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Some Results on Eigenvalues of Nonnegative Matrices

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As is well known, nonnegative matrices and closely related M-matrices play an important role in numerical mathematics. In particular, knowledge about eigenvalues of such matrices has applications in analyzing convergence of numerical processes, in estimates of eigenvalues of general matrices etc.

We shall be interested here in conditions, necessary or sufficient, for n numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, that there exists a nonnegative n by n matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

For sake of brevity, we shall denote by $\mathscr{P}_n$ the set of all (unordered) n-tuples which form the spectrum of some nonnegative n by n matrix, and by $\widehat{\mathscr{P}}_n$ a similar set corresponding to spectra of nonnegative n by n symmetric matrices.

To obtain some necessary conditions for $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}_n$, observe that if $A \geqq 0$ has the eigenvalues $\lambda_1, \ldots, \lambda_n$ then $\operatorname{tr} A^k = \sum_{i=1}^{n} \lambda_i^k$. Since $A^k \geqq 0$ for $k = 1, 2, \ldots$,

$$\sum_{i=1}^{n} \lambda_i^k \geqq 0, \quad k = 1, 2, \ldots \tag{1}$$

is a set of necessary conditions for $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}_n$.

Suleimanova, in a slightly different formulation, conjectured [11] that (1) is also sufficient for $(\lambda_1, \ldots, \lambda_n)$ to belong to $\mathscr{P}_n$.

Salzmann [10] disproved this conjecture for $n \geqq 5$ by showing that $(3, 3, -2, -2, -2) \notin \mathscr{P}_5$ although (1) is satisfied. His simple and elegant proof uses the fact that, according to the well known Perron-Frobenius theorem, an irreducible nonnegative matrix has its spectral radius (Perron root) as a simple eigenvalue.

Since $\mathscr{P}_n$ is easily seen to be a closed cone in the space of all complex n-tuples, it follows that for any sufficiently small positive number ε, $(3, 3-\varepsilon, -2+\varepsilon, -2, -2)$ does not belong to $\mathscr{P}_5$ as well. In this case, however, Salzmann's proof does not apply. This observation suggested that some quantitative extension of the Perron-Frobenius theorem should be looked for.

This was done first in [1] for doubly stochastic matrices, i.e. for nonnegative matrices with all row- and column-sums equal to one. For such an n by n matrix

$A = (a_{ik})$ where $n \geqq 2$, the so called measure of irreducibility was defined by

$$\mu(A) = \min_{\varnothing \neq M \subsetneq N} \sum_{\substack{i \in M \\ k \notin M}} a_{ik} \tag{2}$$

where $N = \{1, 2, \ldots, n\}$.

It was shown that if $\mu(A) > 0$ (i.e. if A is irreducible) then the circle in the complex plane

$$|1 - \lambda| < 2\left(1 - \cos \frac{\pi}{n}\right) \mu(A) \tag{3}$$

contains exactly one eigenvalue of A, the number 1.

The theory of the so called additive compound matrices allowed then to prove in [3] that if A is doubly stochastic and symmetric with eigenvalues $1 = \lambda_1 \geqq \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$ then

$$\lambda_1 - \lambda_2 + 2\left(1 - \cos \frac{\pi}{n}\right) \left(\sum_{i \in N} \lambda_i - \min_{\substack{\varnothing \neq M \subsetneq N, \\ \sum_{i \in M} \lambda_i \geq 0}} \sum_{i \in M} \lambda_i\right) \geq 0 \tag{4}$$

where $N = \{1, \ldots, n\}$.

This inequality already eliminates Salzmann type counterexamples for symmetric stochastic matrices.

For a general nonnegative n by n matrix $A = (a_{ik})$, the measure of irreducibility $m(A)$ was defined in [2] as follows:

$$m(A) = 0 \text{ if } A \text{ is reducible; if } A \text{ is irreducible,}$$

$$m(A) = \frac{n}{\sum_{i \in N} u_i v_i} \min_{\varnothing \neq M \subsetneq N} \sum_{\substack{i \in M \\ k \notin M}} a_{ik} u_k v_i$$

where $u = (u_i) > 0$ and $v = (v_i) > 0$ are eigenvectors of A and A^T corresponding to the Perron root $p(A)$ of A.

A generalization of a new proof for the inequality (3) developed in [4] enabled then to prove the following theorem in [2]:

If A is an n by n irreducible nonnegative matrix, $p(A)$ its Perron root and $m(A)$ its measure of irreducibility then the circle

$$|p(A) - \lambda| < \frac{4}{n(n-1)} m(A) \tag{5}$$

contains exactly one eigenvalue of A, namely $p(A)$.

We should mention that the multiplicative constants $2\left(1 - \cos \frac{\pi}{n}\right)$ in (3) and $4/(n(n-1))$ in (5) are best possible.

It is an open problem to find a similar inequality to (4) for the case of a general nonnegative symmetric matrix or even for the case of a general nonnegative matrix.

Let us turn now to sufficient conditions for $(\lambda_1, \ldots, \lambda_n)$ to belong to $\mathscr{P}_n$. Sulejmanova [12] announced and H. PERFECT [9] proved essentially that if $\lambda_1, \ldots, \lambda_n$ are real such that

$$\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n \tag{6}$$

and

$$\lambda_1 + \sum_{j, \lambda_j < 0} \lambda_j \geqq 0 \tag{7}$$

then $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}_n$.

SALZMANN improved this result by showing [10] that if (6) and

$$\tfrac{1}{2}(\lambda_i + \lambda_{n+1-i}) \leqq \tfrac{1}{2} \sum_{k=1}^{n} \lambda_k, \quad i = 2, \ldots, \left[\frac{n+1}{2}\right], \ \sum_{k=1}^{n} \lambda_k \geqq 0 \tag{8}$$

are fulfilled then $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}_n$.

The following result was in a slightly different form proved by KELLOGG [8]:

If (6) is fulfilled, if a subset L of N and an index m are defined by

$$L = \left\{ i \in \left\{ 2, \ldots, \left[\frac{n+1}{2}\right] \right\} \,\middle|\, \lambda_i \geqq 0 \ \ \& \ \ \lambda_i + \lambda_{n+2-i} < 0 \right\}$$

and

$$m = \max \{ j \mid 1 \leqq j \leqq n \ \ \& \ \ \lambda_j \geqq 0 \}$$

then

$$\lambda_1 + \sum_{\substack{i \in L \\ i < k}} (\lambda_i + \lambda_{n+2-i}) + \lambda_{n+2-i} \geqq 0, \quad k \in L,$$

$$\lambda_1 + \sum_{i \in L} (\lambda_i + \lambda_{n+2-i}) + \sum_{i=m+1}^{n-m+1} \lambda_j \geqq 0 \tag{9}$$

implies $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}_n$.

I was able to show:

If (6) and

$$\lambda_1 + \lambda_n + \sum_{i=1}^{n} \lambda_i \geqq \tfrac{1}{2} \sum_{i=2}^{n-1} |\lambda_i + \lambda_{n+1-i}| \tag{10}$$

are fulfilled then $(\lambda_1, \ldots, \lambda_n) \in \widehat{\mathscr{P}}_n$, i.e. there exists a symmetric n by n matrix with eigenvalues λ_i.

Recently, it was shown in [5] that the following is true:

If $\lambda_1, \ldots, \lambda_n$ satisfy (6) then

$$(7) \Rightarrow (8) \Rightarrow (10) \Rightarrow (9)$$

and (9) implies already $(\lambda_1, \ldots, \lambda_n) \in \widehat{\mathscr{P}}_n$.

It seems interesting that this theorem is valid in any ordered field which is Euclidean, i.e. which contains the square roots of each its positive element.

There is still a wide gap between known necessary and known sufficient conditions for $(\lambda_i) \in \mathscr{P}_n$ or $(\lambda_i) \in \widehat{\mathscr{P}}_n$. Indeed, in the case $\sum_{i=1}^{n} \lambda_i = 0$ the conditions (7) through (10) can be fulfilled only if at least $\tfrac{1}{2}n$ of these numbers are negative. However, it is possible to construct nonnegative, even symmetric matrices of order n with trace zero which have no more than $[2\sqrt{n}] - 1$ negative eigenvalues.

Let us turn now to a different but related problem: to find necessary or sufficient conditions for $2n$ real numbers $\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_n$ and $a_1 \geqq a_2 \geqq \dots \geqq a_n \geqq 0$ that there exists an n by n symmetric nonnegative matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and diagonal entries $a_1, \dots, a_n$.

It is well known that if nonnegativity of the matrix is not required, necessary and sufficient conditions have been obtained by HORN [7] and are as follows:

$$\sum_{i=1}^{s} \lambda_i \geqq \sum_{i=1}^{s} a_i, \quad s = 1, \dots, n-1,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i. \tag{11}$$

For the case of nonnegative matrices, some necessary and some sufficient conditions were proved in [5].

A necessary condition is the following:

$$\lambda_1 \geqq a_1, \quad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,$$

$$\sum_{i=1}^{s} \lambda_i + \lambda_k \geqq \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k$$

for all s and k, $1 \leqq s < k \leqq n$. This condition is even sufficient for $n \leqq 3$.

A sufficient condition is given by

$$\sum_{i=1}^{s} \lambda_i \geqq \sum_{i=1}^{s} a_i, \quad s = 1, \dots, n-1,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,$$

$$\lambda_k \leqq a_{k-1}, \quad k = 2, \dots, n-1.$$

Both these conditions are valid in any ordered Euclidean field. As was shown in [6], the same is true about Horn's system (11).

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