL)^2} \end{aligned} \tag{2.11}$$

which are intereesting for many nuclear calculations. We shall use the notion of the channel potential also for the triplet uncoupled state, where it is given by:

$$V^{(j=l)} = V_2 + V_3 = V_c - V_{SL} + 2V_T + 4V_{(SL)^2} \qquad j \neq 0 \tag{2.12}$$

We can now write for fixed j the Schroedinger equations for the radial wave functions. In the singlet state and in the uncoupled triplet they have the form

$$\left(\frac{d^2}{dx^2} + k^2 - \frac{j(j+1)}{x^2} - \frac{1}{\varrho x} - v\right) u = 0 \qquad j = l \tag{2.13}$$

similarly in the ${}^3P_0$ state

$$\left(\frac{d^2}{dx^2} + k^2 - \frac{2}{x^2} - \frac{1}{\varrho x} - v_{0,1}\right) u = 0 \qquad j = 0 \tag{2.14}$$

and in the triplet coupled state

$$\begin{aligned} &\left(\frac{d^2}{dx^2} + k^2 - \frac{j(j-1)}{x^2} - \frac{1}{\varrho x} - v_{j,j-1}\right) u_1 - t_j u_2 = 0 \\ &\left(\frac{d^2}{dx^2} + k^2 - \frac{(j+1)(j+2)}{x^2} - \frac{1}{\varrho x} - v_{j,j+1}\right) u_2 - t_j u_1 = 0 \qquad j = 1, 2, 3, \ldots \end{aligned} \tag{2.15}$$

Here, the system of natural units is taken ($\hbar = 1$, $c = 1, f^{-1} = 197.32891$ MeV [33]), the mass of the nucleon is $m_N = 4.757\, f^{-1}$ of the pion $\mu = .707\, f^{-1}$. The quantity $x = \mu r$ where r is the actual internucleon distance, the momentum k is measured in pion mass units

$$k^2 = \frac{m_N E_{\text{rel}}}{\mu^2} = m_N \frac{E_{\text{lab}}}{2\mu^2} \tag{2.16}$$

E_{lab} is the energy of incident particle, E_{rel} is the relative energy. The potentials etc. are defined by

$$v = m_N V \frac{1}{\mu^2} \qquad t_j = m_N T_j \frac{1}{\mu^2} \tag{2.17}$$

where V stands for V_0, $V^{(j)}$ etc. When the Coulomb field is present

$$\frac{1}{\varrho} = \frac{m_N e^2}{\mu} \qquad e^2 = \frac{1}{137{,}1} \tag{2.18}$$

otherwise $1/\varrho = 0$. If the bound state is to be considered the substitution

$$-k^2 \to \varkappa^2 = m_N \frac{E_{\text{bound}}}{\mu^2} \tag{2.19}$$

where E_{bound} is the energy of the bound state, has to be performed.

III. The Two-nucleon data and the Deuteron Constants

a) The phase shifts

The *JINR* group [14] and the Livermore group [15] have presented very precise sets of the phase shift data, which almost coincide. In both cases the phase shift analysis is performed under the assumption, that the interaction of the two nucleons in higher orbital states is governed purely by one pion exchange. The maximum value of the total angular momentum j for which the interaction can deviate from the one pion exchange law is now usually taken equal to 5.

The JINR group considers the nuclear proton-proton and proton-neutron phase shifts as equivalent, the Livermore group attempted to distinguish between them at least for the $^1S_0$ state. Moreover, the Livermore group has given as well as the usual energy independent, an energy dependent analysis which has both an "experimental solution" and a "constrained solution". The latter represents a solution of some problems connected with the behaviour of phase shifts in the triplet even state (see Sec. V.f.)

Both groups use the Stapp representation [16] of the scattering matrix in the coupled triplet state. In this representation the S matrix has the form

$$S_j = \begin{pmatrix} \cos 2\varepsilon_j e^{2i\delta_{j,j-1}} & i \sin 2\varepsilon_j\, e^{i(\delta_{j,j-1}+\delta_{j,j+1})} \\ i \sin 2\varepsilon_j\, e^{i(\delta_{j,j-1}+\delta_{j,j+1})} & \cos 2\varepsilon_j\, e^{2i\delta_{j,j+1}} \end{pmatrix} \tag{3.1}$$

where $\delta_{j,j-1}$ and $\delta_{j,j+1}$ are the phase shifts and ε_j is the mixing parameter. For $\varepsilon_j = 0$

the corresponding triplet state is decoupled, for $\varepsilon = \frac{\pi}{4}$ maximum coupling occurs. It is easy to show (see Sec. IV.a) that in the case of maximum coupling the phase shifts are generally discontinuous even for real momenta k, but the sum $\delta_{j,j-1} + \delta_{j,j+1}$ remains continuous. The discontinuity is due only to the parametrisation of the scattering matrix and it does not affect the continuity of the S matrix elements on the real momentum axis. But this effect shows that the energy dependent phase shift analysis, where the notion of the continuity of the phase shifts as a function of momentum k is applied, prefers only certain type of phase shifts. That is why we only use in this study the results of the energy independent solutions of the phase shift analysis.

On the other hand the solutions of the energy independent phase shift analysis are not generally unique. Only such solutions, which are considered by the JINR or Livermore group as unique or likely to be unique are used in this paper.

The complete set of the data used is contained in the reference [34].

In the singlet even state we differentiate between *p-p* and *n-p* scattering. There are not enough reliable data on *n-n* phase shifts and therefore one cannot estimate the degree of symmetry or asymmetry of the *p-p*, *n-p* and *n-n* nuclear phase shifts.

One remarkable feature of the nucleon-nucleon phase shift analysis is the assumption that the interaction in higher orbital states (usually for $j > 5$) is described by a one pion exchange, which implies that the nucleon-nucleon interaction is dependent on the total angular momentum J. Our second assumption (A2) accepts this idea and tries to describe the dependence of the potential on the total angular momentum — such dependence is generally and implicitly contained in the potential (2.1) or (2.9) by a linear approximation in $\mathcal{J}^2$. It leads to a decomposition of every potential term $v(\mathcal{J})$ into two terms

$$v(\mathcal{J}) = v' + \mathcal{J}^2 v'' \tag{3.2}$$

where v and v′ are $\mathcal{J}^2$ independent.

b) The effective range approximation

The effective range approximation [1] and the measurement of the nucleon-nucleon scattering parameters at very low energies enable the phase shifts and their derivatives or certain functions of the phase shifts to be represented as power series in momentum k for $k \to 0$. For the ${}^1S_0$ *n-p* or *n-n* state and for $k \to 0$

$$k \cot \delta = -\frac{1}{a_s} + \frac{1}{2} r_{0,s} k^2 - P_s r_{0,s}^3 k^4 + \ldots \tag{3.3}$$

where a_s is the scattering length, $r_{0,s}$ the effective range and P_s the dimensionless shape parameter. For the ${}^1S_0$ *p-p* state and $k \to 0$

$$kC^2 \cot \delta + 2k\eta\, h(\eta) = -\frac{1}{a_s} + \frac{1}{2} r_{0,s} k^2 - P_s r_{0,s}^3 k^4 + \ldots \tag{3.4}$$

where

$$\eta = \frac{1}{2k\varrho} \qquad C^2 = 2\pi\eta(e^{2\pi\eta} - 1)^{-1}$$

$$h(\eta) = - \cdot 57721 - \ln \eta + \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + \eta^2)} \tag{3.5}$$

and for the $^3S_1$ state and $k \to 0$

$$k \cot \delta = - \frac{1}{a_t} + \frac{1}{2} r_{0,t} k^2 - P_t r_{0,t}^3 k^4 + \ldots \tag{3.6}$$

where the meaning of individual parameters is obvious.

In the effective range approximation another equation, which connects the scattering parameters and the bound state characteristics is quite often used [1]. But its validity cannot be generally proved without assuming a special analytic structure for the S matrix [12]. Therefore, let us limit the discussion only to equations (3.3)–(3.6).

When the momentum is measured in the dimensionless units (2.16) a_s and $r_{o,s}$ in (3.3)–(3.6) are given as dimensionless quantities i.e.

$$a_s = \mu(a_s)_{\text{exp}} \tag{3.7}$$

etc.

The scattering lengths are measured directly and practically at zero energy and they are known with high accuracy. There is a very interesting and not yet satisfactorily explained fact, that the a_s for p-p state and n-p state differ and there is also quite strong evidence [17] that the scattering length for $^1S_0$ n-n state is different from both of these.

If follows from (3.3)–(3.6) that the phase shift is an odd function of k, so that in all cases

$$a = - \left(\frac{\partial \delta(k)}{\partial k} \right)_{k=0} \tag{3.8}$$

holds, whereas the parameters r_0 and P are defined by much higher derivatives of the phase shifts. That is why in practice they can only be obtained at non-zero- energies and actually they are deduced as secondary quantities from such measurements. If all these quantities are known one can easily get the phase shifts for the $^1S_0$ p-p and n-p and for $^3S_1$ states at very low energies.

In complete analogy with the experimental situation the scattering lengths have been used as primary experimental points during the present calculation. Their values have been taken as averaged from the data given in references [18] and the corresponding energy has been put strictly equal to zero. The values used are collected in the Table I., where the effective range and shape parameters are also given. Using all of them the phase shifts for the low energy region in the $^1S_0$ p-p and n-p and for $^3S_1$ states were calculated and used as experimental data of the same weight as the phase shift analysis data. From the theoretically calculated phase shifts in the low energy region and from the theoretically calculated scattering lengths one can then rederive the theoretical value of the effective range and shape parameters.

Tab. I. The experimental and calculated values of the effective range parameters are presented as dimensionless quantities (see (3.7)). To get the value of a and r_0 in f, they should be divided by the pion mass $.707 f^{-1}$. The value of the shape parameter in the $^1S_0$ state is strongly dependent on the reference energies chosen for the calculation. See also[18].

		Scattering length	Effective range	Shape Parameter
$^1S_0$	exp	$-5.527 \pm .012$	$1.99 \pm .02$	$.025 \pm .010$
p-p	theor	-5.528	$2.16 \pm .09$	$.18 \pm .20$
$^1S_0$	exp	$-16.7400 \pm .0020$	$1.77 \pm .12$	$.014 \pm .042$
n-p	theor	-16.7402	$1.98 \pm .05$	$.02 \pm .01$
$^3S_1$	exp	$3.825 \pm .003$	$1.22 \pm .02$	$.055 \pm .07$
n-p	theor	3.827	$1.43 \pm .03$	$-.03 \pm .01$

Here again we meet certain indications that the nuclear interaction in p-p, n-p and n-n states may not be completely symmetric in isospace. If there is a dependence of nuclear forces on the third component of the isospin $\vec{I}$, i.e. if the Hamiltonian is not a scalar in the isospin space, then the phenomenological potential has a more complicated form than (2.1). It is evident, that in such a situation the potential can again be split uniquely in the isotriplet state with the help of the corresponding isospin projection operators. Without doing it explicitly it is possible to consider the existence of different potentials for p-p, n-p and n-n interaction [19].

c) The bound state of the two-nucleon system and the deuteron constants

The basic deuteron constants — the binding energy, the magnetic moment and the quadrupole moment — represent another set of very well known experimental data. If the potential was a purely non-relativistic quantity and the phase shifts for the triplet even state with $j = 1$ were known for all real momenta k, then the potential for this state could not be uniquely determined in the inverse problem of scattering theory, unless the binding energy and two normalization constants were added to the data [11]. In principle, when the analytic structure of the potential is known, the normalization constants are expressible as functions of the binding energy, magnetic moment and quadrupole moment of the deuteron. This means that these constants must be included in any realistic calculation of a phenomenological two-nucleon potential. The addition of the deuteron constants to the other two nucleon data may decrease the ambiguities in the potential caused by other effects.

There is no two nucleon bound state known other than the deuteron. This fact puts some limits on the phase shifts. If the condition (1.1) is fulfilled and the S matrix is known for all real momenta k (in the non-relativistic theory), then the Levinson theorem says [11] that in the singlet and uncoupled triplet state

$$\delta(k)_{k=0} - \delta(k)_{k=\infty} = n\pi \qquad n = 0, 1, 2, \ldots \tag{3.9}$$

where n is the number of possible bound states. Equation (3.9) is correct, if there is no resonance (or virtual state) for $k = 0$. For the triplet coupled state the generalised Levinson theorem gives in the Stapp representation of the S matrix (see (3.1)) in full analogy with the former case [11]

$$(\delta_{j,j-1} + \delta_{j,j+1})_{k=0} - (\delta_{j,j-1} + \delta_{j,j+1})_{k=\infty} = n\pi\,, n = 0, 1, \ldots \tag{3.10}$$

It can be shown [12] that in non-relativistic scattering theory $\delta(k)_{k=\infty} \to 0$. It follows then from (3.9) that in the singlet case and in the triplet uncoupled state

$$\delta(0) = 0 \tag{3.11}$$

must hold in the two-nucleon problem and from (3.10) in the coupled triplet state of the two nucleon

$$\begin{aligned} \delta_{j,j-1}(0) + \delta_{j,j+1}(0) &= \pi \qquad j = 1 \\ \delta_{j,j-1}(0) + \delta_{j,j+1}(0) &= 0 \qquad j \neq 1 \end{aligned} \tag{3.12}$$

If we assume on the basis of experimental evidence that the ${}^3S_1$ state interaction is mainly responsible for the existence of the deuteron, then

$$\delta_{j,j-1}(0) = \pi \qquad j = 1 \tag{3.13}$$

The Levinson theorem is the only one in non-relativistic theory which connects the scattering data and the bound state parameters. Additional information about the relationship between the bound state constants and scattering data can only be obtained if the potential is known a priori or if special assumptions about the analycity of the S matrix in the complex momentum plane are made.

For the solution of the inverse problem only the three main bound state data should be taken as information additional to the phase shift data and the equations (3.11) and (3.12) should then be used to ensure the existence of the deuteron and no other bound state of the two-nucleon system.

Again we express the binding energy, the magnetic moment and the quadrupole moment in dimensionless units. For the binding energy the experimental value [20] $E_{\text{bound}} = (2.2247 \pm .0001)$ MeV is taken. It gives (see (2.19))

$$\varkappa = \sqrt{\frac{m_N E_{\text{bound}}}{\mu^2}} = .32756 \tag{3.14}$$

The magnetic moment is usually defined in nuclear magnetons. The quantity which should be reproduced is the difference between the deuteron magnetic momentum M_d and the sum of the magnetic momentum of the proton M_p and the neutron M_n

$$\Delta M = M_d - (M_p + M_n) = .0223 \pm .0001 \text{ [21]} \tag{3.15}$$

The difference ΔM can be calculated if the two nucleon potential is known and both components u and w of the radial wave function are determined.

If

$$\frac{1}{\mu}\int_0^\infty (u^2 + w^2)\,\mathrm{d}x = 1 \tag{3.16}$$

then

$$\Delta M = -\frac{3}{2}\left((M_p + M_n - \frac{1}{2}\right)\frac{1}{\mu}\int_0^\infty w^2 \mathrm{d}x + \overline{M} \tag{3.17}$$

where the first term is known and denoted usually as

$$-\frac{3}{2}\left(M_p + M_n - \frac{1}{2}\right) P_D \tag{3.18}$$

$-P_D$ is the D state probability — and the second term arises from the transformation of the Hamiltonian when the external magnetic field is switched on. This transformation substitutes [22] in the Hamiltonian for the relative motion of the deuteron

$$H = \frac{1}{m_N} P^2 + V(\vec{r}, \vec{p}) \tag{3.19}$$

the momentum $\vec{p}$ by:

$$\vec{p} \to \vec{p} + \frac{e}{8}(\vec{r} \,.\, \vec{H}) \tag{3.20a}$$

and adds to the Hamiltonian the interaction term

$$\frac{e}{2m_N}(M_n + M_p)(\vec{s} \,.\, \vec{H}) \tag{3.20b}$$

where $\vec{H}$ is the intensity of the external magnetic field. The transformation (3.20) gives rise to the occurence of the additional term $\overline{M}$ in (3.17) if the potential is dependent on the momentum $\vec{p}$, i.e. in the representation used (see (2.2), (2.6)) on $(\vec{s} \,.\, \vec{L})$ and $\vec{J}^2$. The contributions from the individual potential terms to $\overline{M}$ are given in [34].

The quadrupole moment $Q_{\exp} = (.279 + .014) f^2$ [23] gives in dimensionless units

$$Q = \mu^2 Q_{\exp} = .139 \pm .007 \tag{3.21}$$

When the deuteron eigen-function is known Q is given by

$$Q = \frac{1}{\mu}\frac{\sqrt{2}}{10}\int_0^\infty x^2\left(uw - \frac{1}{2\sqrt{2}} w^2\right)\mathrm{d}x \tag{3.22}$$

IV. The Solution of the Inverse Problem

Several mathematical methods for the solution of the inverse problem are available. One of the very well known and often used is based on the Schwinger variation principle for the phase shifts [24] and on the usual variational approach to the solution of the bound state problem. This method, similar to the one which will be described later has to assume a general analytic form for the potential.

The next method described in detail in reference [11] calculates the potential straightforwardly if condition (1.1) is fulfilled and the S matrix known for all real momenta. Evidently, it needs an assumption to be made about the behaviour of the phase shifts at higher energies. It is difficult to suggest the non-relativistic form of the phase shifts in the relativistic region. It can be shown that a slight change of the phase shift at higher energies may change the potential considerably and give it an unphysical shape [25]. Hence the practical application of the method also needs certain additional assumptions about the analytic form of the potential, e.g. the assumption that the potential decreases more rapidly than an exponential for large x.

The method which has been used in this paper is the so called phase function or variable phase method developed simultaneously by several authors, nowadays extensively described in the case of the singlet state by Calogero [26] and in the case of the triplet state by Babikov [27]. Both authors give surveys of other related studies. This method is relatively simple for the computer, because it is based on a solution of first order differential equations. Although they are non-linear, their physical meaning is transparent. For the practical application of this method one has to assume a general analytical form of the potential.

Our third assumption (A3) concerns the analytic form of the radial part of the two-nucleon potential. First, we accept the generally recognised fact that the radial part of the two-nucleon potential behaves at large distances from the origin as a simple Yukawa potential, i.e.

$$v(x) = \text{const}\,\frac{e^{-x}}{x} \qquad x \to \infty \tag{4.1}$$

The next step is more severe. To preserve as large a mathematical simplicity for the potential as possible we take the radial part of the potential as a function for which

$$\lim_{x\to 0} xv(x) = \alpha \qquad |\alpha| < \infty \tag{4.2}$$

This condition has been used in many theoretical studies, because it allows the scattering and bound state problem to be investigated in depth. It is also supported by studies on the analytic properties of the S matrix and by the practical use of this assumption in derivations of the two-nucleon potential [6].

Weaker assumptions can be formulated in this way:

$$\lim x^n v(x) = \alpha \qquad |\alpha| < \infty \qquad n \geq 2 \tag{4.3}$$

If $n = 2$ the Schroedinger equation for the relative motion of the two nucleons remains of the Fuchs type near the origin but the singularity allowed in the potential can interfere with the action of the classical orbital term $\mathcal{J}^2/x^2$ and the physical picture deteriorates. This fact can also cause considerable difficulties in the calculations, when a variational method is used to determine the potential parameters. If $n > 2$ one meets mathematical difficulties, which in many practical cases can be overcome by a cut-off of the potential near the origin, i.e. by reducing its singularity.

Now, if the conditions (4.1) and (4.2) hold the potential can always be represented as a series

$$v(x) = \frac{1}{x} \sum_{n=1}^{\infty} a_n e^{-nx} \tag{4.4}$$

i.e. as a superposition of Yukawa potentials. The functions e^{-nx} are not orthonormal but the series can be easily rearranged to give a series of orthonormal functions. The expression (4.4) is more convenient, because it corresponds directly to the superposition of Yukawa terms and may be considered as an interaction due to one pion, two-pion etc. exchange contributions. Here we will use (4.4) as a mathematical description of the potential.

If condition (4.2) is violated by the potential which is found during the solution of the iverse problem the quantity

$$\alpha = \sum a_n \tag{4.5}$$

will indicate the degree of insufficiency of the potential (4.4).

a) The phase function method at non-zero energy.

Let us briefly recapitulate the system of equations for the phase functions. For singlet and for uncoupled triplet states in which the Coulomb field is absent the following equations:

$$\delta_e'(x) = -\frac{v(x)}{k} [\cos \delta_e(x) \, . \, j_e(kx) - \sin \delta_e(x) \, n_e(kx)]^2 \qquad l = j \tag{4.6}$$

where the phase functions have the limits

$$\lim \delta_e(x) = \delta_e \tag{4.7}$$

and where $j_e(z)$ and $n_e(z)$ are the spherical or Riccati-Bessel functions [26] are used. In contrast to the normal formulation we take for the singlet p-p state a modified form [28], in which the phase function obeys the equation

$$\delta_e'(x) = -\frac{-v(x) - \dfrac{l(l+1)+\eta^2}{x^2}}{k - \dfrac{\eta}{x}} \sin^2 [kx - \eta \ln 2kx + \sigma_e + \delta_e(x) - \pi l] + \\ + \frac{\dfrac{\eta}{x^2}}{2\left(k - \dfrac{\eta}{x}\right)} \sin 2 \left[kx - \eta \ln 2kx + \sigma_e + \delta_e(x) - \frac{\pi l}{2}\right] \tag{4.8}$$

where $\eta = 1/(2\,k\varrho)$ (see (2.18)), σ_e is the Coulomb phase shift defined by

$$e^{2i\sigma_e} = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \tag{4.9}$$

The phase function again fulfils (4.7).

Table 11

$G(u_s, u_B)$ Function

u_s	$\gamma = 4.85 \cdot 10^{+8}$		$\gamma = 1.78 \cdot 10^{+8}$		u_s	$\gamma = 4.85 \cdot 10^{+8}$		$\gamma = 1.78 \cdot 10^{+8}$	
	Y	$u_B = 20$	Y	$u_B = 19$		Y	$u_B = 20$	Y	$u_B = 19$
—40	—60	9.70037 +08	—59	9.70157 +08	1	—19	4.05669 —06	—18	6.89562 —06
—39	—59	5.88242 +08	—58	5.88361 +08	2	—18	1.53862 —06	—17	2.62061 —06
—38	—58	3.56672 +08	—57	3.56791 +08	3	—17	5.84737 —07	—16	9.98220 —07
—37	—57	2.16217 +08	—56	2.16336 +08	4	—16	2.22733 —07	—15	3.81239 —07
—36	—56	1.31027 +08	—55	1.31146 +08	5	—15	8.50659 —08	—14	1.46052 —07
—35	—55	7.93575 +07	—54	7.94764 +07	6	—14	3.25886 —08	—13	5.61568 —08
—34	—54	4.80183 +07	—53	4.81369 +07	7	—13	1.25302 —08	—12	2.16869 —08
—33	—53	2.90105 +07	—52	2.91287 +07	8	—12	4.83901 —09	—11	8.42005 —09
—32	—52	1.74824 +07	—51	1.75999 +07	9	—11	1.87876 —09	—10	3.29086 —09
—31	—51	1.04914 +07	—50	1.06077 +07	10	—10	7.34290 —10	—9	1.29696 —09
—30	—50	6.25293 +06	—49	6.36742 +06	11	—9	2.89392 —10	—8	5.16591 —10
—29	—49	3.68507 +06	—48	3.79659 +06	12	—8	1.15267 —10	—7	2.08530 —10
—28	—48	2.13224 +06	—47	2.23898 +06	13	—7	4.65293 —11	—6	8.55578 —11
—27	—47	1.19777 +06	—46	1.29695 +06	14	—6	1.90905 —11	—5	3.57424 —11
—26	—46	6.42364 +05	—45	7.29857 +05	15	—5	7.97522 —12	—4	1.51611 —11
—25	—45	3.21866 +05	—44	3.92522 +05	16	—4	3.38291 —12	—3	6.43002 —12
—24	—44	1.47908 +05	—43	1.97465 +05	17	—3	1.43473 —12	—2	2.59422 —12
—23	—43	6.23405 +04	—42	9.11761 +04	18	—2	5.78850 —13	—1	8.32672 —13
—22	—42	2.46680 +04	—41	3.85970 +04	19	—1	1.85794 —13	0	0.00000 —00
—21	—41	9.43142 +03	—40	1.53194 +04	20	0	0.00000 —00	1	—4.16423 —13
—20	—40	3.55137 +03	—39	5.86813 +03	21	1	—9.29166 —14	2	—6.51677 —13
—19	—39	1.32929 +03	—38	2.21235 +03	22	2	—1.45408 —13	3	—7.84173 —13
—18	—38	4.96521 +02	—37	8.28889 +02	23	3	—1.74972 —13	4	—8.61159 —13
—17	—37	1.85360 +02	—36	3.09881 +02	24	4	—1.92150 —13	5	—9.06792 —13
—16	—36	6.92020 +01	—35	1.15785 +02	25	5	—2.02332 —13	6	—9.34153 —13
—15	—35	2.58435 +01	—34	4.32665 +01	26	6	—2.08437 —13	7	—9.50656 —13
—14	—34	9.65524 +00	—33	1.61734 +01	27	7	—2.12120 —13	8	—9.60642 —13
—13	—33	3.60894 +00	—32	6.04858 +00	28	8	—2.14348 —13	9	—9.66691 —13
—12	—32	1.34964 +00	—31	2.26329 +00	29	9	—2.15698 —13	10	—9.70359 —13
—11	—31	5.05012 —01	—30	8.47390 —01	30	10	—2.16516 —13	11	—9.72583 —13
—10	—30	1.89078 —01	—29	3.17471 —01	31	11	—2.17012 —13	12	—9.73931 —13
—9	—29	7.08375 —02	—28	1.19021 —01	32	12	—2.17313 —13	13	—9.74750 —13
—8	—28	2.65572 —02	—27	4.46549 —02	33	13	—2.17496 —13	14	—9.75246 —13
—7	—27	9.96385 —03	—26	1.67673 —02	34	14	—2.17606 —13	15	—9.75547 —13
—6	—26	3.74130 —03	—25	6.30144 —03	35	15	—2.17674 —13	16	—9.75729 —13
—5	—25	1.40604 —03	—24	2.37045 —03	36	16	—2.17714 —13	17	—9.75840 —13
—4	—24	5.28919 —04	—23	8.92640 —04	37	17	—2.17739 —13	18	—9.75907 —13
—3	—23	1.99175 —04	—22	3.36527 —04	38	18	—2.17754 —13	19	—9.75948 —13
—2	—22	7.50894 —05	—21	1.27032 —04	39	19	—2.17763 —13	20	—9.75972 —13
—1	—21	2.83448 —05	—20	4.80201 —05	40	20	—2.17769 —13	21	—9.75987 —13
0	—20	1.07147 —05	—19	1.81808 —05					

Table 12

$G(u_s, u_B)$ Function

u_s	$\gamma = 6.57 \cdot 10^{+7}$		$\gamma = 2.41 \cdot 10^{+7}$		u_s	$\gamma = 6.57 \cdot 10^{+7}$		$\gamma = 2.41 \cdot 10^{+7}$	
	Y	$u_B = 18$	Y	$u_B = 17$		Y	$u_B = 18$	Y	$u_B = 17$
—40	—58	9.70227 +08	—57	9.70269 +08	1	—17	1.17447 —05	—16	2.00497 —05
—39	—57	5.88432 +08	—56	5.88474 +08	2	—16	4.47371 —06	—15	7.65739 —06
—38	—56	3.56862 +08	—55	3.56904 +08	3	—15	1.70859 —06	—14	2.93353 —06
—37	—55	2.16407 +08	—54	2.16449 +08	4	—14	6.54560 —07	—13	1.12794 —06
—36	—54	1.31217 +08	—53	1.31259 +08	5	—13	2.51677 —07	—12	4.35594 —07
—35	—53	7.95471 +07	—52	7.95890 +07	6	—12	9.71942 —08	—11	1.69121 —07
—34	—52	4.82075 +07	—51	4.82493 +07	7	—11	3.77360 —08	—10	6.60987 —08
—33	—51	2.91991 +07	—50	2.92409 +07	8	—10	1.47486 —08	—9	2.60502 —08
—32	—50	1.76700 +07	—49	1.77117 +07	9	—9	5.81260 —09	—8	1.03760 —08
—31	—49	1.06774 +07	—48	1.07190 +07	10	—8	2.31520 —09	—7	4.18844 —09
—30	—48	6.43648 +06	—47	6.47781 +06	11	—7	9.34567 —10	—6	1.71847 —09
—29	—47	3.86456 +06	—46	3.90551 +06	12	—6	3.83443 —10	—5	7.17907 —10
—28	—46	2.30523 +06	—45	2.34555 +06	13	—5	1.60186 —10	—4	3.04520 —10
—27	—45	1.36042 +06	—44	1.39974 +06	14	—4	6.79477 —11	—3	1.29150 —10
—26	—44	7.88943 +05	—43	8.26650 +05	15	—3	2.88173 —11	—2	5.21064 —11
—25	—43	4.44802 +05	—42	4.70956 +05	16	—2	1.16265 —11	—1	1.67246 —11
—24	—42	2.39904 +05	—41	2.71112 +05	17	—1	3.73178 —12	0	0.00000 —00
—23	—41	1.21189 +05	—40	1.46660 +05	18	0	0.00000 —00	1	—8.36409 —12
—22	—40	5.62382 +04	—39	7.44057 +04	19	1	—1.86628 —12	2	—1.30893 —11
—21	—39	2.39184 +04	—38	3.47111 +04	20	2	—2.92061 —12	3	—1.57505 —11
—20	—38	9.52501 +03	—37	1.48369 +04	21	3	—3.51442 —12	4	—1.72968 —11
—19	—37	3.65611 +03	—36	5.93001 +03	22	4	—3.85945 —12	5	—1.82134 —11
—18	—36	1.38028 +03	—35	2.28142 +03	23	5	—4.06396 —12	6	—1.87629 —11
—17	—35	5.17700 +02	—34	8.62618 +02	24	6	—4.18658 —12	7	—1.90944 —11
—16	—34	1.93734 +02	—33	3.23933 +02	25	7	—4.26054 —12	8	—1.92950 —11
—15	—33	7.24596 +01	—32	1.21358 +02	26	8	—4.30529 —12	9	—1.94165 —11
—14	—32	2.71044 +01	—31	4.54407 +01	27	9	—4.33241 —12	10	—1.94901 —11
—13	—31	1.01428 +01	—30	1.70176 +01	28	10	—4.34884 —12	11	—1.95348 —11
—12	—30	3.79767 +00	—29	6.37620 +00	29	11	—4.35881 —12	12	—1.95619 —11
—11	—29	1.42279 +00	—28	2.39055 +00	30	12	—4.36486 —12	13	—1.95783 —11
—10	—28	5.33415 —01	—27	8.96910 —01	31	13	—4.36852 —12	14	—1.95883 —11
—9	—27	2.00129 —01	—26	3.36780 —01	32	14	—4.37075 —12	15	—1.95943 —11
—8	—26	7.51460 —02	—25	1.26567 —01	33	15	—4.37209 —12	16	—1.95980 —11
—7	—25	2.82411 —02	—24	4.76118 —02	34	16	—4.37291 —12	17	—1.96002 —11
—6	—24	1.06236 —02	—23	1.79291 —02	35	17	—4.37341 —12	18	—1.96016 —11
—5	—23	4.00053 —03	—22	6.75933 —03	36	18	—4.37371 —12	19	—1.96024 —11
—4	—22	1.50821 —03	—21	2.55152 —03	37	19	—4.37389 —12	20	—1.96029 —11
—3	—21	5.69321 —04	—20	9.64510 —04	38	20	—4.37400 —12	21	—1.96032 —11
—2	—20	2.15211 —04	—19	3.65172 —04	39	21	—4.37407 —12	22	—1.96034 —11
—1	—19	8.14809 —05	—18	1.38502 —04	40	22	—4.37411 —12	23	—1.96035 —11
0	—18	3.09040 —05	—17	5.26364 —05					

This system is free of divergences and gives

$$\lim_{x\to\infty} \operatorname{tg} \xi(x) = -a_t \tag{4.21}$$

i.e. the triplet scattering length a_t.

c) The eigen-function of the deuteron

The "phase function" method can be formulated in a certain sense also for the deuteron [28], but it does not give the eigen-function of the deuteron, which should be known for many reasons. Therefore, in this study, the deuteron eigen-function, the radial components of which are denoted as u and w have been derived directly from the system (2.15) where $j = 1$ and $1/\varrho = 0$ and the substitution

$$k^2 \to -\varkappa^2 \tag{4.22}$$

is performed.

The eigen-function is normalized according to (3.16) and with the help of it the magnetic and quadrupole moments are calculated (see III. c). There are two solutions of the system (2.15) which are regular under the present conditions at $x = 0$ and two singular solutions at this point. The boundary conditions for the deuteron wave function are

$$\begin{gathered} u(x) = 0 \qquad w(x) = 0 \qquad x = 0 \\ u(x) = e^{-\varkappa x} \qquad w(x) = K e^{-\varkappa x}\left(1 + \frac{3}{\varkappa x} + \frac{3}{\varkappa^2 x^2}\right) \qquad x \to \infty \end{gathered} \tag{4.23}$$

and define the eigen-function as a special superposition of the two regular solutions. The eigen-function, if it is to be obtained by direct integration of the system (2.15) must be constructed from the two regular solutions satisfying the boundary conditions at $x = 0$ and from the solution fulfilling the conditions at infinity being matched at a certain point within the interval $(0, \infty)$. The matching at this point is defined by the continuity of the eigen-function and its first derivative. The method gives the correct mixture of the two regular solutions, the constant K, which defines the asymptotic ratio of the S and D states in deuteron and the characteristic value of $\varkappa$, i.e. of the binding energy.

d) The variational method

For a fit to the two-nucleon data the potential (2.6) is now written in dimensionless units (2.17).

$$v = v_0 \qquad \text{if } c = 0 \text{ i.e. in the singlet case} \tag{4.24}$$

$$v = v_1 a + v_2 b + v_3 + v_4 a^2 \qquad \text{if } c = \text{ i.e. in the triplet case} \tag{4.25}$$

where each term v_i is expressed (see (4.4) and (3.2) in the form

$$v_i = \frac{1}{x}\sum_{n=1}^{\infty} (a_n^{(i)} + \mathcal{J}^2 b_n^{(i)})\, e^{-nx} \qquad \mathcal{J}^2 \to j(j+1) \tag{4.26}$$

The variational method applied is the usual statistical χ^2 method. The calculated phase shifts δ_{e-j}, $\delta_{j,j\pm1}$, the calculated mixing parameters ε_j, the scattering lengths and the deuteron constants — let us denote this set by δ_N, where N is the total number of experimental points — are functions of the coefficients $a_n^{(i)}$ and $b_n^{(i)}$

$$(\delta_N)_{\text{theor}} = \delta_N(a_r^{(i)}, b_n^{(i)}) \tag{4.27}$$

Defining the functional

$$S = \frac{1}{2} \sum_N \left\{ \frac{(\delta_N)_{\text{theor}} - (\delta_N)_{\text{exp}}}{\sigma_N} \right\}^2 \tag{4.28}$$

where $(\delta_N)_{\text{exp}}$ and σ_N represent the experimental set of data and their errors, we may find the coefficients $a_n^{(i)}$ and $b_n^{(i)}$ from its stationarity.

If we denote the number of coefficients $a_n^{(i)}$ and $b_n^{(i)}$ as N_c, χ^2 is given by

$$\chi^2 = 2S \tag{4.29}$$

and the quantity

$$\chi_n^2 = \chi^2/(N - N_c) \tag{4.30}$$

tests the adequacy of the χ^2 obtained.

The optimum number N_c of the coefficients was found by trial so as to satisfy three conditions. The number N_c should be small enough in comparison with N to preserve as many degrees of freedom as possible and high enough to give a minimum value of χ_n^2. The third condition follows from the known feature of this statistical method. By increasing the number N_c the quantity χ_n^2 may decrease but the errors in the calculated coefficients may increase considerably. If they are to be relatively small, the number N_c should not exceed a certain limit.

The errors of the coefficients $a_n^{(i)}$ and $b_n^{(i)}$ — let us denote them by $\Delta a_n^{(i)}$ and $\Delta b_n^{(i)}$ — are strongly correlated. It is impossible to change only one of the coefficients without affecting the quality of the fit. If necessary, all coefficients have to be changed in a proper way.

To estimate the quality of errors two quantities are introduced, the averaged relative error

$$P_{N_c} = \frac{1}{N_c} \sum_{n,i} \left(\left| \frac{\Delta a_n^{(i)}}{a_n^{(i)}} \right| + \left| \frac{\Delta b_n^{(i)}}{b_n^{(i)}} \right| \right) \tag{4.31}$$

and the maximum value of the relative error

$$q = \operatorname*{Max}_{i,n} \left(\left| \frac{\Delta a_n^{(i)}}{a_n^{(i(} \right|, \left| \frac{\Delta b_n^{(i)}}{b_n^{(i)}} \right| \right) \tag{4.32}$$

The condition, which is appropriate to the quality of experimental data used in the present calculation, reads

$$P_{N_c} < .3 \qquad q < 1.0 \tag{4.33}$$

For the given set of experimental data the optimum numbers

$$N_c \sim 8 \text{ in the singlet even state}$$
$$N_c \sim 5 \text{ in the singlet odd state}$$
$$N_c \sim 20 \text{ in the triplet odd state}$$
$$N_c \sim 15 \text{ in the triplet even state}$$

have been obtained.

The variational method for the deuteron is, in principle, the same as for the phase shifts and scattering lengths. The computation may be simplified a little by noticing that any of the potential coefficients may play the role of the deuteron eigen-value instead of the energy, i.e. one may fix the binding energy to its experimental value during the solution of the deuteron wave equation and consider the set of the coefficients as a representation of the eigen-value of the given problem. This procedure has been carried out during the present calculations, i.e. the quantity $\varkappa$ has been kept fixed and equal to its experimental value (3.14).

V. The Results

The sets of experimental data contained in [34] have been simultaneously used for the derivation of the two-nucleon potential for the singlet even, singlet odd and triplet odd and even states. The simultaneous fitting of the data gives certain correlations, which can be, as we shall see, quite large for phase shifts of different angular momenta.

a) The singlet even proton-proton nuclear potential

The difference between the scattering lengths for *p-p* and *n-p* interactions indicates a difference between the basic *p-p* and *n-p* nuclear interactions. We shall therefore distinguish between these two interactions. For the description of the proton-proton nuclear interaction we use the potential (2.7) written in the form given in (3.2) and (4.4) i.e.

$$v_0(x) = \frac{1}{x} \sum_n \left[a_n^{(0)} + j(j+1)\, b_n^{(0)} \right] e^{-nx} \qquad j = l = 0, 2, 4.$$
$$a_n^{(0)} = b_n^{(0)} = 0 \qquad n > 4 \tag{5.1}$$

where the number N_c of the non-zero coefficients a_n and b_n was chosen to be optimum (see IVd).

By using the variational method the coefficients a_n and b_n have been obtained. Their values are given in the Table II. They are dimensionless, to get their value in MeV the simple multiplicative transformation $(a_n)\text{MeV} = 20.76\, a_n$ should be performed.

The corresponding χ_n^2 which is χ^2 divided by the number of degrees of freedom is equal to 1.30. The errors of the coefficients a_n and b_n are small. The new values of the coefficients a_n and b_n do not differ significantly from values published earlier [6] so that the results of calculations [30], where the potential [6] was applied to the investigation of some off-shell processes remain unchanged as well as the result of other calculations

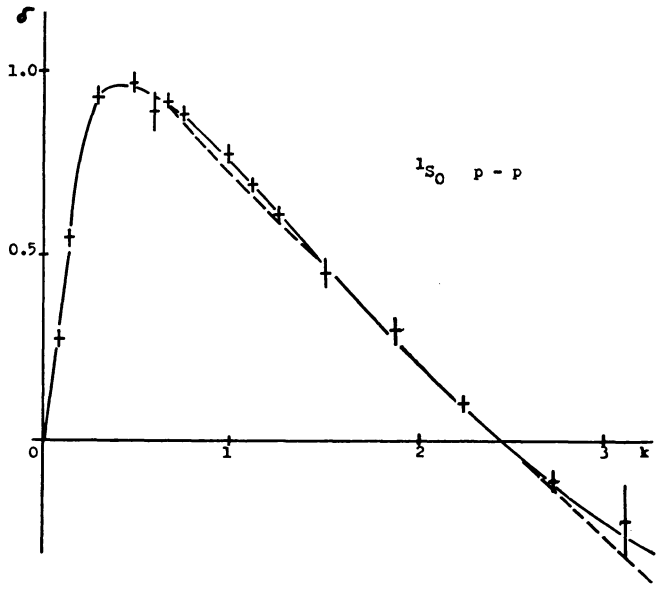

Fig. 2

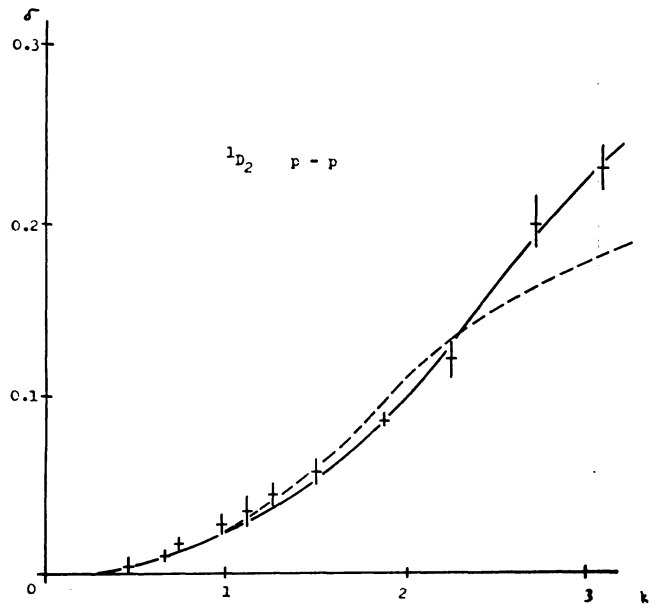

Fig. 3

Tab. II.–XII. and Tab. XIV. The potential coefficients are dimensionless. To obtain their value in MeV the transformation (2.17) should be performed. It gives (a_n) MeV $= 20.76 a_n$.

Table II

n	$a_n^{(0)}$	$b_n^{(0)}$
1	− 2.1205	.1706
2	38.636	− 4.4435
3	− 211.20	24.481
4	246.11	− 31.421

where it was used. The shape of the potential and the behaviour of the phase shifts is only changed slightly. The calculated phase shifts are plotted in Fig. 2–4 where the experimental points and the results of the Livermore [15] energy dependent phase shift analysis (dashed line) are also given. The directly calculated scattering length a_s and the effective range and shape parameters rederived from the phase shifts computed at the energies $k = .1$ and $k = .3$ are given in Table I.

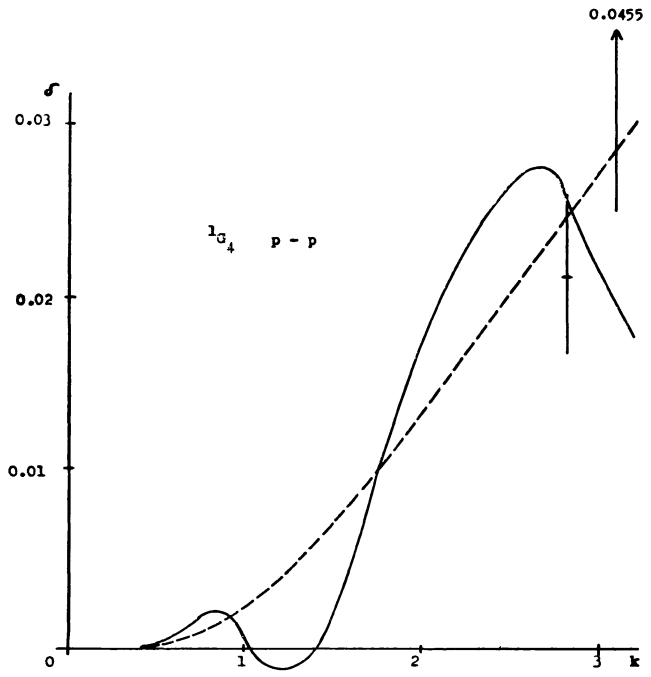

Fig. 4

Fig. 2–4. The singlet even p-p calculated phase shifts — solid lines. The Livermore phase shifts obtained from the energy dependent phase shift analysis — dashed lines. The experimental points used in the calculation are denoted by crosses. The phase shifts are given in radians as a function of momentum k, $k = .1553 \sqrt{E(\text{MeV})}$

The experimental data are very well reproduced. The resulting phase shifts coincide within the error bands with the results of the energy dependent phase shift analysis. The deviations between the results of these two independent calculations occur mainly in the higher energy region, $k \sim 3$, where relativistic effects play their role. This conclusion is supported by the strong correlation between the deviation of the ${}^1G_4$ phase shift at higher energies and the behaviour of the phase shifts for lower angular momenta. A potential

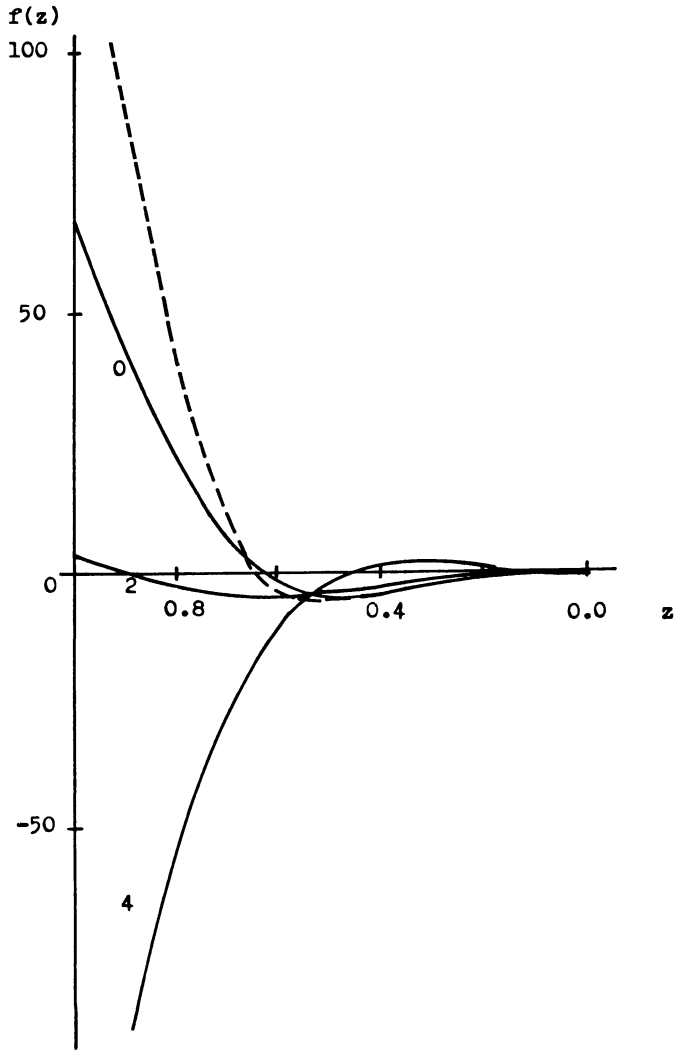

Fig. 5. The $f(z)$ representation of the singlet even proton-proton potential (the Coulomb interaction is not included). $f(z) = x\, e^x\, v(x)$, $x = \mu r$ where $\mu = .707 f^{-1}$ and r is the actual internucleon distance. The solid curves denoted by 0, 2, 4 correspond to the potential acting in $j = l = 0, 2, 4$ orbital states. The dashed line is the $f(z)$ representation of the Reid potential [5] for the ${}^1S_0$ state

which gives the $^1G_4$ phase shift very close to that obtained from the energy dependent phase shift analysis can be easily obtained, but in such a case the description of the other two phase shifts is worse — the corresponding $\chi_n^2 = 2.36$. This value is almost twice as large as for the optimum potential. In both cases i.e. for the potential defined in Table II and for that just mentioned the deviation of the $^1G_4$ phase shift from the values given by the energy dependent analysis at low energies may be attributed to the lack of experimental data in this region.

One may conclude from this result that the JINR and Livermore independent energy phase shifts data for the singlet even p-p state are equivalent and that the energy dependent and independent Livermore analysis are nearly identical in the region below 400 MeV.

The ratio of the coefficients b_n and a_n for a given n is of order .1. Its value support the validity of a linear approximation for the j^2 dependence of the derived potential $v_0(x)$. The values

$$\alpha_j = \sum [a_n^{(0)} + j(j+1)\, b_n^{(0)}], \qquad \alpha_0 = 71.43, \qquad \alpha_2 = 4.154, \qquad \alpha_4 = -152{,}8$$

show that the singlet even potential has a soft core in the $^1S_0$ and $^1D_2$ states and that the condition (4.2) is satisfactorily fulfilled. In Fig. 5 the functions.

$$f_j(z) = \sum [a_n^{(0)} + j(j+1)\, b_n^{(0)}]\, z^{n-1} = x v_0(x) e^x \qquad (5.2)$$
$$z = e^{-x} \qquad j = l$$

are plotted for $j = 0, 2, 4$. They represent the potential curves, the region of repulsion, $f(z) > 0$ and attraction, $f(z) < 0$. The corresponding function $f(z)$ for the Reid potential [5] is drawn as a dashed line in this figure. It is seen that the Reid potential is steeper near the origin and its soft core more pronounced than in the present case.

b) The singlet even neutron-proton nuclear potential

The analytic form of the potential is in this case taken again in the form (5.1) i.e.

$$v_0 = 1 \sum [a_n^{(0)} + j(j+1)\, b_n^{(0)}]\, e^{-nx} \qquad j = l = 0, 2, 4. \qquad (5.3)$$
$$a_n^{(0)} = 0 \qquad b_n^{(0)} = 0 \qquad n > 4$$

and the total number N_c of the coefficients is kept equal to 8, although the less accurate data for the n-p interaction would require a slightly smaller number. The reason for $N_c = 8$ is to have the possibility of comparing the resulting potential with that obtained for the p-p interaction.

Table III

n	$a_n^{(0)}$	$b_n^{(0)}$
1	− 1.7364	.1071
2	31.3968	− 2.5496
3	− 184.48	12.759
4	219.61	− 12.986

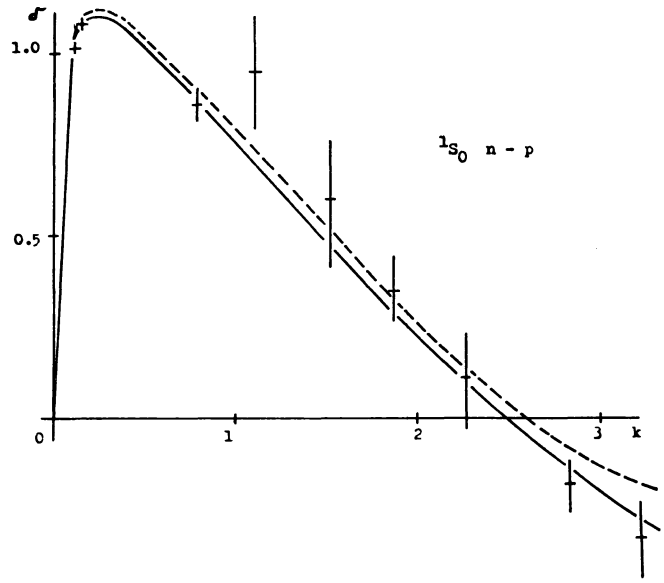

Fig. 6

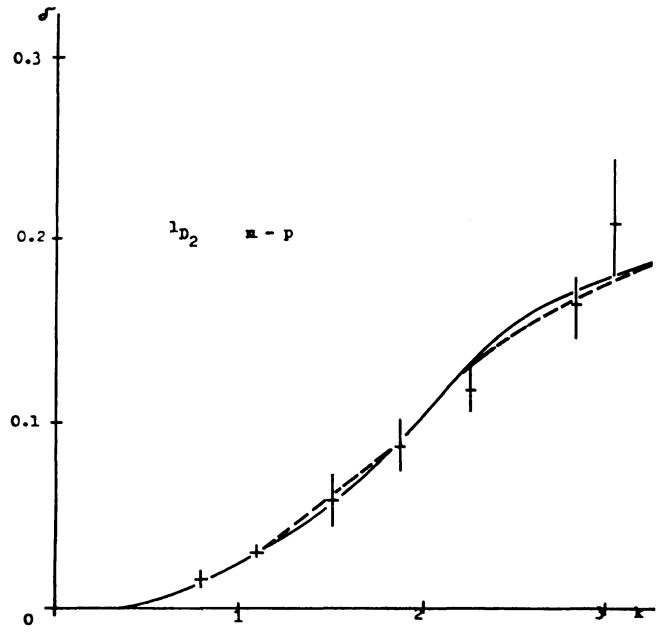

Fig. 7

In this case the effective range data (Table I) and for all states the Livermore phase shift data have been used. The χ^2 method gives the values of the coefficients which are presented in Table III.

The $\chi_n^2 = .94$ but the errors of the coefficients a_n and b_n are higher than in the preceeding case. The condition (4.33) is still well satisfied.

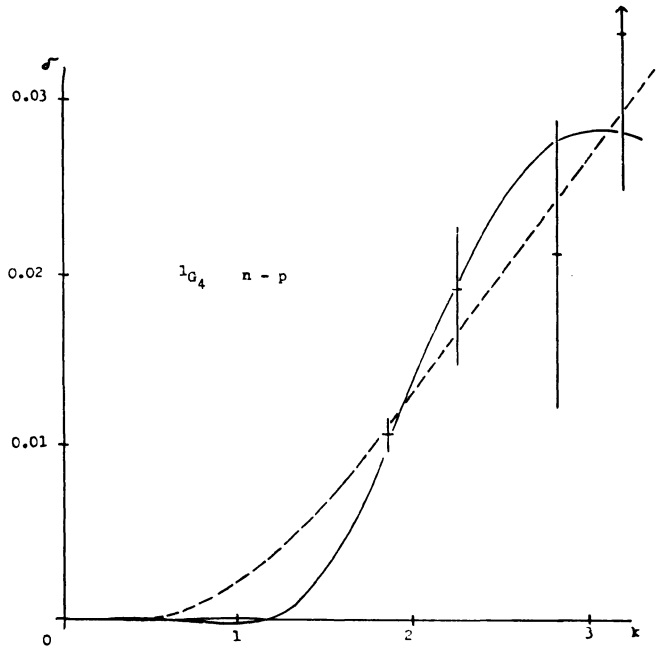

Fig. 8

Fig. 6–8. The singlet even neutron-proton calculated phase shifts — solid lines. The Livermore phase shifts obtained from the energy dependent analysis — dashed lines. For further explanation see caption to Fig. 2–4

The scattering length obtained directly and the effective range and shape parameters derived from the calculated phase shifts at energies $k = .1$ and $k = .23$ are presented in Table I. The calculated phase shifts are illustrated in Fig. 6–8.

The description of all experimental data is good and again it coincides practically with the Livermore energy dependent phase shift analysis. The $^1S_0$ *n-p* phase shift lies a little lower than that given by the Livermore analysis and the $^1G_4$ phase shift again shows the same kind of deviations as in the *p-p* case.

It should be pointed out that the differences in the calculated phase shifts for *p-p* and *n-p* states are principally due to the effective range parameters and to the difference in the experimental $^1S_0$ *n-p* and *p-p* phase shifts.

The potential for the $^1S_0$ state has the same general features as the potential for the corresponding *p-p* state. The major difference between the *n-p* and *p-p* singlet even poten-

tials is found at higher angular momenta, where the repulsive action of the soft core is also seen in the $^1G_4$ state. The $f(z)$ representation (5.2) of the n-p potential is given in the Fig. 9. In this case there is no remarkable difference between the new $^1S_0$ n-p potential and that derived in the reference [6] but there are some differences in the higher orbital states.

For nuclear calculations, with the exception to be described later, both potentials in the $^1S_0$ n-p and p-p and in the $^1D_2$ n-p and p-p states can be considered as equivalent.

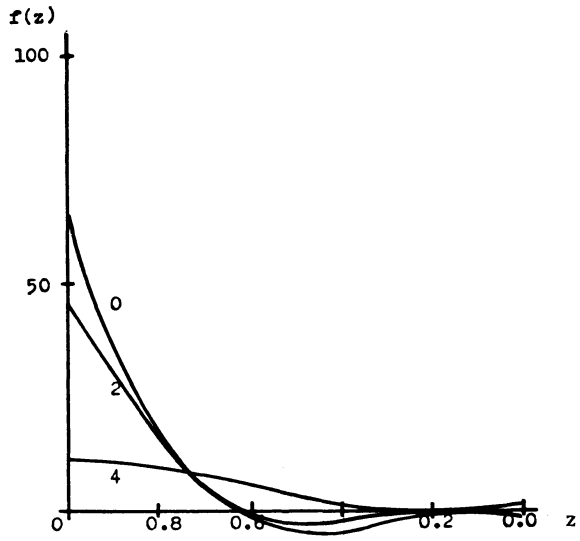

Fig. 9. The $f(z)$ representation of the singlet even neutron-proton potential (see caption to Fig. 5). The solid lines denoted by 0, 2, 4 describe the potential acting in $j = l = 0, 2, 4$ orbital states

The p-p and n-p potentials (Tab. II and III) give different effective range parameters and especially different p-p and n-p scattering lengths. The p-p potential cannot be used for calculation of n-p scattering lengths and vice versa. The p-p potential gives the n-p scattering length $a'_s = 11.9$ in dimensionless units i.e. $16.7f$. This value coincides with the n-n scattering length [17] and may be interpreted as an indication of n-n and p-p symmetry in the nuclear interaction but the calculated value disagrees with the value of the n-p scattering length. An incorrect result also would be obtained when the n-p potential is applied to the calculation of the p-p scattering length.

c) The singlet odd nucleon-nucleon potential

The optimum number N_c of potential coefficients was found to be about 6 for the singlet odd two-nucleon potential (see Sec. IIId). They were equally divided between the $\mathcal{J}^2$ independent and $\mathcal{J}^2$ dependent terms and the potential was written in the usual form:

$$v_0(x) = \frac{1}{x} \sum_n [a_n^{(0)} + j(j+1)\, b_n^{(0)}]\, e^{-nx} \qquad j = l = 1, 3, 5. \tag{5.4}$$
$$a_n^{(0)} = b_n^{(0)} = 0 \qquad n > 3$$

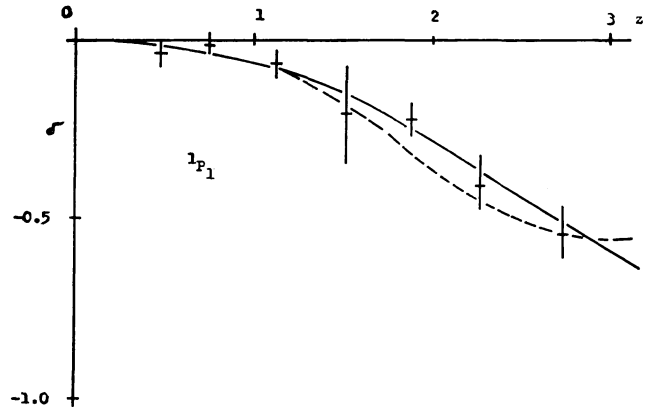

Fig. 10

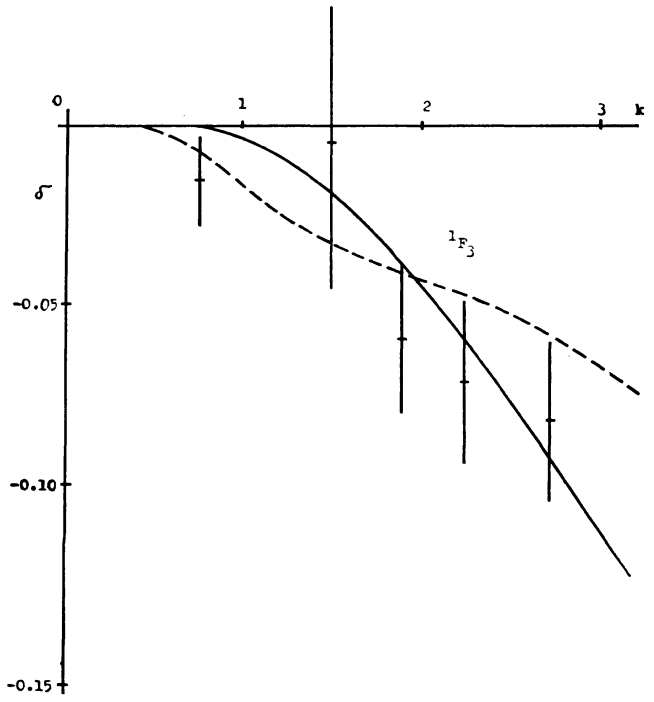

Fig. 11

The stationary value of the functional (4.28) was found for the values of the coefficients a_n and b_n which are given in Table IV.

Table IV

n	$a_n^{(0)}$	$b_n^{(0)}$
1	1.9968	− 0.1644
2	− 31.973	3.2606
3	90.186	− 6.7926

The corresponding $\chi_n^2 = 1.20$ shows that the description of the data is good. Here, the condition (4.33) is also well satisfied.

This result, as well as the phase shift curves plotted in Fig. 10–12 where they are compared with the phase shifts obtained from the Livermore energy dependent phase shift analysis (dashed lines), again show that the *JINR* energy independent and Livermore energy dependent phase shift analysis data are practically equivalent. The deviations of the calculated phase shifts from the Livermore energy dependent solution are now caused

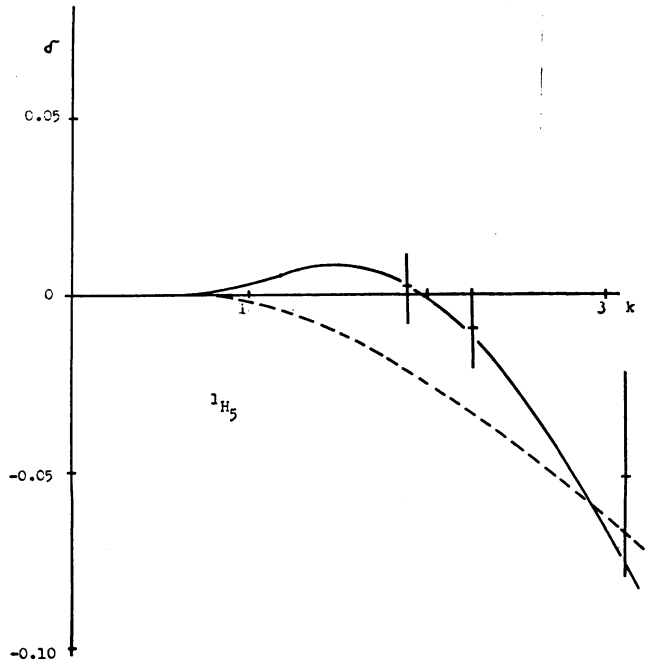

Fig. 12

Fig. 10–12. The singlet odd calculated phase shifts — solid lines. The Livermore phase shifts obtained from the energy dependent analysis — dashed line. (see caption to Fig. 2–4)

by both the effects which we met in the singlet even state, namely, by relativistic effects and a certain lack of experimental data in the n-p interaction.

The potential acting in the singlet odd state is weaker than the singlet even potential, it is repulsive in the $j = 1$ and $j = 3$ state and "attractive" in the $j = 5$ state. A clear picture of its behaviour is given in Fig. 13, where the function

$$(z) = xe^x v_0(x) = \sum_n [a_n^{(0)} + j(j+1)\, b_n^{(0)}]\, z^{n-1} \qquad (5.5)$$

$$z = e^{-x}$$

is drawn and compared with the corresponding $f(z)$ for the Reid potential in the $^1P_1$ state.

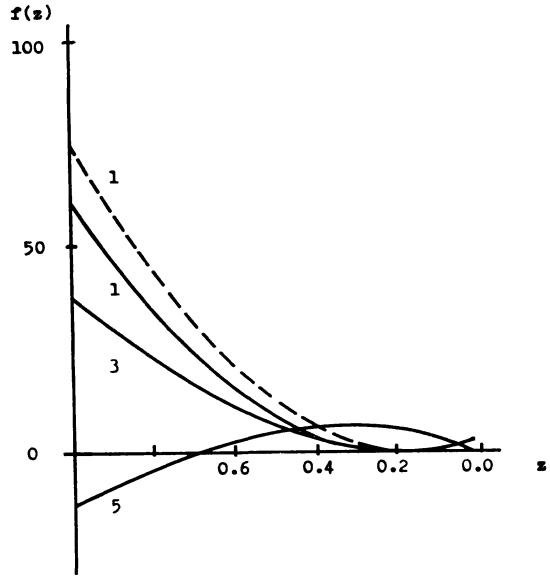

Fig. 13. The $f(z)$ representation of the singlet odd potential (see caption to Fig. 5). The solid lines 1, 3, 5 describe the potential acting in $j = l = 1, 3, 5$ orbital states. The dashed line is the $f(z)$ representation of the Reid potential in $^1P_1$ state [5]

d) The triplet odd two-nucleon potential

The potential for the triplet state is taken in the form (2.8) (see also (2.17))

$$v = v_1 a + v_2 b + v_3 + v_4 a^2 \qquad (5.6)$$

An equivalent expression for it reads

$$v = v_c + v_{SL}\,(\vec{S}\,.\,\vec{L}) + v_T S_{12} + v_{(SL)^2}\,(\vec{S}\,.\,\vec{L})^2 \qquad (5.7)$$

where v_c, v_{SL} etc. are linear functions of v_i (see (2.10), (2.17)) and where v_c denotes the central, v_{SL} the spin-orbit, v_T the tensor and $v_{(SL)^2}$ the quadratic spin-orbit potential

term. For a given j and l we can define the channel potentials (see (2.11),)2.12) and also (2. 17)), so that in the uncoupled triplet (odd) state we have

$$v^{(j)} = v_2 + v_3 \qquad j = l = 1, 3, 5 \,. \tag{5.8}$$

and in the coupled triplet (odd) state

$$\begin{aligned} v_{j,j-1} &= jv_1 + v_2 \frac{1}{2j+1} + v_3 + j^2 v_4 && j = 2, 4 \,. \\ t_j &= \frac{2\sqrt{j(j+1)}}{2j+1} v_2 && j = 2, 4 \,. \\ v_{j,j+1} &= -(j+1) v_1 - v_2 \frac{1}{2j+1} + v_3 + (j+1)^2 v_4 && j = 0, 2, 4. \end{aligned} \tag{5.9}$$

The coupled triplet odd state for $j = 6$, for which only two experimental points at high — practically relativistic — energies are available has not been considered in this study.

The $\mathcal{J}^2$ dependence of the potential (3.2) is again expressed by splitting every term v_i into two terms

$$v_i = v_i' + \mathcal{J}^2 v_i'' \to v_i' + j(j+1) v_i'' \tag{5.10}$$

where v_i' and v_i'' are now only functions of the radial internucleon distance. Hence, altogether 8 functions v_i' and v'' should be found in a simultaneous fit of the triplet odd scattering data. This number is a little high. When one tries to fit only the channel potentials $v_{j,j-1}$, t_j, and $v_{j,j+1}$ (5.9) and $v^{(j)}$ (5.8) to the scattering data in a given channel, as is usually done, the task is considerably simplified, because only three functions, i.e. $v_{j,j-1}$, t_j and $v_{j,j+1}$ in the coupled state and only one function $v^{(j)}$ in the uncoupled state are unknown. Let us therefore examine some possible simplifications of the problem in the general case. If we found the potentials only for the coupled states, where the channel potentials are given by (5.9), then we could use a transformation, which is based on the property of the operator a. This operator in the coupled triplet state, i.e. in the two-row representation (2.5) fulfils the valuable relation

$$a^2 + a = j(j+1) \tag{5.11}$$

which leads, when it is inserted in (5.6), to

$$v = (v_1 - v_4)\, a + v_2 b + v_3 + v_4 j(j+1) \tag{5.12}$$

The new potential consists of the spin-orbit term $(v_1 - v_4)$ tensor term v_2 and a central $\mathcal{J}^2$ dependent term $(v_3 + \mathcal{J}^2 v_4)$. The quadratic spin-orbit term is removed. It means that by fitting the channel data in the coupled triplet state no more than three independent functions $v_{j,j-1}$, t_j, $v_{j,j+1}$ or $(v_1 - v_4)$, and $(v_3 + \mathcal{J}^2 v_4)$ can be obtained. Although such transformation cannot be generally performed, i.e. outside the two-row representation, it suggest that the potential can be simplified by excluding the quadratic spin-orbit term because the $\mathcal{J}^2$ dependence of the other terms in the potential can partially substitute for

the action of this particular term. We try to put $v_4 = 0$ in the following and deal only with the potential

$$v = v_1 a + v_2 b + v_3 \tag{5.13}$$

where, of course, every term is written in the form (5.10).

The optimum number N_c of the potential coefficients for the triplet odd two-nucleon interaction was found to be about 20. To divide the 20 coefficients among the individual terms v_i' and v_i'' we assume that each term v_i' and v_i'', $(i = 1\ 2\ 3)$, has four, generally non-zero, coefficients with the exception of v_1' where we put $v_1'' = 0$.

Table V

n	$a_n^{(1)}$	$b_n^{(1)}$	$a_n^{(2)}$	$b_n^{(2)}$	$a_n^{(3)}$	$b_n^{(3)}$
1	.1270	.0	1.5398	.0077	.1545	− .0567
2	− 4.6337	.0	− 12.572	− .4733	— 1.1852	1.7567
3	38.334	.0	82.185	5.5287	— 46.588	− 11.175
4	− 90.398	.0	− 99.558	− 13.4653	132.25	17.605

These two simplifications need to be verified by a fit to the experimental data. The values of the coefficients for the χ^2 minimum are given in Tab. V, VI, VII and VIII. The Tab. V, contains the coefficients for the potential (5.6), (5.13) where each v_i is given as a superpositions (see (5.10))

$$v_i = \frac{1}{x} \sum_n [a_n^{(i)} + j(j+1)\, b_n^{(i)}]\, e^{-nx} \qquad i = 1, 2, 3.$$
$$a_n^{(i)} = b_n^{(i)} = 0 \qquad n > 4 \tag{5.14}$$
$$b_n^{(1)} = 0$$

Table VI

n	$a_n^{(c)}$	$b_n^{(c)}$	$a_n^{(SL)}$	$b_n^{(SL)}$	$a_n^{(T)}$	$b_n^{(T)}$
1	0.7948	− .0541	.1270	.0	.5133	.0026
2	− 10.010	1.5984	− 4.6337	.0	− 4.1906	− .1578
3	19.141	− 9.3319	38.334	.0	27.395	1.8429
4	8.6690	13.167	− 90.398	.0	− 33.186	− 4.4884

Tab. VI presents the corresponding coefficients for the same potential expressed in the form (5.7), where again

$$v_i = \frac{1}{x} \sum_n [a_n + j(j+1)\, b_n]\, e^{-nx} \qquad i = c, SL, T$$
$$a_n^{(i)} = b_n^{(i)} = 0 \qquad n > 4 \tag{5.15}$$
$$b_n^{(SL)} = 0.$$

Table VII

j	n	$a_n^{(j,j-1)}$	$a_n^{(tj)}$	$a_n^{(j,j+1)}$
0	1	.0	.0	− 1.5123
	2	.0	.0	16.020
	3	.0	.0	− 167.11
	4	.0	.0	322.21
2	1	.3855	1.5538	− .8838
	2	− 2.9947	− 15.100	26.338
	3	− 13.898	113.03	− 251.71
	4	21.017	− 176.15	545.15
4	1	− .2833	1.6829	− 1.8025
	2	12.466	− 21.900	59.566
	3	− 95.331	191.57	− 483.17
	4	81.774	− 366.58	977.32

Table VIII

j	n	$a_n^{(j)}$
1	1	1.5963
	2	− 11.190
	3	24.306
	4	40.974
3	1	1.1061
	2	1.6442
	3	− 32.155
	4	82.371
5	1	.2237
	2	24.746
	3	− 133.78
	4	156.88

Tab. VII and VIII give the channel coefficients. The channel potentials $v^{(j)}$ (5.8) and $v_{j,j-1}$, t_j, $v_{j,j+1}$ (5.9) are for a given j written in the form

$$v_{j,j-1} = \frac{1}{x} \sum a_n^{(j,j-1)} e^{-nx} \qquad \text{etc.} \tag{5.16}$$

The corresponding $\chi_n^2 = 1.62$ and the errors of the coeffiencients $a_n^{(i)}$ and $b_n^{(i)}$ are such that the condition (4.33) is satisfied. They give $p_{N_c} = .29$ and $q = .72$.

The reproduction of the two-nucleon data in the triplet odd state is good. It can be seen from the Fig. 14–19 where the phase shifts $\delta_{j,j-1}$, $\delta_{j,j+1}$ and the mixing parameter

Fig. 14–19. The triplet odd calculated phase shifts — solid lines. The Livermore phase shifts obtained from the energy dependent analysis — dashed lines. The crosses and the dots represent the experimental points used in the present work. $k = .1553\sqrt{E(\text{MeV})}$, where E is the energy of the incident particle in the laboratory system. The phase shifts are given in radians

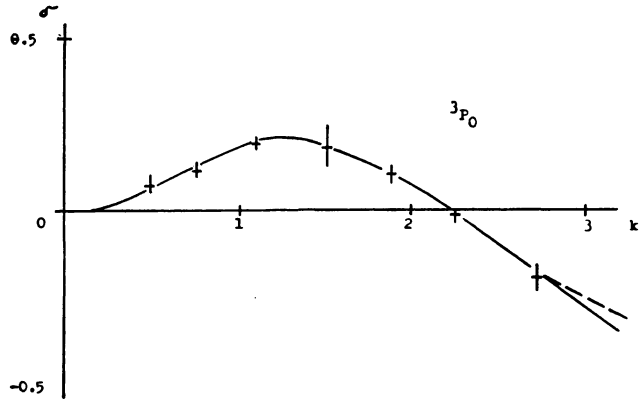

Fig. 14

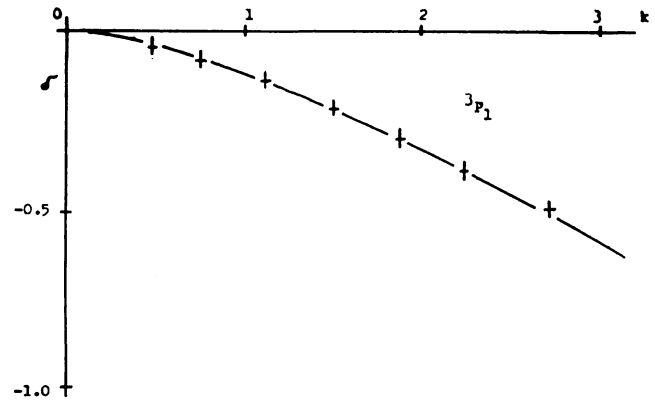

Fig. 15

ε_j (in the Stapp representation) in the coupled and the phase shifts δ_j in the uncoupled triplet state are plotted and compared with the results of the Livermore energy dependent phase shift analysis (dashed line). There are no deviations for the $^3P_1$ state and they are negligible or small for the other states. The differences occur in the transition — quasirelativistic region — and at places where the number of experimental data is small. Some of the deviations may be attributed to correlations among the phase shifts for different angular momenta.

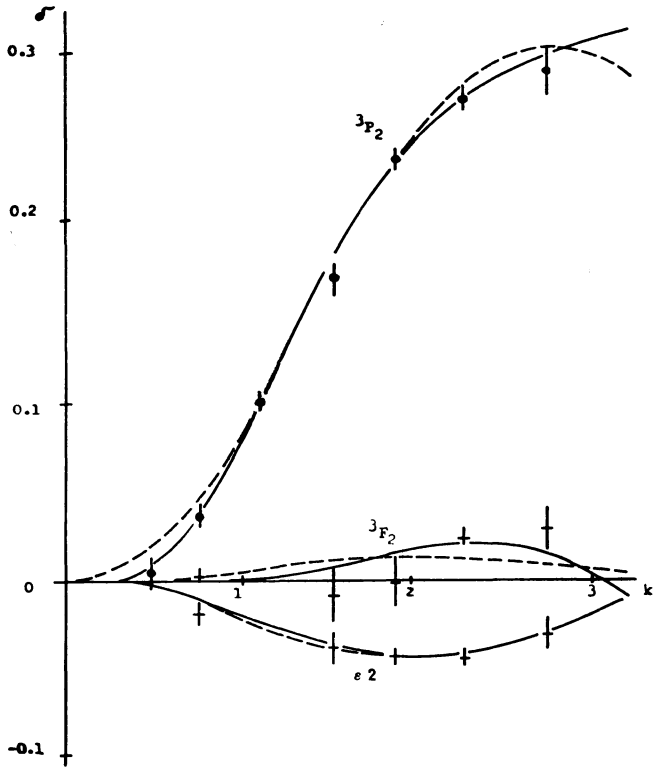

Fig. 16

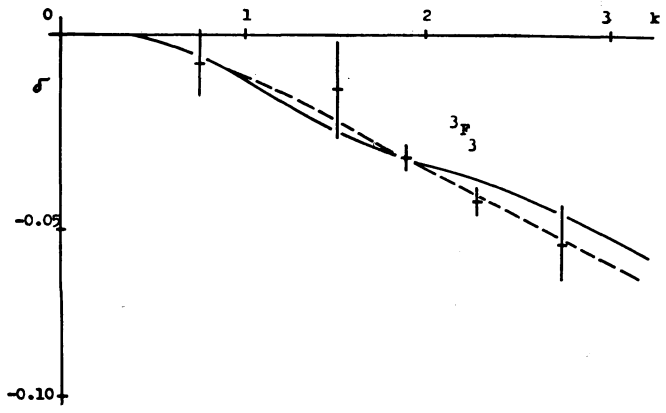

Fig. 17

Here again we can say that the *JINR* and Livermore energy independent and energy dependent phase shift analysis give coincident results.

The behaviour of the triplet odd potential differs now from the potential found earlier [30]. In reference [30] the potential was obtained for individual channels, the

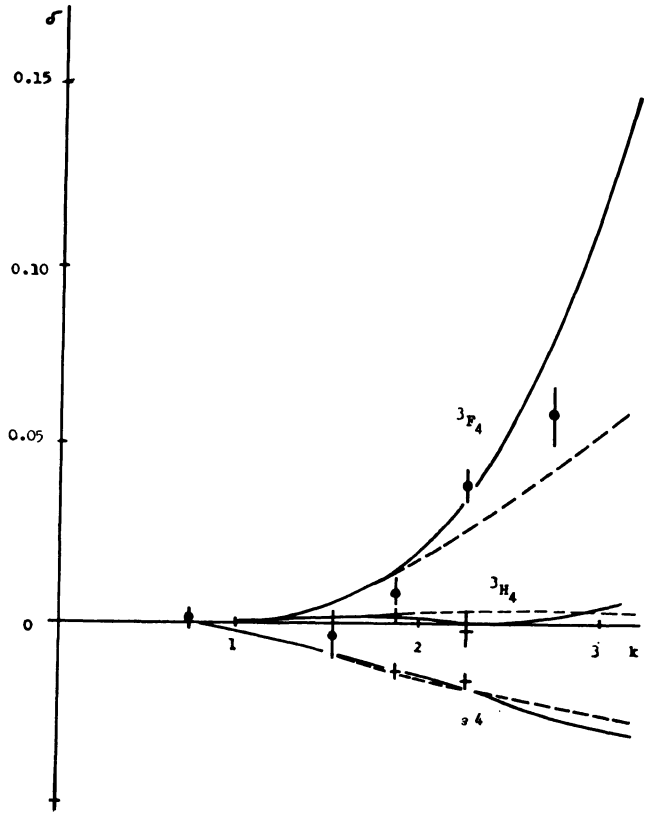

Fig. 18

quadratic spin-orbit term was not excluded neither was the $\mathcal{J}^2$ dependence of the spin-orbit term. In the present study the simultaneous use of the phase shift data enabled us to construct a much simpler potential and the quality of the fit to the data shows that *it* is unnecessary to include more complicated terms in the analytic expression for the two-nucleon potential. The ratio of the coefficients a_n and b_n in potentials v_2 and v_3 also shows that a linear approximation of the $\mathcal{J}^2$ dependence (3.2) is acceptable.

The potential satisfies also the condition (4.2) which proves that the singularity of the potential at the origin is not larger than $1/x$. The corresponding quantity α has the absoluted value

$$|\alpha| < 552 \tag{5.17}$$

for the channel potentials in the triplet odd state. If we look at the individual potentials terms we see that they are relatively weak. In Fig. 20 the central potentials v_c is drawn in the $f(z)$ representation for different angular momenta j. The function $f(z)$ is defined in

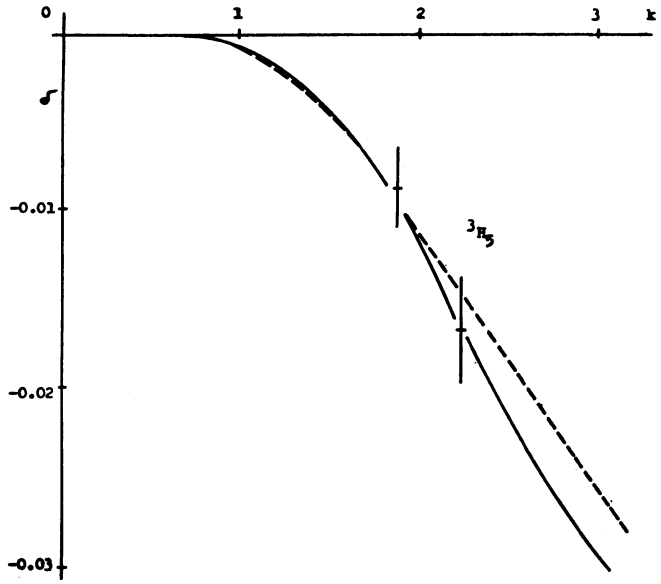

Fig. 19

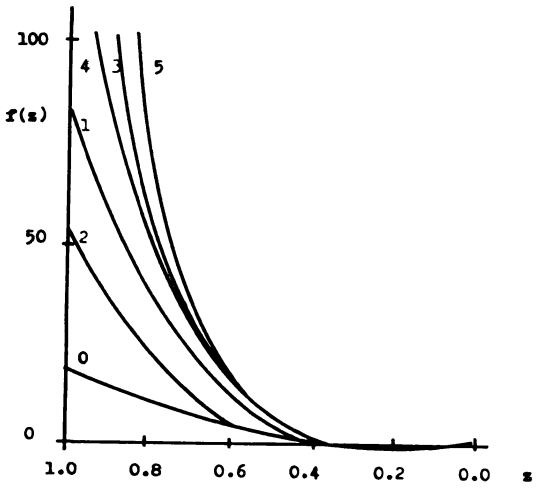

Fig. 20. The $f(z)$ representation (see caption to Fig. 5) of the triplet odd central potential. The curves denoted by 0, 1, 2, ... 5 correspond to the potential acting in $j = 0, 1, 2, \ldots 5$ orbital states

full analogy to the former cases

$$f_c(z) = xe^x v_c = \sum_n [a_n^{(c)} + j(j+1)\, b_n^{(c)}]\, z^{n-1} \tag{5.18}$$

The central potential is "repulsive". On the contrary, the spin-orbit and the tensor potentials are "attractive" as can be seen from Fig. 21 and 22, where the corresponding $f(z)$ are drawn. These qualitative features are similar to those found in the special cases ($j = 0, 1, 2$) in reference [5].

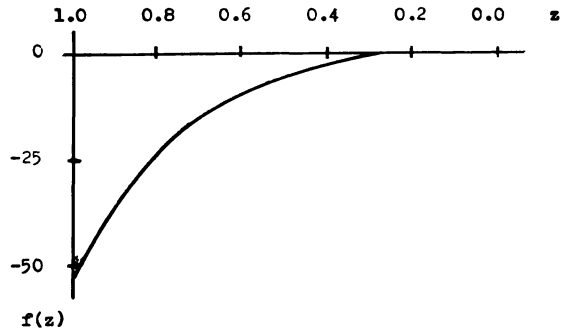

Fig. 21. The $f(z)$ representation (see caption to Fig. 5) of the spin-orbit potential, which is independent on $\mathcal{J}^2$, for the triplet odd state

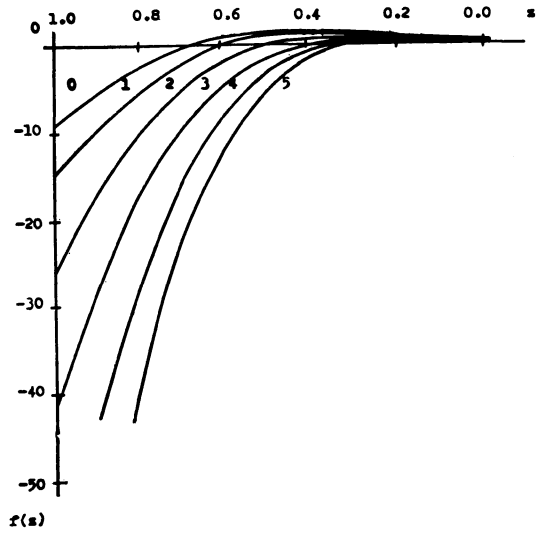

Fig. 22. The $f(z)$ representation (see caption to Fig. 5) of the triplet odd tensor potential. The curves denoted by 0, 1, 2, ... 5 belong to the potential acting in $j = 0, 1, 2, \ldots 5$ orbital states

e) The triplet even nucleon-nucleon potential

The triplet even nucleon-nucleon state significantly differs from the previous cases by the existence of the bound state, which — as it follows from the generalised Levinson theorem — leads to the non zero value of the phase shift $\delta_{j,j-1}$, $j = 1$, for $k = 0$ (see (3.10), (3.13)). Because there is only one bound state then

$$\delta_{j,j-1}(0) = \pi \qquad j = 1 \tag{5.19}$$

The deuteron data should be added to the scattering data, to obtain a solution of the inverse problem. In this case the quality of the scattering data is poorer than for the triplet odd state. Therefore, it is now more important than in the preceding cases to solve the inverse problem simultaneously for all available data in the non-relativistic region.

The same form of the potential as in the triplet odd state has been used although some other forms have also been investigated. The final potential is written (compare with (5.8)) for the uncoupled triplet (even) state as

$$v^{(j)} = v_2 + v_3 \qquad j = 2, 4. \tag{5.20}$$

and for the coupled triplet (even) state (compare (5.9), (5.13))

$$\begin{aligned} v_{j,j-1} &= jv_1 + \frac{v_2}{2j+1} + v_3 \\ t_j &= \frac{2\sqrt{j(j+1)}}{2j+1} v_2 \qquad j = 1, 3, 5. \\ v_{j,j+1} &= -(j+1)\, v_1 - \frac{v_2}{2j+1} + v_3 \end{aligned} \tag{5.21}$$

where every term v_i, $i = 1, 2, 3$ is given as a superposition

$$\begin{gathered} v_i = \frac{1}{x} \sum [a_n^{(i)} + b_n^{(i)} j(j+1)] \qquad i = 1, 2, 3 \\ b_n^{(1)} = 0 \end{gathered} \tag{5.22}$$

The quadratic spin-orbit term has been again excluded and the $\mathbf{j}^2$ dependence of the spin-orbit term suppressed.

The optimum number N_c of the potential coefficients has been found to be about 15. The simplest possibility has been chosen

$$a_n^{(i)} = b_n^{(i)} = 0 \qquad n > 3 \tag{5.22a}$$

The values of the coefficients $a_n^{(i)}$ and $b_n^{(i)}$ defined in (5.22) have been obtained by minimisation of χ^2. The results are presented in Tab. IX.

The next Table (Tab. X) contains the coefficients of the central — c, spin-orbit — SL, and tensor — T terms for the same potential (see (5.7)).

Table XI and XII give the channel coefficients for the corresponding channel potentials defined in (5.20) and (5.21) (see also (5.16)).

Table IX

n	$a_n^{(1)}$	$b_n^{(1)}$	$a_n^{(2)}$	$b_n^{(2)}$	$a_n^{(3)}$	$b_n^{(3)}$
1	− .2660	.0	− 6.0336	.2189	4.7811	− .2172
2	4.8104	.0	24.536	− 3.0393	− 62.887	3.0507
3	− 18.914	.0	− 97.074	9.0636	143.26	− 4.6193

Table X

n	$a_n^{(c)}$	$b_n^{(c)}$	$a_n^{(SL)}$	$b_n^{(SL)}$	$a_n^{(T)}$	$b_n^{(T)}$
1	2.5039	− .1442	− .2660	.0	− 2.0112	.0730
2	− 49.898	2.0376	4.8104	.0	8.1786	− 1.0131
3	91.987	− 1.5981	− 18.914	.0	− 32.358	3.0212

Table XI

j	n	$a_n^{(j,j-1)}$	$a_n^{(tj)}$	$a_n^{(j,j+1)}$
1	1	2.2155	− 5.2757	6.7440
	2	− 45.822	17.401	− 72.558
	3	88.791	− 74.431	198.16
3	1	0.8901	− 3.3719	3.7255
	2	− 13.553	− 11.8134	− 43.815
	3	32.756	11.569	161.81
5	1	− 3.0161	.5309	− 0.1871
	2	46.626	− 66.367	5.8291
	3	− 73.994	174.11	102.27

Table XII

j	n	$a_n^{(j)}$
2	1	− 1.2422
	2	− 38.282
	3	72.851
4	1	− 1.2184
	2	− 38.124
	3	135.07

The potentials are generally weaker than in the triplet odd state so that the condition (4.2) is better satisfied. The central, spin-orbit and tensor potentials are drawn in the $f(z)$ representation (5.18) in Fig. 23 and 24. The central potential is repulsive near the origin and slightly attractive at larger distances. The attractive part is most pronounced in the

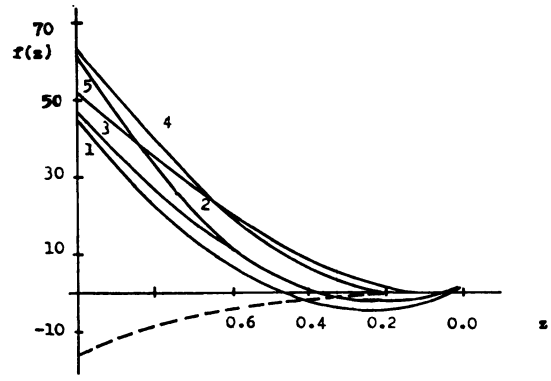

Fig. 23. The $f(z)$ representation (see caption to the Fig. 5) of the triplet even central potential acting in $j = 1, 2, 3, 4, 5$ orbital states. The corresponding solid lines are denoted by the value of j. The dashed line is the $f(z)$ representation of the spin-orbit potential for the triplet even state

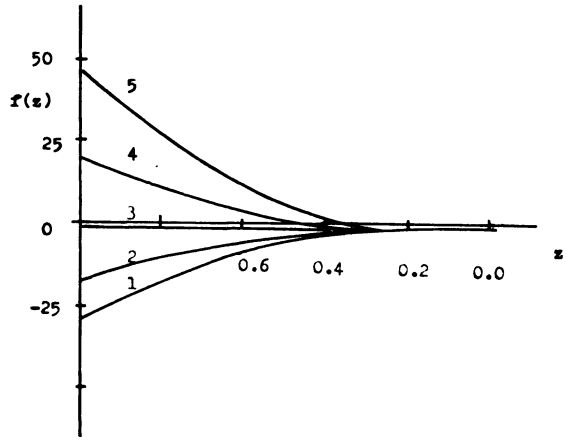

Fig. 24. The $f(z)$ representation (see caption to the Fig. 5) of the triplet even tensor potential for the $j = 1, 2, 3, 4, 5$ orbital states. The corresponding lines are denoted by the value of j

$j = 1$ state, which gives the bound state of the deuteron. The spin-orbit term as well as the tensor potential is attractive for $j = 1, 2$ and 3. For $j = 1$ the potentials found here share the same qualitative features as the Reid potentials, [5] which were computed for this channel. They differ from the potentials obtained in our earlier studies [31], partly because of their modified structure. The corresponding χ^2 divided by the number of degrees of

freedom i.e. the quantity χ_n^2 (the experimental and calculated scattering data and the quadrupole and magnetic moments are compared) gives 1.62. The calculated errors of the coefficients give $p_{N_c} = .14$ and $q = .30$ (see (4.33)).

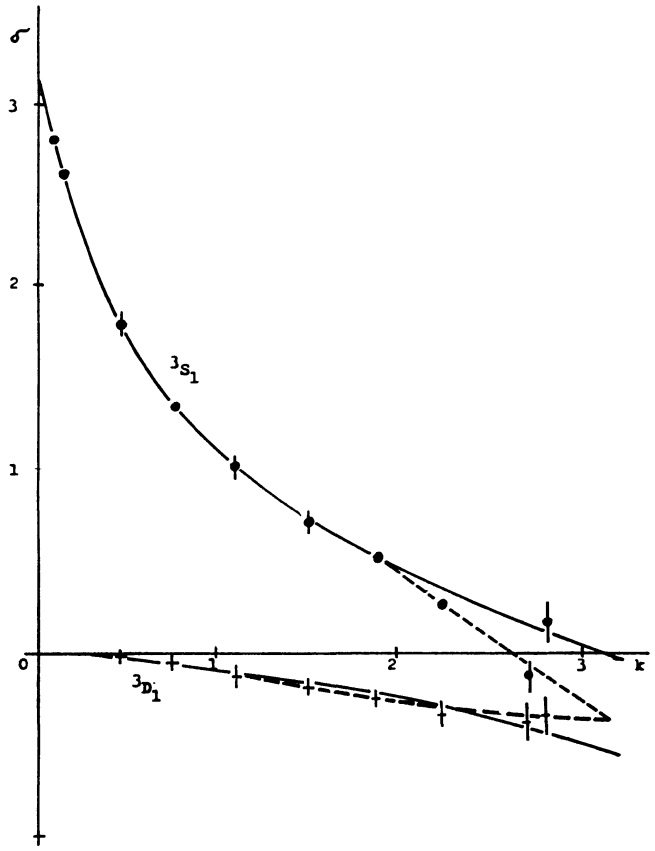

Fig. 25

Fig. 25–29. The triplet even calculated phase shifts — solid lines. The phase shifts resulting from the Livermore energy dependent (experimental) solution — dashed lines. The crosses and dots denote the experimental points used in the derivation of the potential. $k = .1553\sqrt{E(\text{MeV})}$, where E is energy of the incident particle in the laboratory system. The phase shifts are given in radians

The directly calculated triplet scattering length and the effective range parameters derived from the phase shifts and calculated at the energies $k = .1$ and $k = .3$ are contained in the Tab. I. The phase shift curves are plotted — with the exception of the mixing parameter ε_1 (see Sec. V.f) — in Fig. 25–29, where they are compared with the results of the Livermore energy dependent phase shift analysis (dashed line). In this case, the differences are larger as would be expected but with the exception of the mixing parameter

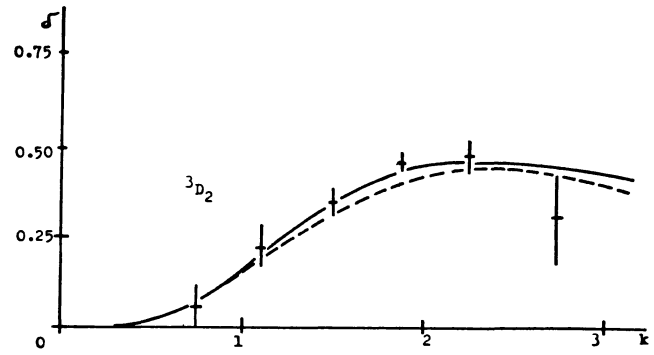

Fig. 26

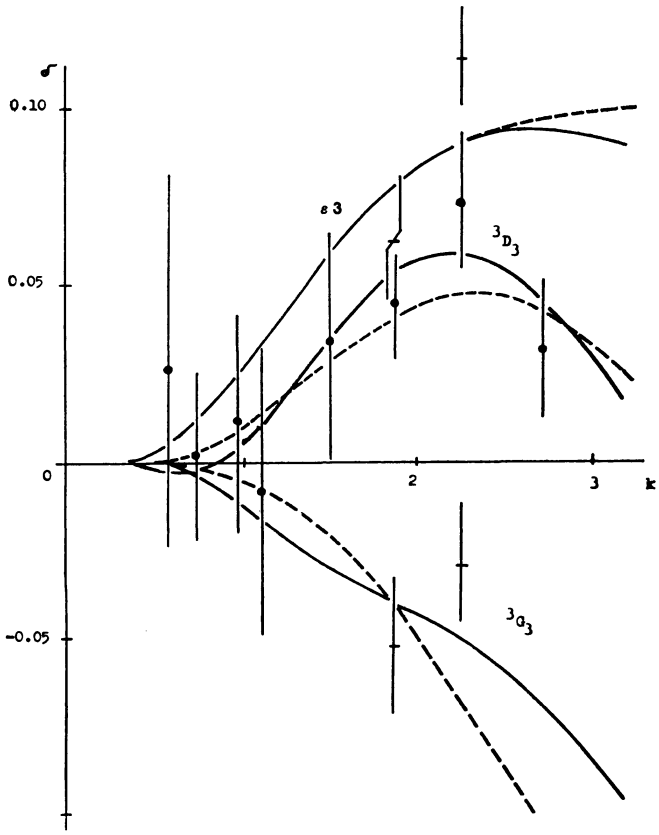

Fig. 27

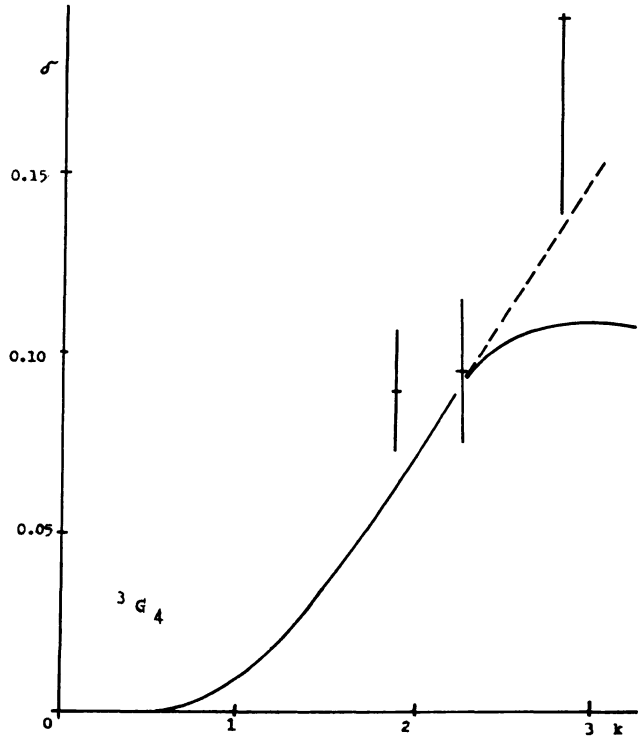

Fig. 28

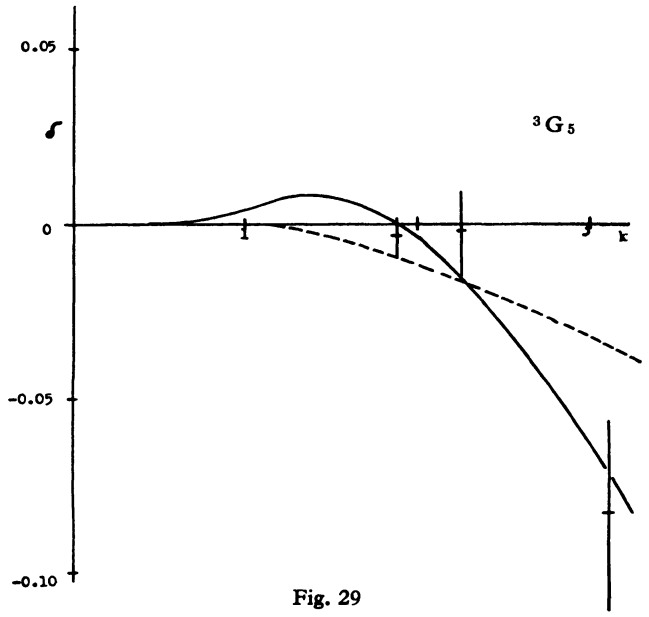

Fig. 29

ε_1, the calculated phase shifts are very close to the curves from the Livermore energy dependent analysis (to the so called experimental solution). Deviations are found in the transition region to relativistic energies and at the places where either the number of experimental data is small or their errors are large.

As described in Sec. III.c. the deuteron binding energy was fixed during the calculation of the potential coefficients $a_n^{(i)}$ and $b_n^{(i)}$. Its value in the dimensionless units and in MeV reads

$$\varkappa = .32756 \qquad E_{\text{bind}} = 2.2247 \text{ MeV} \tag{5.23}$$

The quadrupole moment and the deviation of the deuteron magnetic moment from the sum of the proton and neutron magnetic momenta have been calculated. Their values

$$\begin{aligned} Q_{\text{theor}} &= .1433 \text{ (dimensionless) i.e.} \quad Q_{\text{theor}} = .2861 f^2 \\ & \qquad\qquad\qquad\quad Q_{\text{exp}} = (.279 \pm .014) f^2 \\ \Delta M_{\text{theor}} &= .02234 \text{ (nuclear magneton)} \quad M_{\text{exp}} = (.02231 \pm .00012) \end{aligned} \tag{5.24}$$

agree with the experimental results.

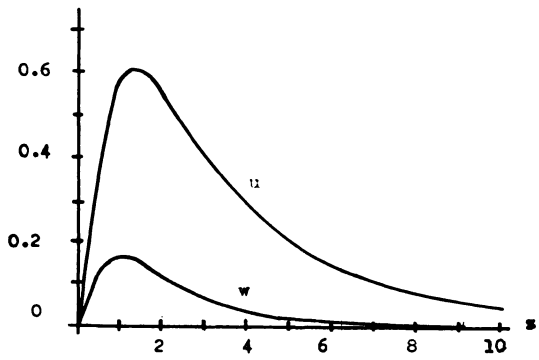

Fig. 30. The normalized deuteron wave function. Its first component — u, the second — w; $x = \mu r$, $\mu = .707 f^{-1}$, r is the internucleon distance

The corresponding normalized deuteron wave function is plotted in Fig. 30 and tabulated in Tab. XIII.

The constant K (4.23), which defines the asymptotic ratio of the S and D states in deuteron is equal to .027350, and the D state probability $P_D = .047$.

f) The ε_1-mixing parameter problem

The triplet even potential (Table IX–XII) gives the mixing parameter drawn in Figure 31 (solid line). Its shape is not quite as expected. The ε_1 is compared in this figure with the results of the Livermore energy dependent analysis [15]. There are two Livermore solutions, the first, called experimental is obtained without any particular assumption about the form of ε_1 (dashed line 1), the second, called constraint (dashed line 2) is a result

Tab. XIII. The components u and w of the normalized deuteron wave function are given as function of the parameter $x = \mu r$ where $\mu = .707f^{-1}$ is the pion mass and r is the internucleon distance.

x	u	w
0.01	0.6351 −03	0.6941 −04
0.02	0.1504 −02	0.2621 −03
0.03	0.2611 −02	0.5699 −03
0.04	0.3959 −02	0.9904 −03
0.05	0.5553 −02	0.1523 −02
0.06	0.7397 −02	0.2168 −02
0.07	0.9493 −02	0.2925 −02
0.08	0.1184 −01	0.3795 −02
0.09	0.1444 −01	0.4778 −02
0.10	0.1731 −01	0.5872 −02
0.11	0.2042 −01	0.7078 −02
0.12	0.2380 −01	0.8393 −02
0.13	0.2742 −01	0.9816 −02
0.14	0.3130 −01	0.1134 −01
0.15	0.3542 −01	0.1298 −01
0.16	0.3978 −01	0.1471 −01
0.17	0.4438 −01	0.1655 −01
0.18	0.4921 −01	0.1847 −01
0.19	0.5427 −01	0.2049 −01
0.20	0.5955 −01	0.2260 −01
0.21	0.6504 −01	0.2479 −01
0.22	0.7074 −01	0.2706 −01
0.23	0.7664 −01	0.2939 −01
0.24	0.8272 −01	0.3182 −01
0.25	0.8898 −01	0.3430 −01
0.26	0,9542 −01	0.3683 −01
0.27	0.1020 00	0.3941 −01
0.28	0.1087 00	0.4206 −01
0.29	0.1156 00	0.4474 −01
0.30	0.1227 00	0.4747 −01
0.31	0.1298 00	0.5022 −01
0.32	0.1371 00	0.5300 −01
0.33	0.1445 00	0.5576 −01
0.34	0.1520 00	0.5861 −01
0.35	0.1596 00	0.6142 −01
0.36	0.1673 00	0.6433 −01
0.37	0.1750 00	0.6715 −01
0.38	0.1828 00	0.6994 −01
0.39	0.1907 00	0.7258 −01
0.40	0.1986 00	0.7537 −01
0.41	0.2066 00	0.7802 −01
0.42	0.2146 00	0.8072 −01
0.43	0.2226 00	0.8313 −01
0.44	0.2305 00	0.8631 −01

Table XIII contd.

x	u	w
0.45	0.2384 00	0.8843 −01
0.46	0.2463 00	0.9134 −01
0.47	0.2543 00	0.9346 −01
0.48	0.2623 00	0.9584 −01
0.49	0.2702 00	0.9717 −01
0.50	0.2777 00	0.1000 00
0.60	0.3554 00	0.1273 00
0.70	0.4250 00	0.1475 00
0.80	0.4837 00	0.1614 00
0.90	0.5305 00	0.1695 00
1.00	0.5657 00	0.1729 00
1.10	0.5903 00	0.1725 00
1.20	0.6059 00	0.1695 00
1.30	0.6138 00	0.1646 00
1.40	0.6156 00	0.1584 00
1.50	0.6125 00	0.1516 00
1.60	0.6056 00	0.1444 00
1.70	0.5958 00	0.1371 00
1.80	0.5840 00	0.1299 00
1.90	0.5706 00	0.1228 00
2.00	0.5563 00	0.1161 00
2.10	0.5413 00	0.1096 00
2.20	0.5260 00	0.1034 00
2.30	0.5105 00	0.9759 −01
2.40	0.4951 00	0.9206 −01
2.50	0.4799 00	0.8685 −01
2.60	0.4648 00	0.8193 −01
2.70	0.4501 00	0.7731 −01
2.80	0.4357 00	0.7296 −01
2.90	0.4217 00	0.6887 −01
3.00	0.4081 00	0.6502 −01
3.10	0.3949 00	0.6141 −01
3.20	0.3820 00	0.5801 −01
3.30	0.3696 00	0.5482 −01
3.40	0.3576 00	0.5182 −01
3.50	0.3459 00	0.4900 −01
3.60	0.3347 00	0.4635 −01
3.70	0.3238 00	0.4386 −01
3.80	0.3133 00	0.4151 −01
3.90	0.3031 00	0.3931 −01
4.00	0.2932 00	0.3724 −01
4.10	0.2837 00	0.3528 −01
4.20	0.2745 00	0.3344 −01
4.30	0.2656 00	0.3171 −01
4.40	0.2570 00	0.3008 −01

Table XIII contd.

x	u	w
4.50	0.2486 00	0.2855 −01
4.60	0.2406 00	0.2710 −01
4.70	0.2328 00	0.2573 −01
4.80	0.2252 00	0.2445 −01
4.90	0.2179 00	0.2323 −01
5.00	0.2109 00	0.2208 −01
5.10	0.2041 00	0.2100 −01
5.20	0.1975 00	0.1998 −01
5.30	0.1911 00	0.1901 −01
5.40	0.1849 00	0.1810 −01
5.50	0.1789 00	0.1723 −01
5.60	0.1732 00	0.1642 −01
5.70	0.1676 00	0.1564 −01
5.80	0.1622 00	0.1491 −01
5.90	0.1569 00	0.1422 −01
6.00	0.1519 00	0.1356 −01
6.10	0.1470 00	0.1294 −01
6.20	0.1422 00	0.1235 −01
6.30	0.1376 00	0.1178 −01
6.40	0.1332 00	0.1125 −01
6.50	0.1289 00	0.1075 −01
6.60	0.1247 00	0.1027 −01
6.70	0.1207 00	0.9817 −02
6.80	0.1168 00	0.9384 −02
6.90	0.1130 00	0.8973 −02
7.00	0.1094 00	0.8581 −02
7.10	0.1059 00	0.8209 −02
7.20	0.1024 00	0.7855 −02
7.30	0.9918 −01	0.7517 −02
7.40	0.9599 −01	0.7196 −02
7.50	0.9289 −01	0.6890 −02
7.60	0.8990 −01	0.6599 −02
7.70	0.8700 −01	0.6321 −02
7.80	0.8419 −01	0.6056 −02
7.90	0.8148 −01	0.5803 −02
8.00	0.7885 −01	0.5562 −02
8.10	0.7631 −01	0.5332 −02
8.20	0.7385 −01	0.5112 −02
8.30	0.7147 −01	0.4903 −02
8.40	0.6916 −01	0.4703 −02
8.50	0.6693 −01	0.4511 −02
8.60	0.6478 −01	0.4329 −02
8.70	0.6269 −01	0.4154 −02
8.80	0.6067 −01	0.3987 −02
8.90	0.5871 −01	0.3828 −02

Table XIII contd.

x	u	w
9.00	0.5682 −01	0.3675 −02
9.10	0.5499 −01	0.3529 −02
9.20	0.5321 −01	0.3389 −02
9.30	0.5150 −01	0.3256 −02
9.40	0.4984 −01	0.3128 −02
9.50	0.4823 −01	0.3005 −02
9.60	0.4668 −01	0.2888 −02
9.70	0.4517 −01	0.2776 −02
9.80	0.4372 −01	0.2668 −02
9.90	0.4231 −01	0.2565 −02
10.00	0.4094 −01	0.2466 −02

of including a special condition, which keeps ε_1 positive. The reason for this condition is hidden in an expected connection between the scattering and bound state characteristics of the two-nucleon system. For example, the positive value of the quadrupole moment is often related to the positive value of ε_1. It has already been pointed out that there is no theoretical relation, which connects the bound state and scattering properties other than the Levinson theorem (see Sec. III.c). Other connections may be looked for on a basis of special and limiting assumptions about the analytic properties of the S matrix. Moreover, it is easy to find a channel potential for the $j = 1$ triplet even state, which describes quite well the scattering data and the bound state characteristics giving the quadrupole moment $Q_{\text{theor}} = .152$, i.e. a value, which is only slightly higher than the experimental (5.24). The $^3S_1$ and $^3D_1$ phase shifts practically coincide with those plotted in the Fig. 25, the corresponding ε_1 which is negative at low energies, is drawin in Fig. 31 (dotted line 1) and the corresponding channel coefficients are tabulated in Tab. XIV.

Table XIV

j	n	$a_n^{(j,j-1)}$	a_n^{tj}	$a_n^{(j,j+1)}$
1	1	5.4485	− 7.1870	3.6047
	2	− 74.711	32.403	− 30.132
	3	139.89	− 97.675	97.349

There is also a potential with nine coefficients giving the $^3S_1$ and $^3D_1$ phase shifts very close to those from Fig. 25 and ε_1 almost coinciding with the Livermore constrained solution. The corresponding $\chi_n^2 = 1.8$. It follows from these two examples and from other potentials describing the $j = 1$ triplet even state of the two nucleon system that the experimental data in this channel would admit quite a large set of different potentials enabling them to be constructed in almost any desired form.

A simultaneous solution of the inverse problem significantly limits this ambiguity. If the potential for the triplet even state is derived using a solution of the inverse problem from the scattering data only but for all angular momenta $j = 1 \div 5$ then it takes a form which is very close to the final shape obtained by considering the complete set of available non-relativistic data. This result shows the strong influence of the correlations between the phase-shifts and is quite independent on the actual method of solution.

In Fig. 31 a direct fit to the *JINR* data for the ε_1 is also plotted (the dotted line 2). This fit also takes into account the *JINR* measurements at higher energies up to 600 MeV. It seems that the computed mixing parameter ε_1 shares in the low energy region certain common features with the Livermore constrained solution and that at higher energies it tends to its asymptotic form of the fit (given by dotted line 2). As a result the relative high

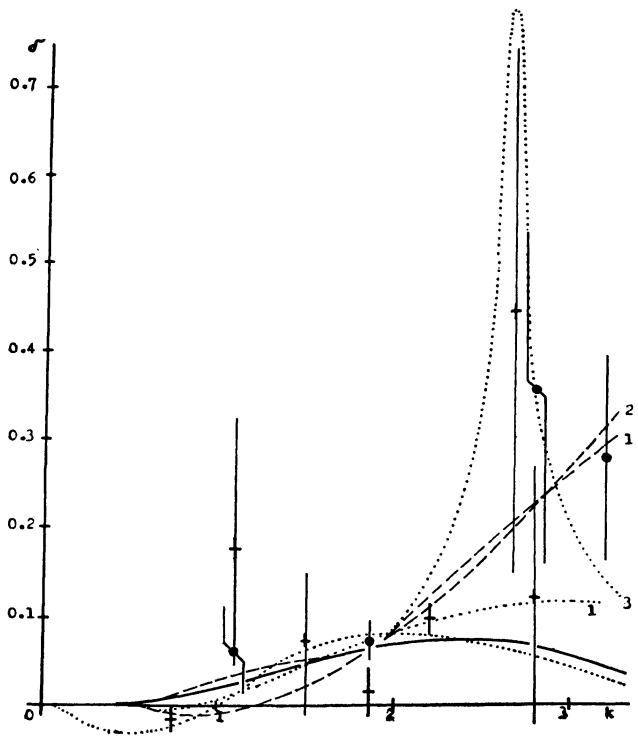

Fig. 31. The calculated mixing parameter ε_1 (triplet even $j = 1$ state) — solid line. The crosses are experimental points used in the computation of the two-nucleon potential. The dots represent the other available (Livermore) data [15], which were not used. The dashed line 1 — the Livermore experimental solution, the dashed line 2 — the Livermore constrained solution; the dotted line 1 — the mixing parameter ε_1 for a channel potential for which $\varepsilon_1 < 0$ for small momenta k and the deuteron quadrupole moment is positive. The dotted line 2 — the direct fit of ε_1 to the *JINR* data, the dotted line 3 — the mixing parameter ε_1 for a channel potential, which gives unbounded phase shifts $\delta_{j,j-1}$ and $\delta_{j,j+1}$ ($j = 1$). $k = .1553 \sqrt{E(\text{MeV})}$, the parameter ε_1 is given in radians

experimental values of the mixing parameter ε_1 at energies 300–400 MeV ($k \sim 3$) may perhaps not be correct. If they are, they may be connected with the discontinuity of the phase shifts $\delta_{j,j-1}$ and $\delta_{j,j+1}$ described in Sec. IV.a. Although ε_1 does not achieve the critical value $\pi/4$, it is quite near to it at $k \sim 3$ and this fact — if it is true — may induce some irregularities in the behaviour of these two phase shifts at this energy. Again it is not difficult for this case to find a relatively good channel-potential ($j = 1$) [32], which gives ε_1 as plotted in Fig. 31 (dotted line 3). The corresponding phase shifts $\delta_{j,j-1}$ and $\delta_{j,j+1}$ are discontinuous at $k = 2.7$. Such potential however as well as giving discontinuities seems to be excluded by the simultaneous solution of the inverse scattering problem.

Conclusion

1. By a simultaneous fit to the nucleon-nucleon scattering data in the energy region 0–330 MeV and to the basic deuteron data a two-nucleon potential has been derived for the singlet even and odd and for the triplet odd and even states for all total angular momenta $j \leq 5$. The χ^2 test gives $\chi^2_n \leq 1.62$. The Livermore scattering data alone are also fairly well reproduced by this potential [34].
2. The potential is independent of the radial velocity but has central spin-orbit and tensor terms and is a linear function of the square of the total angular momentum $\mathcal{J}$. It has not been found necessary to include the quadratic spin-orbit interaction or radial velocity dependence.

   The radial part of the potential is expressed as a sum of Yukawa terms, i.e. the singularity of the potential is small.
3. The most recent *JINR* energy independent phase shift analysis data together with the corresponding new Livermore data have been used for the solution of the inverse problem and it has been shown that these two sets of data are in principle equivalent to the results of the Livermore energy dependent phase shift analysis. The only discrepancy in this one to one correspondence seems to occur in the description of the mixing parameter $\varepsilon_1(j = 1)$
4. A simultaneous fit of the potential to the two nucleon data indicates quite strong correlations among the phase shifts for different angular momenta. The most pronounced correlation appears in the triplet even state.
5. The zero energy data and the difference in the $^1S_0$ *p-p* and *n-p* nuclear phase shift leads to slightly different potentials acting in the singlet even *p-p* and *n-p* states.
6. The potentials have generally both attractive and repulsive parts and in some cases the repulsive part near the origin of the co-ordinate system can be interpreted as a soft core.

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