Acta Universitatis Carolinae. Mathematica et Physica

Ivan Hlaváček
On a conjugate semi-variational method for parabolic equations

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 15 (1974), No. 1-2, 43--45

Persistent URL: http://dml.cz/dmlcz/142324

1974 ACTA UNIVERSITATIS CAROLINAE - MATHEMATICA ET PHYSICA NO. 1-2 PAG. 43-45

On a Conjugate Semi-variational Method for Parabolic Equations

I. HLAVÁČEK
Mathematical Institute, Czechoslovak Academy of Sciences, Prague

Formulating the heat conduction problem in terms of the heat flux vector, a particular case of a general parabolic equation with positive definite and positive semi-definite operator coefficients is obtained. To such problem, the so-called semi-variational method can be applied, yielding a sequence of approximations with an increasing accuracy. Even the first approximation shows favourable numerical results in comparison with the corresponding procedure for the original problem.

Let us consider the following mixed problem

$$\frac{\partial u}{\partial t} - \Delta u = f \,,$$

$$x = (x_1, x_2, \dots x_n) \in \Omega \,, \quad 0 < t \leq T < \infty \,, \tag{1}$$

$$u(.\,, 0) = \varphi \,, \tag{2}$$

$$u = g \quad \text{on} \quad \Gamma_u \times (0, T\rangle \,, \tag{3}$$

$$\frac{\partial u}{\partial \nu} = P \quad \text{on} \quad \Gamma_h \times (0, T\rangle \,, \tag{4}$$

$$\alpha u + \frac{\partial u}{\partial \nu} = P \quad \text{on} \quad \Gamma_v \times (0, T\rangle \,, \tag{5}$$

where Ω is a bounded domain with a Lipschitz boundary Γ, $\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x_i} \nu_i$, repeated Latin index implies summation over the range $1, 2, \dots, n$, ν_i denote the components of the unit outward normal to Γ. The boundary consists of four mutually disjoint parts Γ_u, Γ_h, Γ_v and Γ_0. Each of Γ_u, Γ_h, Γ_v is either open in Γ or empty, mes Γ_0 is zero. We assume

$$0 < \alpha_0 \leqq \alpha(x) \leqq \alpha_1 < \infty \,, \qquad x \in \Gamma_v \,.$$

Let us denote

$$\frac{\partial u}{\partial x_i} = h_i \,, \qquad \Delta u = \frac{\partial h_i}{\partial x_i} = \operatorname{div} \mathbf{h} \,, \qquad h_i \nu_i = h_\nu \,. \tag{6}$$

The approach can be applied to more general parabolic problems (cf. [2](b)), e.g. to the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0 u = f\,.$$

Differentiating (1) with respect to x_k and using (6), we are led to problem

$$\frac{\partial}{\partial t} h_k - \frac{\partial}{\partial x_k} \operatorname{div} \mathbf{h} = \frac{\partial f}{\partial x_k}\,, \quad k = 1, 2, \ldots, n\,, \tag{7}$$

$$\mathbf{h}(\,.\,, 0) = \operatorname{grad} \varphi\,, \tag{8}$$

$$f + \operatorname{div} \mathbf{h} = \frac{\partial g}{\partial t} \quad \text{on} \quad \Gamma_u \times (0, T\rangle\,, \tag{9}$$

$$h_\nu = P \quad \text{on} \quad \Gamma_h \times (0, T\rangle\,, \tag{10}$$

$$\alpha(f + \operatorname{div} \mathbf{h}) + \frac{\partial}{\partial t} h_\nu = \frac{\partial}{\partial t} P \quad \text{on} \quad \Gamma_v \times (0, T\rangle\,. \tag{11}$$

The problem (7) — (11) will be called *conjugate* to the original problem (1) — (5) (cf. [1]). Henceforth, we set $P = 0$ on $\Gamma_h \times (0, T\rangle$ for brevity.

Let us define the following linear spaces and bilinear forms:

$$H_B = \{\boldsymbol{\chi} \in [L_2(\Omega)]^n,\ \chi_\nu \in L_2(\Gamma_v)\}, \quad \overline{H}_A = \{\boldsymbol{\chi} \in [L_2(\Omega)]^n,\ \operatorname{div} \boldsymbol{\chi} \in L_2(\Omega)\},$$

$$H_A = \{\boldsymbol{\chi} \in \overline{H}_A,\ \chi_\nu = 0 \quad \text{on} \quad \Gamma_h\}, \quad \mathscr{V} = H_A \cap H_B\,,$$

$$(\varphi, \psi) = \int_\Omega \varphi\psi \,\mathrm{d}x, \quad (\varphi, \psi)_{\Gamma_S} = \int_{\Gamma_S} \varphi\psi \,\mathrm{d}\Gamma, \ \Gamma_S = \Gamma_u \quad \text{or} \quad \Gamma_v,$$

$$B(\mathbf{h}, \boldsymbol{\chi}) = (h_i, \chi_i) + (\alpha^{-1} h_\nu, \chi_\nu)_{\Gamma_v}\,, \quad \mathbf{h}, \boldsymbol{\chi} \in H_B\,,$$

$$A(\mathbf{h}, \boldsymbol{\chi}) = (\operatorname{div} \mathbf{h}, \operatorname{div} \boldsymbol{\chi}), \quad \mathbf{h}, \boldsymbol{\chi} \in \overline{H}_A,$$

$$B(\mathbf{h}, \mathbf{h}) = \|\mathbf{h}\|_B^2\,, \quad A(\mathbf{h}, \mathbf{h}) = \|\mathbf{h}\|_A^2\,, \quad \|\mathbf{h}\|_A^2 + \|\mathbf{h}\|_B^2 = \|\mathbf{h}\|_{\mathscr{V}}^2\,.$$

Let $L_2(\langle 0, T\rangle, H)$ denote the space of measurable mappings $u(t)$ of $\langle 0, T\rangle$ into a normed space H such that

$$\int_0^T \|u(t)\|_H^2 \,\mathrm{d}t < \infty\,.$$

Assume that the data satisfy the following conditions: $\operatorname{grad} \varphi \in H_B$, $f(\,.\,, t) \in L_2(\Omega)$, $\frac{\partial}{\partial t} g(\,.\,, t) \in W_2^{(1)}(\Omega)$, $\frac{\partial}{\partial t} P(\,.\,, t) \in L_2(\Gamma_v)$, $t \in \langle 0, T\rangle$.

We say that $\mathbf{h}(x, t)$ is a weak solution of the conjugate problem (7) through (11), if $\mathbf{h} \in L_2(\langle 0, T\rangle, \mathscr{V})$, $\frac{\partial \mathbf{h}}{\partial t} \in L_2(\langle 0, T\rangle, H_B)$,

$$B\left(\frac{\partial \mathbf{h}}{\partial t}, \boldsymbol{\chi}\right) + A(\mathbf{h}, \boldsymbol{\chi}) = -(f, \operatorname{div} \boldsymbol{\chi}) + \left(\frac{\partial g}{\partial t}, \chi_\nu\right)_{\Gamma_u} + \left(\alpha^{-1}\frac{\partial P}{\partial t}, \chi_\nu\right)_{\Gamma_v}\,,$$

$$0 < t \leqq T\,, \quad \boldsymbol{\chi} \in \mathscr{V}\,,$$

$$B(\mathbf{h}(\,.\,, 0) - \operatorname{grad} \varphi, \boldsymbol{\chi}) = 0, \quad \boldsymbol{\chi} \in \mathscr{V}\,. \tag{12}$$

The definition (12) corresponds with the equation

$$B \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} + A\mathbf{h} = \bar{f} \tag{13}$$

in $\mathscr{V}$, where B is a positive definite and A a positive semi-definite operator. Hence the semi-variational method [2a] can be applied. If $\{\chi^1, \chi^2, \dots, \chi^N\}$ form a basis of a subspace $\mathscr{M} \subset \mathscr{V}$ and $\tau = T/M$ is the time-increment, then e.g. the first approximation $\mathbf{h}^{(1)}(., m\tau)$ (so-called CRANK-NICOLSON-GALERKIN) is given by

$$\begin{gathered}
\mathbf{h}^{(1)}(., m\tau) = \sum_{j=1}^{N} w_j^m \chi^j, \qquad m = 0, 1, 2, \dots M, \\
\mathscr{B}\mathbf{w}^0 = \omega, \qquad \mathbf{w}^{m+1} = 2\,\mathbf{a}^m - \mathbf{w}^m, \\
\left(\mathscr{B} + \frac{1}{2}\tau\mathscr{A}\right)\mathbf{a}^m = \mathscr{B}\mathbf{w}^m + \frac{1}{4}\tau(\mathbf{F}(m\tau) + \mathbf{F}(m\tau + \tau)),
\end{gathered} \tag{14}$$

$$\begin{gathered}
\mathscr{B}_{ij} = B(\chi^i, \chi^j), \qquad \mathscr{A}_{ij} = A(\chi^i, \chi^j), \\
\omega_j = B(\operatorname{grad}\varphi, \chi^j), \quad F_j(m\tau) = \langle \bar{f}(m\tau), \chi^j \rangle,
\end{gathered} \tag{15}$$

by $\langle \bar{f}(t), \chi \rangle$ denoting the right hand side of (12).

Using finite elements with the diameter δ in $[W_2^{(1)}(\Omega)]^n \subset \bar{H}_A \cap H_B$, the estimate of the type

$$\|\mathbf{z}_m\|_B + \left(\sum_{p=0}^{m} \tau \left\| \frac{1}{2}(\mathbf{z}_p + \mathbf{z}_{p+1}) \right\|_A^2 \right)^{\frac{1}{2}} = O(\delta^k + \tau^2)$$

can be proved for $\mathbf{z}_m = \mathbf{h}(., m\tau) - \mathbf{h}^{(1)}(., m\tau)$ and any m.

The second approximation (cf. [2]) is proved to be fourth order correct in τ.

Numerical examples of the first approximation were calculated for the case $n = 1$, $\Omega = (0,1)$, $T = 1{,}6$, $\Gamma = \Gamma_u$, $g = 0$, $\varphi = 0$, the solution of which is $u = te^{-t} \,.\, x^2(1 - x)^2$ with the use of cubic elements on $\langle 0,1 \rangle$. Comparing the derivatives $\frac{\partial u}{\partial x}(0, t)$, we obtain 10 till 10^3 times smaller errors by the conjugate method than by the original one, though the gross amounts of the computing work (the size of matrices) were the same.

References

[1] AUBIN, J. P., BURCHARD, H. G.: Some Aspects of the Method of Hypercircle. Numer. Sol. Part. Dif. Eqs. II. Synspade, pp. 1—67 (1970).

[2] HLAVÁČEK, I.: (*a*) On a Semi-variational Method for Parabolic Equations. Aplikace matematiky 17, (1972), 5, 327—321 and 18, (1973), 1, 43—64; (*b*) On a Conjugate Semi-variational Method for Parabolic Equations. Aplikace matematiky 18, (1973), 6, 434—444.