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Universal Approximations of Certain Functionals in Banach Spaces

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Some older results of the author [1], [2] are modified for Banach spaces with bases. Simultaneous approximations of coefficient functionals associated to a given basis generating a system of Banach spaces are treated. The conditions on this system sufficient for the existence of universal approximation are given. In a special case an optimal universal approximation is constructed.

Let $\mathscr{C}$ be a class of Banach spaces generated by a common Schauder basis $\{x_j\}$. Our problem is to find $F_j(x), j = 1, 2, \ldots, r$ where $x = \sum_{j=1}^{\infty} F_j(x)\, x_j$. Suppose $r > 1$ and let us know that x is an element of an indefinite space $E \in \mathscr{C}$. We shall approximate the vector $[F_j]$ of functionals from $E^\star$ by another vector $[G_j]$, $G_j \in E^\star$ and we suppose that

$$G_j(x) = \sum_{k=1}^{n} a_k(x)\, g_k(j)$$

where $1 \leq n < r$, a_k are linear functionals defined on every $E \in \mathscr{C}$ and g_k are complex-valued functions of an integer argument j.

For a given n, denote M_n the set of all the above vector approximations. The error of the approximation is defined as

$$\omega_E(G_j) \equiv \max_{j=1,2,\ldots,r} \|F_j - G_j\|_{E^\star}.$$

Let $M \subset M_n$. The best approximation from the set M (if it exists) has the error

$$\Omega_E(M) \equiv \inf_{[G_j]\in M} \omega_E(G_j)$$

which can be shown to be positive. Now, we define universal approximations as those approximations $[G_j] \in M_n$ for which

$$\omega_E(G_j) \leq D \,.\, \Omega_E(M_n)$$

for any $E \in \mathscr{C}$ and D is a constant independent of E.

It is clear that for a sufficiently small class $\mathscr{C}$ every approximation from M_n will be universal. On the other hand, for wide classes of spaces a universal approimation need not exist. Thus it is desirable to have some (as general as possible) conditions on $\mathscr{C}$ that would guarantee the existence of a universal approximation.

For brevity we must limit ourselves to the classes of Banach spaces with monotone bases [3]. The sufficient condition we look for is then that the elements of the basis can be assigned subscripts in such a way that

$$||x_1|| \leq ||x_2|| \leq \dots \leq ||x_r||$$

in every $E \in \mathscr{C}$. We shall call such classes $\mathscr{C}$ conservative. Under these conditions (which can be much weakened) an example of a universal approximation is the approximation $[B_j]$ with $a_k = F_k$, $g_k(j) = \delta_{kj}$. Here the matrix $[g_k(j)]$ has the form $[\mathbf{I}, \mathbf{O}]$ where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{O}$ the $n \times (r - n)$ null matrix.

Ask now what the matrices $[g_k(j)]$ of universal approximations look like. To exclude from our considerations the classes $\mathscr{C}$ with respect to which any approximation from M_n is universal it is necessary to suppose that $\mathscr{C}$ is wide enough. For example, if we suppose that $\mathscr{C}$ is conservative and for any D it contains a space in which $||x_{n+1}|| > D \, . \, ||x_n||$ then it can be proved that the matrix $[g_k(j)]$ of a universal approximation necessarily has the form $[\mathbf{A}, \mathbf{O}]$ where $\mathbf{A}$ is a regular $n \times n$ matrix.

Denote U_n the subset of all universal approximations from M_n now. It can be shown that for the above introduced approximation $[B_j]$

$$\omega(B_j) \leq 2\Omega(U_n)$$

and that in the case of an orthogonal basis [3] this $[B_j]$ is even an optimal universal approximation.

Since it was more lucid we spoke about approximating functionals by functionals i.e. more or less theoretically. However, if we replace the above "theoretical" approximations by convergent sequences of functionals we can easily obtain analogous asymptotic results of more practical importance.

References

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[2] PŘIKRYL, P.: On the Best Universal Approximations of Fourier Coefficients (Russian). In: Primenenie funkcional'nych metodov k kraevym zadačam matematičeskoĭ fiziki. Institut matematiki SOAN SSSR, Novosibirsk (1972), pp. 191—198.

[3] SINGER, I.: Bases in Banach Spaces I. Springer-Verlag, Berlin-Heidelberg-New York (1970).