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The Colimits in the Generalized Algebraic Categories

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The aim of this paper is to discuss the cocompleteness of a certain class of categories, the generalized algebraic categories. These categories are a natural generalization of the categories of universal algebras — they were first defined in [1] by Trnková and Goralčík in connection with Wyler's paper [2]. This class of categories contains not only all the categories of universal algebras, but also some categories of topological and convergent spaces and other well-known categories.

A generalized algebraic category, denoted by $A(F, G)$, where F and G are set functors (i.e. functors from the category of sets into itself), is the category, the objects of which are pairs (X, ω), X a set, ω a mapping from FX to GX; the morphisms from (X, ω) to (X', ω') are all the mappings $f : X \to X'$ such that the diagram, consisting of Ff, ω, Gf, ω' is commutative. In particular in the covariant case (both F and G covariant) we have

$$\begin{tikzcd} FX \arrow[r, "\omega"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] \\ FX' \arrow[r, "\omega'"] & GX' \end{tikzcd}$$

in the contravariant case (both functors contravariant)

$$\begin{tikzcd} FX \arrow[r, "\omega"] & GX \\ FX' \arrow[r, "\omega'"] \arrow[u, "Ff"] & GX' \arrow[u, "Gf"] \end{tikzcd}$$

We shall deal only with these $A(F, G)$ which have a common variance of F and G; the case of different variances is the subject of another paper [6].

Several papers study the question of the existence of limits and colimits in $A(F, G)$ in connection with the choice of the two set functors ([1], [3] – [6]). In the covariant case this problem was fully solved for sums, [3] and coequalizers, [4], and so, by adding cosingleton in the current paper, we are able to give a necessary and sufficient condition for the cocompleteness and the finite cocompleteness. We also turn our attention to the preservation of the colimits by the natural forgetful functor (i.e. to the colimits which in underlying sets and mappings coincide with the colimits in the category of sets). The contravariant case proves to be much simpler. Here we even give a necessary and sufficient condition for the existence of colimits of all diagrams over any given diagram scheme. Moreover we show that whenever colimits exist they are preserved by the forgetful functor.

Let us remark that originally the generalized algebraic categories were defined in a bit different way, as categories $A(F, G, \delta)$ where $\delta = \{\alpha_i, i \in I\}$ is a type (i.e. α_i are ordinals) with objects $(X, \{\omega_i, i \in I\})$, where $\omega_i : (FX)^{\alpha_i} \to GX$, and morphisms $f : (X, \{\omega_i, i \in I\}) \to (X', \{\omega_i', i \in I\})$ fulfilling: for every i the diagrame with $(Ff)^{\alpha_i}$, ω_i, Gf, ω_i' is commutative. In this form it is more evident how these categories originated — clearly the category of universal algebras of the type δ is just the category $A(I, I, \delta)$ (I is the identity functor). Clearly for every $A(F, G, \delta)$ there exists a functor $F^\star$ such that $A(F^\star, G)$ is isomorphic with $A(F, G, \delta)$ — in this sense the two definitions are equivalent.

We use this oportunity to thank Dr. V. Trnková who adviced us during our whole work on this problem.

I. Preliminaries

I. 1. Convention

As usual we denote by $\mathfrak{K}^\circ$ the class of objects of a category $\mathfrak{K}$, by $\mathfrak{K}(a, b)$ the set of morfisms from a to b, $a, b \in \mathfrak{K}^\circ$.

We shall use the usual terminology from the theory of categories: diagram in a category (i.e. a functor from a small category into this one-covariant or contravariant), bound and cobound, limit and colimit etc. The limit of the empty diagram in $\mathfrak{K}$ (the empty functor to $\mathfrak{K}$) is called the *singleton* of $\mathfrak{K}$ — it is such an object, to which there leads just one morphism from any other object; analogously the colimit of the empty diagram is *consingleton*, from which there leads always just one morphism.

I. 2. Convention

Denote by *Set* the category of sets and mappings. We work in the Gödel-Bernays set theory, indicating by *GCH* that we assume the generalized continuum hypothesis. The cardinality of a set X is denoted by $|X|$; 0 is the empty set, 1 the standart one-point set $1 = \{0\}$.

We recall, that for arbitrary set diagram $D : \mathfrak{K} \to Set$ the following holds: A bound of D, $(M, \{\varphi_d, d \in \mathfrak{K}^\circ\})$, is its limit iff for every collection $\{x_d, d \in \mathfrak{K}^\circ\}$, $x_d \in Dd$ such that whenever $\delta \in D(d_1, d_2)$, then $D\delta(x_{d_1}) = x_{d_2}$ (if D is covariant) or $D\delta(x_{d_2}) = x_{d_1}$ (if it is contravariant) there exists just one x such that $\varphi_d(x) = x_d$ holds for every $d \in \mathfrak{K}^\circ$.

The singleton in *Set* is any one-point set, the cosingleton is 0.

I. 3. Convention

When we say just functor, without giving its domain and range, we mean a set functor, i.e. a functor from *Set* to *Set*. In parts II, III (the covariant case) it means covariant set functor.

Denote by Q_M (or P_M) the covariant (or conravariant) hom-functor. Denote by C_{MpN} the constant functor ($p : M \to N$ an arbitrary mapping):

$$C_{MpN}\, X = \begin{cases} N & \text{if } X \neq 0 \\ M & \text{if } X = 0 \end{cases} \qquad \text{for arbitrary set } X$$

$$C_{MpN}\, f = \begin{cases} id_N & \text{if } f \text{ is non-empty} \\ p & \text{if } f \neq id_0 \text{ is empty} \\ id_M & \text{if } f = id_0 \end{cases} \qquad \text{for arbitrary mapping } f$$

Put $C_1 = C_{1p1}$, $C_0 = C_{0p0}$, $C_{01} = C_{0p1}$ (p is the only possible mapping).
Denote by $C^\star_{0,M}$ the contravariant constant functor:

$$C^\bullet_{0,M}\, X = \begin{cases} 0 & \text{if } X \neq 0 \\ M & \text{if } X = 0 \end{cases} \qquad C^\bullet_{0,M}\, f = \begin{cases} id_0 & \text{if } f \text{ is non-empty} \\ \text{the empty mapping to } M & \text{if } f \text{ is empty, } f \neq id_0 \\ id_M & \text{if } f = id_0 \end{cases}$$

$\simeq$ denotes the natural equivalence of functors.

I. 4. Note

Let D be a set diagram, $D : \mathfrak{K} \to Set$, let F be a covariant functor. We say that F *preserves the limit of* D if $\lim FD = F \lim D$. We say that F preserves limits over $\mathfrak{K}$ if F preserves the limit of any $D : \mathfrak{K} \to Set$. In particular F preserves sums (*F preserves* $\vee$) if for arbitrary collection of sets $\{X_i, i \in I\}$ we have $F(\bigvee_{i \in I} X) = \bigvee_{i \in I} (Fj_i(FX_i))$, where $j_i : X_i \to \bigvee_{i \in I} X_i$ is the $i - th$ injection. Analogously F preserves unions (*F preserves* $\bigcup$) if $F(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} Fj_i(FX_i)$.

Let $\{f_i, i \in I\}$ be a collection of mappings with common domain X. The co-union of $\{f_i, i \in I\}$ is such an epimorphism f with domain X, that for every i there exists h_i such that $f_i = h_i f$, and whenever g is an epimorphism with domain X such that for every i there exists k_i with $f_i = k_i g$, then there exists h with $g = hf$ (we denote $f = \bigcup_{i \in I}{}^* f_i$). Now we say that F preserves co-unions (*F preserves* $\bigcup^*$) if for every $\{f_i, i \in I\}$ we have $\bigcup^*_{i \in I} Ff_i = F(\bigcup^*_{i \in I} f_i)$. We remark that this definition slightly differs from that, given in [4], where $I \neq 0$ was assumed. Therefore functors, preserving co-unions, are just the functors, which preserve co-unions in the sense of [4] and are connected (i.e. $|F1| = 1$). Really: id_X is the co-union of an empty collection of mappings iff $|X| = 1$.

I. 5. Note

Analogously proceed for contravariant functors: let F be a contravariant functor. We say that F *dualizes the colimit of* D if $\lim FD = F \operatorname{colim} D$.

I. 6. Note

Let F be a contravariant functor, X an arbitrary set. Let $FX = 0$. Then given an arbitrary $Y \neq 0$ there exists a mapping $f : X \to Y$, and as $Ff : FY \to 0$ we have $FY = 0$. Therefore if $F \neq C^*_{M0}$ then $FX \neq 0$ for every X, and for every couple $f, g : X \to Y$ there exists $t \in FY$ with $Ff(t) = Fg(t)$: let $h : X \to 1$, we have $hf = hg$ and so, whenever $t \in imFh$, we have $Ff(t) = Fg(t)$.

I. 7. Note

We recall from [7]: Let F be a covariant functor, denote for an arbitrary set X and arbitrary $x \in FX$, $I^X_F(x) \subseteq \exp X$:

$$I^X_F(x) = \{Y \subseteq X, x \in Fj^X_Y(FY)\}$$

where j^X_Y *denotes the inclusion mapping* from Y to $X(j^X_Y(y) = y)$. The following holds: $I^X_F(x)$ is a filter or for every couple of mappings f, g with common domain X we have $Ff(x) = Fg(x)$. If $f/A = g/A$ and $A \in I^X_F(x)$, then $Ff(x) = Fg(x)$.

I. 8. Note

Let F be a contravariant functor, let $u \in F1$. u is a *distinguished point* of F if for every couple of mappings $f, g : 1 \to X$, where X is arbitrary, we have $Ff(u) = Fg(u)$. If u is a distinguished point, then for a set X denote $u_X = Ff(u)$, $f : 1 \to X$ arbitrary. Notice that if $f : X \to Y$ then $Ff(u_X) = u_Y$.

Whenever $F0 \neq 0$ then F has distinguished points: let $t \in F0$, let $u = Fk(t)$, $k : 0 \to 1$. Then u is distinguished, as $fk = gk$ for every $f, g : 1 \to X$.

I. 9. Theorem

Every covariant functor is naturally equivalent with an inclusions — preserving functor (i.e. such a functor H that whenever $Y \subseteq X$, $Y \neq 0$ then $FY \subseteq FX$ and $Fj^X_Y = j^{FX}_{FY}$).

Proof: Let F be an arbitrary functor. Le $\sim$ be an equivalence on the class of all couples (x, X) — X a set, $x \in FX$ — defined by $(x, X) \sim (y, Y)$ iff $X \subseteq Y$ and $Fj^Y_X(x) = y$ or $Y \subseteq X$ and $Fj^X_Y(y) = x$. Let $\mathfrak{A}$ be a choice-class of $\sim$. Put for an arbitrary set $X \neq 0, HX = \{a \in \mathfrak{A}, (\exists\, x \in FX)\,(a \sim (x, X))\}$; put $H0 = F0$ Further put $\tau^X : FX \to HX$ $\tau^X(x) = a$, where $a \in \mathfrak{A}$ and $a \sim (x, X)$; put $\tau_0 = id_{F0}$.

Clearly τ^X is a bijection and so HX is a set. Let $f : X_1 \to X_2$ be an arbitrary mapping, $X_1 \neq 0$. Put $Hf : HX_1 \to HX_2$:
$Hf(a) = b$ where $a, b \in \mathfrak{A}$ and for some $x \in FX_1$ $a \sim (x, X_1)$ and $b \sim (Ff(x), X_2)$; if $f : 0 \to X$ put $Hf = (\tau^X)^{-1} Ff$.

Clearly Hf is a mapping from HX_1 to HX_2 and in this way we defined a functor H. Let $0 \neq X \subseteq Y$, then clearly $HX \subseteq HY$ and $Hj_X^Y(a) = a$ for every $a \in HX$, i.e. $Hj_X^Y = j_{HX}^{HY}$. We shall conclude the proof by showing that $\tau : F \to H$ is a transformation — then it is necessarily a natural equivalence.

$$\begin{array}{ccc} FX_1 & \xrightarrow{\tau^{X_1}} & HX_1 \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Hf} \\ FX_2 & \xrightarrow{\tau^{X_2}} & HX_2 \end{array}$$

Let $f : X_1 \to X_2, 0 \neq X_1$, let $t \in FX_1$. Then $\tau^{X_2}(Ff(t)) = a$, where $a \in \mathfrak{A}, a \sim (Ff(t), X_2)$ and $Hf(\tau^{X_1}(t)) = Hf(d)$, where $d \in \mathfrak{A}$, $d \sim (t, X_1)$ and we have $Hf(d) = a$. Therefore τ is a transformation.

I. 10. Convention

Clearly if $F \simeq F'$ and $G \simeq G'$ then $A(F, G) \simeq A(F', G')$. Therefore we shall assume throughout the sections II and III (the covariant case) that F preserves inclusions and we shall work with F "up to natural equivalence". On the other hand, from technical reasons, we shall work with the concrete choice of the functor G.

I. 11. Convention

Denote $\square : A(F, G) \to Set$ the forgetful functor, assigning to every (X, ω) its underlying set X and to every morphism its underlying mapping.

I. 12. Note

Let $p : A \to B$ be a mapping. We denote $im\, p = \{p(a), a \in A\}$. We recall from [7] that for an arbitrary covariant, inclusions-preserving functor F we have: $im\, Fp = F\, im\, p$, and arbitrary non-empty $X, Y : F(X \cap Y) = FX \cap FY$.

The constant mapping $p : A \to B$ to $b (b \in B)$ is sometimes denoted by *const b*. The restriction of a mapping $r : A \to B$ to a set $C \subseteq A$ is denoted by $r/C(: C \to B)$.

II. The Covariant Case — Cosingleton

II. 1. Proposition

$A(F, G)$ has a cosingleton preserved by $\square$ iff either F preserves cosingleton (i.e. $F0 = 0$) or $G = C_1$.

Proof: A cosingleton, preserved by $\square$ is an object with empty underlying set. If $F0 = 0$ or $G = C_1$, then there exists just one such object and it can be easily seen that this is the cosingleton.

On the other hand let $A(F, G)$ have a cosingleton $(0, \omega)$ and let $F0 \neq 0$. Then clearly $G0 \neq 0$ (as $\omega : F0 \to G0$) and so either $G = C_1$ or there exists a set X with $|GX| > 1$. If so, let $x, y \in GX, x \neq y$; as $(0, \omega)$ is the consingleton, we have $p : (0, \omega) \to (X, \text{const}\, x)$ and $p : (0, \omega) \to (X, \text{const}\, y)$ where $p : 0 \to X$ is the empty mapping. But this is a contradiction because then for arbitrary $t \in F0$ we have $Gp\, \omega(t) = (\text{const}\, x)\, Fp(t) = x$ and at the same time $Gp\, \omega(t) = y$.

II. 2. Proposition

If $A(F, G)$ has a cosingleton, not preserved by $\square$, then G is equivalent with a covariant homfunctor.

Proof: As a consequence of the preceding proposition $F0 \neq 0$ and so $FX \neq 0$ for all sets X. Let (B, ω_0) be the the cosingleton of $A(F, G)$ $(B \neq 0)$. Let $t \in FB$ be arbitrary, put $\xi = \omega_0(t)$. As we recalled in I. 7., $I_G^B(\xi)$ is a filter or for every couple of mappings f, g with the same range and with the domain B we have $Ff(\xi) = Fg(\xi)$. Let us exclude the latter: clearly $GB \neq 0$ and $G \neq C_1$ and so there exists a set C with $|GC| > 1$, let $x, y \in GC, x \neq y$; as (B, ω_0) is a cosingleton, there exists $f : (B, \omega_0) \to (C, \text{const}\, x)$ and $g : (B, \omega_0) \to (C, \text{const}\, y)$. Hence $Ff\omega_0 = \text{const}\, x$, in particular $Ff\omega_0(t) = Ff(\xi) = x$ and at the same time $Fg(\xi) = y$. Therefore $I_G^B(\xi)$ is a filter. Let us prove, that it is closed under intersections, in other words that $M \in I_G^B(\xi)$, where $M = \bigcap I_G^B(\xi)$. If this were not the case, then clearly for every set $X \in I_G^B(\xi)$ there would exist a point $x \in X$ with $X - \{x\} \in I_G^B(\xi)$. But this leads to a contradiction: There exists a unique $f : (B, \omega_0) \to (B, \text{const}\, \xi)$. As then $Gf\omega_0 = \text{const}\, \xi$ we have $\xi \in im\, Gf = Gj_{imf}^B G(im\, f)$ and so $im\, f \in I_G^B(\xi)$. There exists $x \in im\, f$ with $im\, f - \{x\} \in I_G^B(\xi)$ and (from the same reasons) there exists $y \in im\, f - \{x\}$ with $im\, f - \{x, y\} \in I_G^B(\xi)$. Let $h : B \to B$ be the transposition of x and $yh(x) = y$, $h(y) = x$, otherwise $h = id$). As $h/im\, f - \{x, y\} = id/im\, f - \{x, y\}$ we have $Fh(\xi) = Fid_B(\xi) = \xi$ (see I. 7). Therefore hf is a morphism from (B, ω_0) to $(B, \text{const}\, \xi)$ (as $Gf\omega_0 = \text{const}\, \xi$, $Ff = \text{const}\, \xi$, we have $Gf/im\, \omega_0 = \text{const}\, \xi$ and as $Gh(\xi) = \xi$ we have also $G(fh)\omega_0 = \text{const}\, \xi$). But that is a contradiction with the unicity of f, as $hf \neq f$. Therefore $M \in I_G^B(\xi)$, i.e. there exists $\xi' \in GM$ with $Gj_M^B(\xi') = \xi$. Clearly $I_G^M(\xi') = \{M\}$ (because $M = \bigcap I_G^B(\xi)$).

Let us show that G is equivalent with Q_M. Yoneda lemma guarantees the existence of a transformation $\tau : Q_M \to G$ such that $\tau^M(id_M) = \xi'$; we have then $\tau^X(f) = Gf(\xi')$. Verify that τ is a natural equivalence:

(1) τ is a monotransformation. If not so, then there would exist mappings k, l: $M \to X$, $k \neq l$, with $\tau^X(k) = \tau^X(l)$. As $\tau^X(k) = Gk(\xi')$, $\tau^X(l) = Gl(\xi')$ we would have $Gk(\xi') = Gl(\xi') = \xi^*$. Then $k, l : (M, \text{const}\, \xi') \to (X, \text{const}\, \xi^*)$. Now denote g the unique morphism from (B, ω_0) to $(M, \text{const}\, \xi')$. Then $kg, lg : (B, \omega_0) \to (M, \text{const}\, \xi')$ and necessarily $kg = lg$. But this is a contradiction, as clearly $\xi' \in im\, Gg$ and so $im\, g \in I_G^M(\xi') = \{M\}$ and so g is an epimorphism.

(2) τ is an epitransformation. Let X be an arbitrary set, $x \in GX$. We shall

show that $x \in im\, \tau^X$. Let f be the unique morphism from (B, ω_0) to $(X, \text{const } x)$. Then $\tau^X(fj_M^B) = G(fj_M^B)(\xi') = Gf(\xi) = Gf(\omega_0(t) = (\text{const } x)\ Ff(t) = x$. Therefore τ is a natural equivalence of G and Q_M.

II. 3. Proposition

$A(F, G)$ has a cosingleton as soon as there exists a set D such that $|D| \geq |M| . |FD|$ holds. The cardinality of the underlying set of the cosingleton then equals to the least cardinal $\mathfrak{n}$, for which $\mathfrak{n} = |M| . |F\mathfrak{n}|$ holds.

II. 4. Note

The following proof of II. 3 gives an algorithm for constructing the cosingleton — in fact whenever $F0 \neq 0$ and $A(F, G)$ has a cosingleton, then it can be constructed in this way.

Proof of II. 3.: Consider any set D with $|D| \geq |M| . |FD|$. There clearly exists a mapping $\omega : FD \to Q_M(D)$ such that

(1) for every $t \in FD$ the mapping $\omega(t)$ (from M to D) is injective

(2) for every couple $t_1, t_2 \in FD$, $t_1 \neq t_2$, we have $im\, \omega(t_1) \cap im\, \omega(t_2) = 0$.

If $F0 = 0$, we obtain the proposition as a consequence of II. 1 (put $D = \mathfrak{n} = 0$). If $F0 \neq 0$, then there is a distinguished point $u \in F1$ (see I. 8). Define a transfinite sequence of subsets of D (δ, γ arbitrary ordinals):

$$B_0 = im\, \omega(u_D)$$
$$B_\gamma = \bigcup_{\delta > \gamma} \bigcup_{t \in FB_\delta} im\, \omega(t) \quad \text{if} \quad \gamma > 0.$$

As $\gamma_1 < \gamma_2$ implies $B_{\gamma_1} \subseteq B_{\gamma_2}$ there clearly exists an ordinal γ_0 such that $B_{\gamma_0} = B_{\gamma_0+1}$. Put $B = B_{\gamma_0}$. Then for every $t \in FB$ and every $m \in M$ $\omega(t)(m) \in B$ and so $\omega/FB : FB \to Q_M(B)$. Put $\omega_0 = \omega/FB$. We shall show that (B, ω_0) is the cosingleton.

Let (X, ω_X) be an arbitrary object of $(A(F, G)$. Let us define for every ordinal δ a mapping $f_\delta : B_\delta \to X$:
let $p \in B_0$ (i.e. $p = \omega(u_D)(m)$ for some $m \in M$) then put $f_0(p) = \omega_X(u_X)(m)$; let f_δ be already defined for all $\delta < \gamma$, let $p \in B_\gamma$ (i.e. $p = \omega(t)(m)$, where $t \in B_\beta$, $\beta < \gamma$, $m \in M$) then put $f_\gamma(p) = \omega_X(Ff_\beta(p))(m)$.

As ω has the properties (1) and (2) from above, this definition is correct. Notice that if $\gamma_1 < \gamma_2$ then $f_{\gamma_1} = f_{\gamma_2}/B_{\gamma_1}$. Put $f = f_{\gamma_0}$ $(= f_{\gamma_0+1})$. We shall prove that

(a) f is a morphism from (B, ω_0) to (X, ω_X), i.e. that for every $t \in FB$
$f\omega_0(t) = \omega_X(Ff(t))$

$$\begin{array}{ccc} FB & \xrightarrow{\omega_0} & Q_M(B) \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Q_M f} \\ FX & \xrightarrow{\omega_X} & Q_M(X) \end{array}$$

(b) f is unique.

(a) Let $m \in M$ be arbitrary, we shall show that $f(\omega_0(t)(m)) = \omega_X(Ff(t))(m)$. As $t \in FB = FB_{\gamma_0}$ we have $f(\omega_0(t)(m)) = f_{\gamma_0+1}(\omega_0(t)(m)) = \omega_X(Ff_{\gamma_0}(t))(m) = \omega_X(Ff(t)(m)$.

(b) Let $g : (B, \omega_0) \to (X, \omega_X)$. We shall verify that for every ordinal δ $g/B_\delta = f_\delta = f/B_\delta$. First let $\delta = 0$, $p \in B_0$ (i.e. $p = \omega(u_D)(m)$), then $f_0(\omega(u_D)(m)) \neq \omega_X(u_X)(m)$, on the other hand $\omega_X Fg = Q_M g\, \omega_0$ and so $g(\omega(u_D)(m)) = (Q_M g\, \omega_0)(u_D)(m) = \omega_X(Fg(u_D))(m) = \omega_X(u_X)(m)$; second let $f_\delta = g/B_\delta$ for all $\delta < \gamma$, let $p \in B_\gamma$ (i.e. $p = \omega(t)(m)$, $t \in B_\beta$, $\beta < \gamma$, $m \in M$), then $f_\gamma(p) = \omega_X(Ff_\beta(t))(m) = \omega_X(Fg(t))(m) = (Q_M g\, \omega_0)(t)(m) = g(\omega(t)(m)) = g(p)$. So (B, ω_0) is a cosingleton.

Now let us verify that $|B| = |M| . |FB|$ and whenever $|X| = |M| . |FX|$ then $|X| \geq |B|$. The latter is clear: notice that we can start this proof instead with D, with any X such that $|X| = |M| . |FX|$, then we get a cosingleton (B', ω_0') with $B' \subseteq X$. (Of course $|B'| = |B|$.) Further let us show that $B = \bigcup_{t \in FB} im\, \omega_0(t)$; hence, in virtue of the properties (1) and (2) of ω, we shall obtain, $|B| = |M| . |FB|$. Put $C = \bigcup_{t \in FB} \mathrm{im}\ \omega_0(t)$, put $B_1 = (B \times \{1\}) \cup ((B - C) \times \{0\})$, put $f, g : B \to B_1, f(b) = (b, 1)$, $g(b) = \begin{cases} (b, 1) \text{ if } b \in C \\ (b, 0) \text{ if } b \in B - C. \end{cases}$

As $im f \cap im\, g = C \times \{1\}$ we have $im\, Ff \cap im\, Fg = F(C \times \{1\})$ (see I. 12) and as Ff and Fg are injections and $Ff/FC = Fg/FC (f/C = g/C)$, it follows that whenever $Ff(x) = Fg(y)$ then $x = y$. Therefore it is correct to put $\omega_1 : FB_1 \to Q_M B_1$, $\omega_1(Ff(x)) = f(\omega_0(x))$, $\omega_1(Fg(x)) = g(\omega_0(x))$ for all $x \in FB$, otherwise ω_1 arbitrary. Clearly both f and g are morphisms from (B, ω_0) to (B_1, ω_1) and so $f = g$. This implies $C = B$, which concludes the proof.

II. 5. Theorem

$A(F, G)$ has a cosingleton if and only if
— either F preserves cosingleton (i.e. $F0 = 0$)
— or for some $M, G \simeq Q_M$ and there exists a set D with $|D| = |M| . |FD|$.

Proof: The sufficiency follows from II. 1 and II. 3.

To prove the necessity, it is enough — due to II. 1 and II. 2 — to show that if $A(F, Q_M)$ has a cosingleton (B, ω_0) then $|B| = |M| . |FB|$.

Let F_1 be a subfunctor of F, $F_1 X = \bigcup_{f : B \to X} Ff(FB)$ for every X; if $p : X \to Y$, $F_1 p = Fp/F_1 X$. Clearly $F_1 B = FB$ and so (B, ω_0) is an object in $A(F_1, Q_M)$. There exists a set D with $|D| \geq |M| . |FD|$ (in fact obviously for every set X, $|F_1 X| \leq |X^B| . |FB|$ holds, and so any infinite cardinal bigger then $|2^B|$; $|M|$ and $|FB|$ will do). Therefore, in virtue of II. 3, $A(F_1, Q_M)$ has a consingleton with an underlying set, the cardinality of which $\mathfrak{n}$ fulfils $\mathfrak{n} = |M| . |F\mathfrak{n}|$. Now, it suffices to be shown, that in fact (B, ω_0) is the cosingleton of $A(F_1, Q_M)$, because then $|B| = |M| . |F_1 B| = |M| . |FB|$.

Given (X, ω) an arbitrary object of $A(F_1, Q_M)$, let $\omega' : FX \to Q_M X$ be an arbitrary mapping such that $\omega' / F_1 X = \omega$. Then there exists a unique $f : (B, \omega_0) \to$ $\to (X, \omega')$ in $A(F, Q_M)$. It is easy to see, that $f : (B, \omega_0) \to (X, \omega)$ in $A(F_1, Q_M)$ and that it is unique. Therefore (B, ω_0) is the consingleton of $A(F_1, Q_M)$.

III. The Covariant Case — Cocompletness

First, we recall the main results from the papers [3] and [4]:

III. 1. Definition

A functor F is *excessive* if there exists a cardinal $\mathfrak{n}$ such that if X is a set with $|X| > \mathfrak{n}$ then $|FX| > |X|$.

III. 2. Proposition *(GCH)*

$A(F, G)$ has sums (finite sums) if and only if one of the following cases occurs:

(1) F preserves sums (finite sums resp.).

(2) F preserves unions (finite unions resp.) and either G is naturally equivalent with a hom-functor or it preserves co-unions.

(3) F is non-excessive and G is naturally equivalent with a hom-functor.

(4) Either $G \simeq C_{M1}$ for some M and $F0 = 0$, or $G \simeq C_0, C_1$, or C_{01}.
The sums are preserved by $\square$ just in cases (1) and (4).

III. 3. Proposition

$A(F, G)$ has coequalizers if and only if either F preserves coequalizers or G preserves co-unions. The coequalizers are preserved by $\square$ if and only if F preserves coequalizers or $G \simeq C_{Mp1}$ for some M.

It is well-known that a category is cocomplete iff it has sums, coequalizers and a cosingleton. Thus the results from the part II together with the propositions above enable us to give a necessary and sufficient condition for the cocompletness of the category $A(F, G)$. (The same holds for the finite sums and finite cocompletness.)

To make the condition simpler, we shall assume that both functors F and G are non-constant, though there is no difficulty — of course — in considering also the case of one or both functors being constant (the assumption of non-constantness was never made above).

III. 4. Theorem *(GCH)*

A necessary and sufficient condition for the cocompletness of the category $A(F, G)$ with F, G non-constant is expressed by the following table, where $+$ stands for: it is cocomplete, $\square$ for: it is cocomplete and all colimits are preserved by $\square$ and — for: it is not cocomplete.

		F preserves $\bigvee$	F does not preserve $\bigvee$		
			F preserves $\bigcup$		F does not preserve $\bigcup$
			$FO = 0$	$FO \neq 0$	
$G \cong Q_M$ for some M		$\square$	$+$	$+$ iff $\lvert D\rvert = \lvert M\rvert . \lvert FD\rvert$ for some D	$+$ iff F is non-excessive
$G \not\cong Q_M$ for every M	G preserves $\bigcup^\star$	$\square$	$+$	—	—
	G does not preserve $\bigcup^\star$	$\square$	—	—	—

A necessary and sufficient condition for the finite cocompletness is expressed by the same table, in which "F preserves $\bigvee$ (or $\bigcup$)" would be changed to "F preserves countable $\bigvee$ (or countable $\bigcup$ resp.)."

Proof: We omit the proof because it is an easy but laborious consequence of the preceding results, in case of using the following notes (F is an arbitrary covariant functor):

(1) F preserves coequalizers iff it preserves countable unions (see [8]),

(2) if F preserves $\bigcup$ (or countable $\bigcup$ or finite $\bigcup$) then F preserves $\bigvee$ (or countable $\bigvee$ or finite $\bigvee$) iff F does not have distinguished points (in particular then $F0 = 0$!), see [7],

(3) all Q_M preserve $\bigcup^*$ (see [7]),

(4) if F is non-excessive then for an arbitrary set M there exists a set D such that $|D| = |M| . |FD|$.

III. 5. Note

We recall from [7] and [8]:

(1) F preserves $\bigvee$ iff F is equivalent with a sum of the identity functors.

(2) F preserves $\bigcup$ iff $F \simeq F_1 \vee F_2$ where F_1 preserves $\bigvee$ and F_2 is constant.

(3) All subfunctors of hom-functors preserve $\bigcup^\star$ but there are others, too.

(4) Assuming the non-existence of meassurable cardinals, every functor, which preserves countable $\bigvee$ (or countable $\bigcup$), preserves all $\bigvee$ (all $\bigcup$ resp.).

IV. The Contravariant Case

IV. 1. Theorem

Let F and G be contravariant functors, $F \neq C^*_{0,M}$, let D_0 be a diagram in *Set*. Then $A(F, G)$ has colimits of all diagrams D in $A(F, G)$ for which $D_0 = \square D$ holds if and only if G dualizes the colimit of D_0. The colimits are then preserved by $\square$.

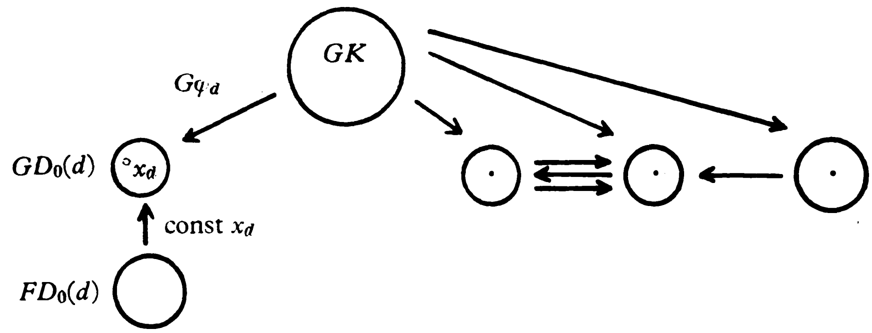
Proof: Let $D_0 : \mathfrak{K} \to Set$ be an arbitrary set diagram; without loss of generality we can assume that D_0 is covariant.

(a) Let $\mathfrak{K}$ is not empty.

First let us prove the necessity:

Denote by $(K, \{\varphi_d, d \in \mathfrak{K}^\circ\})$ the colimit of D_0 in *Set*. Clearly $(GK, \{G\varphi_d, d \in \mathfrak{K}^\circ\})$ is a bound of GD_0 in *Set*. It is our task to show that this is the limit of GD_0. Let $\{x_d, d \in \mathfrak{K}^\circ\}$, $x_d \in GD_0(d)$ be any such collection that for every $\delta \in \mathfrak{K}(d_1, d_2)$ we have $GD_0(\delta)\,(d_2) = d_1$. We are to show that there exists a unique $x \in GK$ such that $G\varphi_d(x) = x_d$ for all $d \in \mathfrak{K}^\circ$ (see I. 2).

Clearly for every $\delta \in \mathfrak{K}(d_1, d_2)$ $GD_0(\delta)$ is a morphism from $(D(d_2)$, const $x_{d_2})$ to $(D(d_1)$, const $x_{d_1})$ and so we may define a diagram $D : \mathfrak{K} \to A(F, G)$ by $D(d) = (D(d).\,\mathrm{const}\, x_d)$ for every $d \in \mathfrak{K}^\circ$ and $\square D = D_0$.



Due to the pressumptions D has a colimit in $A(F, G)$, let it be $((H, \omega), \{\tau_d, d \in \mathfrak{K}^\circ\})$. Then $(H, \{\tau_d, d \in \mathfrak{K}^\circ\})$ is a cobound of D_0 in *Set* and so there exists $h : K \to H$ with $\tau_d = h\varphi_d$. Let $t \in FH$ be arbitrary (as $F \neq C_0^*$, $FH \neq 0$), put $x = Gh(\omega(t))$. Then x is the point we are looking for: for every $d \in \mathfrak{K}^\circ$ $G\varphi_d(x) = G(h\varphi_d)\,\omega(t) = G\tau_d\omega(t) = \mathrm{const}\, x_d\, F\tau_d(t) = x_d$.

$$\begin{array}{ccc} FH & \xrightarrow{\ \omega\ } & GH \\ {\scriptstyle F\tau_d}\downarrow & & \downarrow{\scriptstyle G\tau_d} \\ FD(d) & \xrightarrow{\ \mathrm{const}\, x_d\ } & GD(d) \end{array}$$

Moreover x is unique: Given $y \in GK$ with $G\varphi_d(y) = x$. Then both $((K, \mathrm{const}\, x), \{\varphi_d, d \in \mathfrak{K}^\circ\})$ and $((K, \mathrm{const}\, y), \{\varphi_d, d \in \mathfrak{K}^\circ\})$ form a cobound of D in $A(F, G)$ and so there exist morphisms k_x, k_y from (H, ω) to $(K, \mathrm{const}\, x)$ (or $(K, \mathrm{const}\, y)$, resp.) such that $k_x\tau_d = k_y\tau_d = \varphi_d$ for all $d \in \mathfrak{K}^\circ$. Now k_xh, k_yh are mappings from K to K such that $k_xh\varphi_d = k_x\tau_d = \varphi_d$ (analogously $k_yh\varphi_d = \varphi_d$) and because $(K, \{\varphi_d, d \in \mathfrak{K}^\circ\})$ is the colimit of D_0 clearly $k_xh = k_yh \doteq id_K$. Let ξ be such that $\xi \in FK$ and $Fk_x(\xi) = Fk_y(\xi)$ (see I. 6). Then $x = G(k_xh)\,(x) = GhGk_x\, \mathrm{const}\, x(\xi) = Gh\omega Fk_x(\xi) = Gh\omega Fk_y(\xi) = GhGk_y\, \mathrm{const}\, y(\xi) = y$.

Second let us prove the sufficiency and the preservation by $\square$: Let $D : \mathfrak{K} \to A(F, G)$ be arbitrary diagram with $D_0 = \square D$. We have to show the existence of such a $\omega_0 : FK \to GK$ that $((K, \omega_0), \{\varphi_d, d \in \mathfrak{K}^\circ\})$ is the colimit of D in $A(F, G)$. Denote $D(d) = (X_d, \omega_d)$ for every $d \in \mathfrak{K}^\circ$.

As $(GK, \{G\varphi_d, d \in \mathfrak{K}^\circ\})$ is the limit of GD_0 (due to the pressumptions) there clearly exists for every $u \in FK$ a unique $x_u \in GK$ such that for every $d \in \mathfrak{K}^\circ$ we have $G\varphi_d(x_u) = \omega_d(F\varphi_d(u))$. Put $\omega_0(u) = x_u$. Then clearly $((K, \omega_0), \{\varphi_d, d \in \mathfrak{K}^\circ\})$ is a cobound of D. Let us show that it is a colimit. Let $((L, \omega), \{\psi_d, d \in \mathfrak{K}^\circ\})$ be an arbitrary cobound of D in $A(F, G)$. As $(L, \{\psi_d, d \in \mathfrak{K}^\circ\})$ is a cobound of D_0 there exists a unique $l : K \to L$ with $l\varphi_d = \psi_d$ for all $d \in \mathfrak{K}^\circ$. Clearly for every d, $G\varphi_d\, Gl\omega = \omega_d(F\varphi_d Fl)$, i.e. for every $t \in FL$ $G\varphi_d\, Gl\,\omega(t) = \omega_d F\varphi_d Fl(t)$ — but when we notice how ω_0 was defined, we shall find that then $\omega_0 Fl(t) = Gl\omega(t)$ for all $t \in FL$ and so $\omega_0 Fl = Gl\omega$ and so $l : (K, \omega_0) \to (L, \omega)$. The unicity is clear.

(b) Let $\mathfrak{K}$ is empty.

We are to prove that $A(F, G)$ has a cosingleton iff $|G0| = 1$, and that the cosingleton has empty underlying set.

If $|G0| = 1$, then there exists just one object with underlying set 0 and it is easy to verify that this is the cosingleton.

If $|GO| \neq 1$ then $A(F, G)$ does not have cosingleton — either $G0 = 0$ and then $G = C_0^*$ and so $A(F, G)$ is empty (we assume $F \neq C_0^*$), or $|G0| > 1$ and then let $x_1, x_2 \in G0$, $x_1 \neq x_2$, let (B, ω) be a cosingleton in $A(F, G)$, let f_i: $(B, \omega) \to (0, \text{const } x_i)$ $i = 1,2$ — this would mean that $B = 0$, $f_1 = f_2 = id_0$ and $x_1 = x_2$ — that is a contradiction.

IV. 2. Collorary

$A(F, G)$ with F, G contravariant, $F \neq C_{0,M}^*$, has all colimits of diagrams over a scheme $\mathfrak{K}$ iff G dualizes colimits over $\mathfrak{K}$.

In particular $A(F, G)$ is cocomplete iff G is equivalent to a contravariant hom-functor.

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