Zdeněk Frolík
On fixed sets under continuous transformations

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 3 (1962), No. 1, 5--14

Persistent URL: http://dml.cz/dmlcz/142140

1962 ACTA UNIVERSITATIS CAROLINAE — MATHEMATICA ET PHYSICA NO 1 PAG 5—14

ON FIXED SETS UNDER CONTINUOUS TRANSFORMATIONS

ZDENĚK FROLÍK,

Charles University Prague

(Received May 25th, 1961)

1. PRELIMINARIES

All spaces under consideration are supposed to be T_1-spaces, i. e. finite sets are closed. $\overline{M}^P$, or merely $\overline{M}$, will be used to denote the closure of a subset M of a space P. If $\mathfrak{M}$ is a family of subsets of a space P, then $\overline{\mathfrak{M}}^P$, or merely $\overline{\mathfrak{M}}$, will be used to denote the family of all $\overline{M}^P$, $M \in \mathfrak{M}$. A centred family of sets is a family $\mathfrak{M}$ of sets with the finite intersection property, i. e., the intersection of every finite subfamily of $\mathfrak{M}$ is non-void. $\beta(P)$ will be used to denote the Čech-Stone compactification of a completely regular space P.

A transformation of a space P is a continuous mapping from P to P. If f is a transformation of P, then f^0 denotes the identity transformation of P, i. e. $f^0(x) = x$ for each x in P, and by induction

$$f^{n+1} = f \circ f^n \quad (n = 0, 1, 2, \ldots)$$

Let f be a transformation of a space P. A subset M of P is fixed under f, or merely f-fixed, if $f[M] \subset M$, or equivalently, if the restriction f/M of f to M is a transformation of M.

Definition 1. Let f be a transformation of a space P. A minimal f-fixed set is a non-void closed f-fixed subset F of P which contains no non-void closed f-fixed proper subset, i. e., if $\emptyset \neq F_1 \subset F$ and F_1 is a closed f-fixed set, then $F_1 = F$.

In the present note we shall investigate the existence and properties of minimal f-fixed sets. In the sequel, the following trivial assertions (a) and (b) will be used without references:

(a) Let f be a transformation of a space P. The intersection of f-fixed sets is a f-fixed set. The closure of an f-fixed set is an f-fixed set.

Indeed, if $f[M] \subset M$ for each M in $\mathfrak{M}$, then

$$f[\bigcap \mathfrak{M}] \subset \bigcap \{f[M]; M \in \mathfrak{M}\} \subset \bigcap \mathfrak{M}$$

and if $f[M] \subset M$, then by continuity of f we have

$$f[\overline{M}] \subset \overline{f[M]}$$

and consequently, $f[\overline{M}] \subset \overline{M}$.

(b) Let f be a transformation of P and let x be a point of P. The sets

(1) $$M(f, x) = \{f^n(x);\ n = 0, 1, 2, \ldots\}$$

and

(2) $$F(f, x) = \overline{M(f, x)}$$

are f-fixed. Moreover, if M is f-fixed and $x \in M$, then $M(f, x) \subset M$. If M is closed and f-fixed and if $x \in M$, then $F(f, x) \subset M$.
Thus we have proved.

THEOREM 1. *Let f be a transformation of a space P. A non-void subset F of P is a minimal f-fixed set if, and only if,*

$$x \in F \Rightarrow F(f, x) = F$$

In consequence, if F is a minimal f-fixed set, then $f[F] = F$.

2. EXISTENCE OF MINIMAL FIXED SETS

THEOREM 2. *If f is a transformation of a compact space P, then every non-void closed f-fixed subset of P contains a minimal f-fixed set. In consequence, if $P \neq \varnothing$, then there exists at least one minimal f-fixed set.*

Proof. Let $\mathfrak{M}$ be the family of all non-void closed f-fixed subsets of P. The family $\mathfrak{M}$ is ordered by inclusion. Clearly, minimal elements of $\mathfrak{M}$ are precisely minimal f-fixed sets. Thus we have to prove that for every F in $\mathfrak{M}$ there exists a minimal $F_0 \in \mathfrak{M}$ with $F_0 \in \mathfrak{M}$. It is sufficient to show that every linearly ordered subfamily $\mathfrak{N}$ of $\mathfrak{M}$ has a lower bound. But it is obvious. Indeed, $N_0 = \bigcap \mathfrak{N}$ is f-fixed as an intersection of f-fixed sets and N_0 is non-void, since P is a compact space. Clearly $N_0 \subset N$ for each N in $\mathfrak{N}$.

Let us recall that a Lipschitz transformation of a metric space (P, ϱ) is a transformation f of P such that

$$x, y \in P \Rightarrow \varrho(f(x), f(y)) \leqq \alpha\varrho(x, y) .$$

The following result is well-known.

THEOREM 3. *If f is a Lipschitz transformation of a complete metric space (P, ϱ) with constant $\alpha < 1$, then there exists a point x in P such that $f(x) = x$ and for every y in P*

$$\lim_{n \to \infty} \varrho(f^n(y), x) = 0,$$

that is,

$$y \in P \Rightarrow x \in F(f, y)$$

We shall prove a generalization of the preceding theorem. Clearly, a transformation f of a metric space (P, ϱ) is a Lipschitz transformation with constant α if and only if

$$M \subset P \Rightarrow d(f[M]) \leqq \alpha d(M)$$

where

(3) $$d(M) = \sup\{\varrho(x, y);\ x \in M, y \in M\},$$

$$d(\varnothing) = 0 .$$

Definition 2. A diameter on a space P is a non-negative function d (values of d are non-negative real numbers and ∞) defined for all $M \subset P$ and satisfying the following axioms:

(d1) $M \subset N \subset P \Rightarrow d(M) \leqq d(N)$

(d2) $x \in P \Rightarrow d((x)) = 0$

(d3) $d(M) = \inf \{d(U);\ U \text{ open},\ U \supset M\}$

(d4) $d(\overline{M}) = d(M)$

A d-Cauchy family is a centred family $\mathfrak{M}$ of subsets of P such that

$$\inf \{d(M);\ M \in \mathfrak{M}\} = 0 .$$

A diameter d on P will be called complete, if $\bigcap \overline{\mathfrak{M}} \neq \varnothing$ for every d-Cauchy family $\mathfrak{M}$.

Definition 3. Let d be a diameter on a space P. A transformation f of P will be called a Lipschitz transformation of P with constant α, if

$$M \subset P \Rightarrow d(f[M]) \leqq \alpha d(M)$$

THEOREM 4. *Let d be a complete diameter on a space P. If f is a Lipschitz transformation of P with constant α, then every non-void closed f-fixed set of finite diameter contains a minimal f-fixed set.*

Proof. Let F be a non-void closed f-fixed set and let $d(F) < \infty$. Let us define by induction $F_0 = F$ and

$$F_{n+1} = f[F_n] \quad (n = 0, 1, 2, \ldots)$$

Then $F_n \supset F_{n+1} \neq \varnothing$ for all n and

$$\lim_{n \to \infty} d(F_n) = 0 \tag{4}$$

It is easy to see that

$$K = \bigcap \{\overline{F}_n;\ n = 1, 2, \ldots\}$$

is a compact non-void f-fixed set. K is f-fixed, since every F_n is f-fixed. The diameter d being complete, from (4) it follows at once that K is non-void. To prove compactness of K, let $\mathfrak{M}$ be a centred family of subsets of P. Clearly

$$M \in \mathfrak{M} \Rightarrow d(M) \leqq d(\overline{F}_n) = d(F_n) \leqq \alpha^n d(F_0)$$

and consequently,

$$M \in \mathfrak{M} \Rightarrow d(M) = 0$$

The diameter d being complete, we have

$$\varnothing \neq \bigcap \overline{\mathfrak{M}}^P = \bigcap \overline{\mathfrak{M}}^K .$$

Thus K is a compact f-fixed subspace of P. By Theorem 2 there exists a minimal (f/K)-fixed subset K_0 of K. Clearly K_0 is a minimal f-fixed set and $K_0 \subset \subset K \subset F$. The proof is complete.

It is easy to see that a metric space (P, ϱ) is complete if and only if the diameter d defined by (3) is complete in the sense of Definition (2). A transformation f of (P, ϱ) is Lipschitz with constant α if and only if f is a Lipschitz transformation of (P, d) with constant α.

Now the Theorem 3 follows easily from Theorem 4. Indeed, it is easy to see that

$$d\,(F(f, y)) = d(M(f, y)) < \infty$$

Thus every $F(f, y)$ contains a minimal f-fixed set.

If $d(F) < \infty$ and $f[F] = F$, then $d(F) = 0$ because $d(F) \leqq \alpha d(F)$. But $d(M) = 0$ if and only if M is at most one-point. Thus minimal f-fixed sets are one-point and there exists only one minimal set. The fact

$$\lim_{n \to \infty} \varrho(f^n(y), x) = 0$$

follows from the fact that

$$\lim_{n \to \infty} d(F(f, f^n(y))) = 0\,.$$

Let us recall that a completely regular space P is said to be topologically complete in the sense of E. Čech, or merely complete, if P is a G_δ in the Čech-Stone compactification $\beta(P)$ of P. If P is complete and $P \subset R$, $\overline{P} = R$, then P is a G_δ in R.

THEOREM. 5. *A completely regular space P is complete if and only if there exists a complete diameter on P.*

Proof. In [2] it is proved that if there exists on P a complete diameter satisfying only (d1), (d2) and (d3), then P is a complete space. Conversely, let P be a G_δ in a compactification K of P. Let us choose open sets U_n in K such that $U_n \supset U_{n+1}$ and

$$P = \bigcap \{U_n;\ n = 1, 2, \ldots\}.$$

Let $M \subset P$. If M and $K - U_n$ are completely separated for no n, then we put $d(M) = 1$. (Of course, subsets N_1, N_2 of K are completely separated, if there exists a continuous real-valued function f on K with $f[N_1] = (0)$ and $f[N_2] = (1)$.) In the other case, let
$d(M) = \inf\{1/n;\ M$ and $K - U_n$ are completely separated$\}$. Clearly the conditions (d1)—(d4) are fulfilled. It is sufficient to prove that d is complete. Let $\mathfrak{M}$ be a d-Cauchy family. Let us choose $M_n \in \mathfrak{M}_n$ with $d(M_n) < 1/n$. Clearly we have $\overline{M}_n^K \subset U_n$. It follows

$$\varnothing \neq \bigcap \overline{\mathfrak{M}}^K \subset \bigcap \{\overline{M}_n;\ n = 1, 2, \ldots\} \subset P.$$

But for every $M \subset P$ we have

$$\overline{M}^K \cap P = \overline{M}^P$$

and consequently,

$$\bigcap \overline{\mathfrak{M}}^P = \bigcap \overline{\mathfrak{M}}^K$$

The proof is complete.

Note. To prove Theorem 4, it is sufficient to assume that d satisfies only the conditions (d1) and (d4).

3. A QUOTIENT SPACE ASSOCIATED WITH A GIVEN TRANSFORMATION

Let f be transformation of a space Q. Let $\mathfrak{D}$ be the family of subsets of Q consisting from all minimal f-fixed sets and all one-point sets (x), where x belongs to no minimal f-fixed set. Clearly, $\mathfrak{D}$ is a decomposition of Q. Let us consider the quotient space $K = Q/\mathfrak{D}$. Let π be the projection of Q onto K. Clearly

$$\pi(x) = \pi(y) \Rightarrow \pi(f(x)) = \pi(f(y)).$$

Let us define a transformation Φ of K as follows

$$\Phi\ (\pi(x)) = \pi(f(x)),$$

that is,

$$\Phi \circ \pi = \pi \circ f$$

Evidently, Φ is a mapping from K to K. Both mappings f and π are continuous, and consequently, $\pi \circ f$ and hence $\Phi \circ \pi$ is continuous. π being the quotient-mapping and $\Phi \circ \pi$ being continuous, the mapping Φ is continuous. Thus Φ is a transformation of K.

Proposition 1. If M is a f-fixed subset of Q, then $\pi[M]$ is a Φ-fixed set.

Indeed, clearly

$$\Phi[\pi[M]] = \pi[f[M]] \subset \pi[M].$$

Proposition 2. If N is a Φ-fixed subset of K, then $\pi^{-1}[N]$ is a f-fixed subset of Q.

Proof. Clearly

$$\pi[f[\pi^{-1}[N]]] = \Phi[\pi[\pi^{-1}[N]]] = \Phi[N] \subset N.$$

It follows

$$F = f[\pi^{-1}[N]]] \subset \pi^{-1}[N]$$

because

$$\pi^{-1}[\pi[F]] = F.$$

The proof is complete.

If F is a minimal f-fixed set, then $F \in \mathfrak{D}$ and $f[F] = F$, and consequently

$$\Phi(F) = F.$$

Thus minimal f-fixed sets are fixed points of Φ. Conversely, fixed points of Φ are minimal f-fixed sets.

Thus we have proved the following assertion.

THEOREM 6. *Let f be a transformation of a space Q. There exists a quotient mapping π of Q onto a space K and a transformation Φ of K such that*

(1) *If $F \subset Q$ is a minimal f-fixed set, then $\pi[F]$ is a fixed point of Φ.*

(2) *If $F \subset Q$ is f-fixed, then $\pi[F]$ is Φ-fixed.*

(3) *If $F \subset K$ is Φ-fixed, then $\pi^{-1}[F]$ is f-fixed.*

(4) $\Phi \circ \pi = \pi \circ f$.

4. ORBICULAR SPACES

In this section we shall investigate spaces which are minimal f-fixed sets under a transformation of a space.

Let f be a transformation of a space P and let F be a minimal f-fixed set. If $x \in P$, then $F(f, x) = F$ (See 2).

Definition. An orbicular transformation of a space P is a transformation f of P such that

$$x \in P \Rightarrow F(f, x) = P.$$

A space P will be called orbicular, if there exists an orbicular transformation of P and P has at least two points.

Examples. If a space P has the fixed-point property, that is, if every continuous transformation f has a fixed point, then P is not an orbicular space. For example, a closed interval of real-numbers (i. e. simple arc) and Euclidean cubes are not. On the other hand a simple closed curve (i. e., any space homeomorphical with the circle) is an orbicular space. It is easy to construct an orbicular transformation of the circle $K = \{z;\ |z| = 1\}$ of the complex plane. Let

$$f_\eta(e^{it}) = e^{i(t+\eta)}.$$

If $\eta = \alpha\pi$, α irrational, then f_η is an orbicular mapping of K.

Proposition 3. Let f be an orbicular transformation of a countably compact space P. Let $\{y_n\}$ be a sequence of points in P such that

$$f(y_{n+1}) = y_n \quad (n = 1, 2, \ldots)$$

Then the set Y of all y_n, $n = 1, 2, \ldots$ is dense in P.

Proof. For every $k = 1, 2, \ldots$, let Y_k be the closure of the set $\{y_n;\ n \geqq k\}$. Put

$$F = \bigcap_{k=1}^{\infty} Y_k .$$

The space P being countably compact, the set F is non-void. By our assumption

$$f[Y_{k+1}] \subset Y_k$$

and hence

$$f[F] \subset \bigcap_{k=1}^{\infty} f[Y_{k+1}] \subset \bigcap_{k=1}^{\infty} Y_k = F .$$

Thus F is a non-void closed f-fixed subset of P. f being an orbicular mapping, $F = P$. In consequence, Y is dense in P because $\overline{Y} \supset Y_k \supset F$.

Proposition 4. Let f be an orbicular transformation of an infinite space P. Let $x_0 \in P$, $X_0 = (x_0)$ and

$$X_{n+1} = f^{-1}[X_n] \quad (n = 0, 1, 2, \ldots)$$

Then the sequence $\{X_n\}$ is disjoint.

Proof. Let us suppose that the proposition is not true. Let n be the least integer such that $X_n \cap X_m \neq \varnothing$ for some $m > n$. Let p be the least positive integer, with

$$X_n \cap X_{n+p} \neq \varnothing .$$

If $n > 0$, then

$$X_{n-1} \cap X_{n+p-1} \neq \varnothing$$

since

$$f[X_n] = X_{n-1}, f[X_{n+p}] = X_{n+p-1} .$$

But it is impossible. Thus $n = 0$. Since $X_0 = (x_0)$, we have $x_0 \in X_p$, and by definition of X_k we have $f^p(x_0) = x_0$. In consequence

$$F(f, x_0) = \{x_0, f(x_0), \ldots, f^{p-1}(x_0)\} .$$

The transformation f being orbicular,

$$F(f, x_0) = P ,$$

but it is impossible, since P is an infinite set and $F(f, x_0)$ is a finite set. This contradiction completes the proof.

THEOREM 7. *Let f be an orbicular transformation of a space P. Then either f^2 is an orbicular transformation of P or there exist two disjoint subsets F_1 and F_2 of P such that $F_1 \cup F_2 = P$, $f[F_1] = F_2$, $f[F_2] = F_1$ and f^2/F_i is an orbicular transformation of Fi ($i = 1, 2$). In particular, if P is connected, then f^2 is an orbicular transformation of P.*

Proof. Let us suppose that f^2 is not an orbicular transformation of P. Thus there exists a point x in P such that the set

$$M = M(f^2, x) = \{f^{2n}(x); n = 0, 1, 2, \ldots\}$$

is not dense in P. Consider also the set

$$N = \{f^{2n+1}(x); n = 0, 1, 2, \ldots\}.$$

Since f is an orbicular mapping, $M(f, x)$ is dense in P and hence ($M(f, x) = M \cup N$)

$$\overline{M} \cup \overline{N} = P.$$

Clearly

(3)
$$f[\overline{M}] \subset \overline{N}, \quad f[\overline{N}] \subset \overline{M}$$

and hence

$$f[\overline{M} \cap \overline{N}] \subset \overline{M} \cap \overline{N}.$$

Thus the set $\overline{M} \cap \overline{N}$ is f-fixed. The transformation f being orbicular, we have either $\overline{M} \cap \overline{N} = \varnothing$ or $\overline{M} \cap \overline{N} = P$. By our assumption M is not dense in P, and consequently, $\overline{M} \cap \overline{N} = \varnothing$. Moreover, from (3) it follows that $f^2[\overline{M}] \subset \overline{M}$ and $f^2[\overline{N}] \subset \overline{N}$. It remains to prove, that the restrictions $f^2/\overline{M}$ and $f^2/\overline{N}$ are orbicular. Let y be an element of $\overline{M}$. The set $M(f, y)$ is dense in P and $\overline{M}$ is open in P. Thus

$$\overline{M} \cap M(f, y)$$

is a dense subset of $\overline{M}$. But

$$\overline{M} \cap M(f, y) = M(f^2, y)$$

Thus the restriction $f^2/\overline{M}$ of f to M is an orbicular transformation of $\overline{M}$. Analoguously we can prove that f^2/N is orbicular.

The preceding theorem has the following generalization.

THEOREM 8. *Let f be an orbicular transformation of P and let p be a positive integr. Then either f^p is an orbicular transformation of P or there exist disjoint closed sets $F_0, \ldots, F_{p-1}$ such that*

(a) $F_0 \cup \ldots \cup F_{p-1} = P$

(b) $f[F_0] \subset F_1, \ldots, f[F_{p-2}] \subset F_{p-1}, f[F_{p-1}] \subset F_0$.

(c) The restriction f^p/F_i of f^p to F_i ($i = 0, 1, \ldots, p-1$) is an orbicular transformation of F_i.

In consequence, if P has at most $p-1$ different components, then f^p is an orbicular transformation of P.

As an immediate consequence of Theorem 8 we have the following

THEOREM 9. *If f is an orbicular transformation of a connected space P, then f^p is an orbicular transformation of P for every positive integer p.*

Proof of Theorem 8. Let us suppose that the theorem is not true. Hence, there exists an orbicular transformation f of a space P and a positive integer p such that f^p is not an orbicular transformation of P and for no disjoint closed sets $F_0, \ldots, F_{p-1}$ the conditions (a), (b) and (c) are fulfilled.
Since f^p is not orbicular, there exists a point x in P such that the set

$$M_0 = \{f^{pn}(x);\ n = 0, 1, 2, \ldots\}$$

is not dense in P. Put

$$M_1 = f[M_0] = \{f^{pn+1}(x);\ n = 0, 1, 2, \ldots\}$$

$$M_{p-1} = f[M_{p-2}] = \{f^{p(n+1)-1}(x);\ n = 0, 1, 2, \ldots\}$$

Since

$$M_0 \cup \ldots \cup M_{p-1} = M(f, x)$$

we have

$$\overline{M}_0 \cup \ldots \cup \overline{M}_{p-1} = P.$$

By continuity of f we have

$$f[\overline{M}_0] \subset \overline{M}_1, \ldots, f[\overline{M}_{p-2}] \subset \overline{M}_{p-1}$$

Clearly

$$f[M_{p-1}] \subset M_0$$

and hence also

$$f[\overline{M}_{p-1}] \subset \overline{M}_0.$$

The set

$$F = \overline{M}_0 \cap \ldots \cap \overline{M}_{p-1}$$

is f-invariant, since

$$f[F] \subset \overline{M}_1 \cap \ldots \cap \overline{M}_{p-1} \cap \overline{M}_0 \subset F.$$

Since f is an orbicular transformation of P, we have either $F = \varnothing$ or $F = P$. If $F = P$, then $\overline{M}_0 = F = P$, which is impossible because M_0 is not dense in P (by our assumption). Hence $F = \varnothing$. Let $i + 1$ be the least integer such that

$$\overline{M}_0 \cap \ldots \cap \overline{M}_{i+1} = \varnothing.$$

Put

$$F_0 = \overline{M}_0 \cap \ldots \cap \overline{M}_i \neq \varnothing$$

$$F_1 = \overline{f[F_0]}, \ldots, F_{p-1} = \overline{f[F_{p-2}]}.$$

It is easy to see that the sets $F_0, \ldots, F_{p-1}$ are disjoint and $\overline{f[F_{p-1}]} \subset F_0$. It follows that the set

$$F = F_0 \cup \ldots \cup F_{p-1}$$

is f-fixed (and of course closed) and hence $F = P$. Clearly F_i satisfy the conditions (a) and (b). To prove (c) it is sufficient to notice that F_i are also open in P. The proof is complete.

THEOREM *10. Let f be an orbicular transformation of an infinite Hausdorff compact space K. Let us choose points $x_n \in K$, $n = 0, \pm 1, \pm 2, \ldots$, so that*

$$f(x_n) = x_{n+1}\ (n = 0, \pm 1, \pm 2, \ldots)$$

Let X be the set of all x_n and let $\beta(X)$ be the Čech-Stone compactification of X. Let g be the Čech-Stone continuous mapping of $\beta(X)$ onto K (that is, the restriction of g to X is the identity mapping). Then there exists an orbicular transformation Φ of $\beta(X)$ such that

$$f \circ g = g \circ \Phi \tag{4}$$

The preceding equality uniquely defines continuous Φ, and Φ is the unique continuous extension of f/X to $\beta(X)$.

Proof. It is easy to see that $f[X] = X$ and hence f/X is a continuous mapping from X onto X. By Čech-Stone theorem, there exists a continuous mapping Φ from $\beta(X)$ onto $\beta(f[X]) = \beta(X)$. Thus Φ is a transformation of $\beta(X)$. Now we shall prove (4), i. e. the commutativity of the following diagram:

$$\begin{array}{ccc} K & \xrightarrow{f} & K \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g} \\ \beta(X) & \xrightarrow{\Phi} & \beta(X) \end{array}$$

Since g/X is the identity mapping, we have

$$f \circ g/K = f/X .$$

Since $\Phi/X = f/X$, we have also

$$(g \circ \Phi)/X = g \circ (\Phi/X) = g \circ (f/X) = f/X$$

The last equality follows from the fact that $f[X] \subset X$. Since $f \circ g$ and $g \circ \Phi$ are continuous, X is dense in $\beta(X)$ and the restrictions to X are identical, we have $f \circ g = g \circ \Phi$.

It remains to prove that Φ is an orbicular transformation. Let F be a minimal Φ-fixed set in $\beta(X)$. Then

$$f[g[F]] = g[\Phi[F]] = g[F] .$$

Thus $g[F]$ is a non-void closed f-fixed subset of P, and consequently, $g[F] = K$. In particular, $X \subset g[F]$, and hence, $X \subset F$. X being dense in $\beta(X)$, $F = \beta(X)$.

Corollary. An orbicular space may fail to be a homogeneous space (in any usual sense).

Indeed, if K is a circle, then K is a metrizable space, and in particular, every point of K is of a countable character. Let X be the set from theorem 10. Clearly every point of X has a countable character in $\beta(X)$. On the other hand, no point of $\beta(X) - X$ has a countable pseudocharacter in $\beta(X)$. It can be noticed that in this case the space $\beta(X)$ is totally disconnected.

Problem. I do not know whether there exists a connected (locally connected) orbicular compact space which is not homogeneous.

Note. Let us recall that the character of a point x of a space P is the least cardinal of a local base at x in P. The pseudocharacter is the least cardinal m. such that there exists a family $\mathfrak{M}$ of open subsets of P such that the cardinality of $\mathfrak{M}$ is m and the intersection of $\mathfrak{M}$ is the one-point set (x). Let us recall, if P is a compact space then characters and pseudocharacters of points coincide. For further informations see [3].

O SAMODRUŽNÝCH MNOŽINÁCH PŘI SPOJITÝCH TRANSFORMACÍCH

Souhrn

Transformací prostoru P rozumíme spojité zobrazení prostoru P do sebe. V článku se studují minimální uzavřené invariantní množiny při dané transformaci a dále prostory, které mohou být minimální uzavřenou samodružnou množinou při nějaké transformaci nějakého kompaktního prostoru. Přitom minimální uzavřenou samodružnou množinou transformace f prostoru P se rozumí neprázdná uzavřená množina $F \subset P$, která je samodružná, to zn. $f[F] = F$ a která neobsahuje žádnou neprázdnou uzavřenou samodružnou množinu $F_1 \neq F$. Zřejmě každý pevný bod je minimální uzavřenou samodružnou množinou.

Snadno se nahlédne, že každá transformace kompaktního prostoru má minimální uzavřené samodružné množiny. Dokazuje se, že některé transformace topologicky úplného úplně regulárního prostoru mají minimální uzavřené samodružné množiny. Tato věta obsahuje jako speciální případ známou větu o existenci pevného bodu Lipschitzovských transformací úplného metrického prostoru.

Transformaci f prostoru P nazýváme cirkulární, jestliže pro každé x postupné interace $f(x)$, $f^2(x) = f(f(x))$, ... bodu x tvoří hustou mnežinu v P. Prostor nazýváme cirkulárním, jestliže připouští cirkulární transformaci. Zřejmě právě cirkulární prostory mohou být minimální uzavřenou samodružnou množinou při transformacích prostorů. Například kružnice je cirkulárním prostorem. Dále zřejmě všechny konečné prostory jsou cirkulární. Dá se dokázat, že Cantorovo discontinuum je cirkulárním prostorem. V článku se dokazuje m. j. několik výsledků o mocninách cirkulárních zobrazení.

ОБ ИНВАРИАНТНЫХ МНОЖЕСТВАХ ПРИ НЕПРЕРЫВНЫХ ОТОБРАЖЕНИЯХ

Резюме

Непрерывное отображение f пространства P в себя называется циркулярным, если для всякой точки x пространства P множество всех точек $f(x)$, $f^2(x) = f(f(x))$, ... является плотным в P. Пространство P называем циркулярным, если существует циркулярное отображение P до P. В статье рассматриваются циркулярные пространства и доказывается некоторое обобщение известной теоремы о неподвижной точке отображения Липшица полного метрического пространства для некоторых отображений полных (в смысле Э. Чеха) вполне регулярных пространств.

REFERENCES

[1] ZDENĚK FROLÍK: Generalizations of the G_δ-property of complete metric spaces. Czech. Math. J. 10 (85), 359—379 (1960).

[2] ZDENĚK FROLÍK: Internal characterizations of topologically complete spaces in the sense of E. Čech. Czech. Math. J. 12(87), 445—456 (1962).

[3] MIROSLAV KATĚTOV: Remarks on characters and pseudocharacters, Comm. Math. Univ. Carol. 1,1, 20—25 (1960).

Zdeněk Frolík
Matematicko-fysikální fakulta
Ke Karlovu 3
Praha 2 — Nové Město