Zdeněk Frolík

On the classification of 1-dimensional manifolds

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 3 (1962), No. 1, 1--4

Persistent URL: `http://dml.cz/dmlcz/142137`

1962 ACTA UNIVERSITATIS CAROLINAE — MATHEMATICA ET PHYSICA NO. 1 PAG. 1—14

ON THE CLASSIFICATION OF 1-DIMENSIONAL MANIFOLDS

ZDENĚK FROLÍK

Charles University Prague

(Received October 4, 1961)

1. INTRODUCTION

A topological manifold of dimension n (or merely an n-manifold) is a Hausdorff connected space V such that every point of V has a neighborhood homeomorphic with Euclidean n-space. Obviously, any point of an n-manifold has arbitrarily small open neighborhoods homeomorphic with the Euclidean n-space.

THEOREM 1. *Every compact 1-manifold is homeomorphic with the circle.*

The proof follows at once from the fact that every compact 1-manifold can be covered by a finite number of open sets homeomorphic with Euclidean line (= Euclidean 1-space).

Let V be a 1-dimensional manifold which is not compact. The following cases may happen:

(1) *Every countably compact subspace of V is compact.*

(2) *V is not countably compact but some subspace of V is countably compact but not compact.*

(3) *V is countably compact.*

We shall show that a 1-manifold V satisfies (1) if and only if V contains a countable dense set, or equivalently, V is the Euclidean line. A 1-manifold V satisfies (3) if and only if V is so called transfinite line. Finally, V satisfies (2) if and only if V is the union of an Euclidean half-line V_1 and so called transfinite half-line V_2 such that the intersection of V_1 and V_2 is a one-point set. Further, a 1-manifold satisfying (3) contains a 1-manifold satisfying (2) but the converse is not true. Analoguously, a 1-manifold satisfying (2) contains a 1-manifold satisfying (1) and the converse is not true.

2. NATURAL ORDER OF NON-COMPACT 1-MANIFOLDS.

We shall call an arc every space homeomorphic with a compact interval of real numbers; any space homeomorphic with the Euclidean line will be called an open arc.

Now let V be a non-compact 1-manifold. Let $\mathfrak{A}$ be the class of all arcs $A \subset V$.

Using the fact that V is not compact one can prove at once the following two assertions:

(4) If A and B belong to $\mathfrak{A}$, then the intersection of A and B either belongs to $\mathfrak{A}$ or is at most a one-point set.

(5) If A and B belong to $\mathfrak{A}$ and the intersection of A and B is non-void, then the union of A and B belongs to $\mathfrak{A}$.

Now let z be a fixed point of V. The set of all $A \in \mathfrak{A}$ with an end-point z will be denoted by $\mathfrak{A}(z)$.

By definition of 1-manifolds there exists a neighborhood $I \in \mathfrak{A}$ of z. We can write $I = I_1 \cup I_2$, where $I_i \in \mathfrak{A}$ and $I_1 \cap I_2 = (z)$. Define

(6) $\mathfrak{A}_i(z) = \{A;\ A \in \mathfrak{A}(z),\ A \cap I_i \in \mathfrak{A}\}$

One can prove at once

(7) $\bigcup \{A;\ A \in \mathfrak{A}(z)\} = V$

Indeed, the set on the left side of (7) is both open and closed. From (7) it follows at once the following assertion:

(8) If $x \in V$, $x \neq z$, then there exists an $A(z, x) \in \mathfrak{A}$ such that x is an end-point of $A(z, x)$. Moreover, clearly the set $A(z, x)$ is uniquely determined. Now if $x \in V$, $x \neq z$, then either $A(z, x) \in \mathfrak{A}_1(z)$ or $A(z, x) \in \mathfrak{A}_2(z)$. In the first case we shall write $x > z$ and in the second one $z > x$. If $x > z$, $y > z$, $x \neq y$, then we write $x > y$ if and only if $A(z, x) \supset A(z, y)$. If $x > z$, $y > z$, $x \neq y$, then $x > y$ if and only if $A(z, x) \subset A(z, y)$. Finally, if $x > z$ and $y < z$, then we put $x > y$. One can prove at once that $>$ is a linear order on V and

(9) $[x, y] \doteq \{t;\ t \in V,\ x \leqq t \leqq y\} \in \mathfrak{A}$ for every $x < y$. From (9) it follows at once that the topology of V is the order topology, that is, the family of all open intervals on V is a base for open sets of V. We have proved the following result.

Proposition 1. Let V be a 1-manifold which is not compact. Then there exists a linear order $>$ on V such that (9) is true. The topology of V is the order topology.

THEOREM 2. *The following conditions on a* 1-*manifold which is not compact are equivalent*:

(a) *V is homeomorphic with the Euclidean line.*

(b) *V is σ-compact (a countable union of compact subspaces).*

(c) *V contains a countable dense set.*

Proof. Clearly (a) implies (b) and (b) implies (c). Let us suppose (c). From (7) it follows at once that there exists a sequence $\{A_n\}$ of arcs $A_n \subset V$ such that V is the union of $\{A_n\}$ and A_n is contained in the interior of A_{n+1}. It is easy to construct a homeomorphism of V onto the Euclidean line.

3. CONSTRUCTION OF THE TRANSFINITE LINE

Let T be the set of all countable ordinals. Let $>$ be the lexicographical order on the cartesian product P of T and the half-open interval $[0, 1)$ of real numbers. That means, $(\alpha, x) > (\beta, y)$ if and only if either $\alpha > \beta$ (in T) or $\alpha = \beta$ and $x > y$ (in $[0, 1)$). The order $>$ defines in P the order topology. This space will be called the transfinite half-line. It is easy to see that every closed section $\{p;\ p \in P,\ p \leqq q\}$, $q \in P$, of P is an arc. From this fact it follows at once that

(10) $P - (0, 0)$ is a 1-manifold.

One can prove easily:

(11) Every countable subset of the transfinite half-line is contained in an arc of P.

Indeed, if $N \subset P$ is countable, then there exists (because ω_1 is not cofinal with ω_0)

$$\alpha_0 = \sup \{\alpha; (\alpha, x) \in N\} \in T$$

Clearly N is contained in $[(0, 0), (\alpha + 1, 0))$. From (11) it follows the following assertions:

(12) The transfinite half-line is countably compact.

Now the transfinite line will be defined. Let P_- be the set of all pairs $(p, —)$, $p \in P$. Let us define $(x, —) > (y, —)$ if and only if $y > x$. Let R be the union of P and P_-. Identifying the points $(0, 0)$ and $((0, 0,), —)$ we obtain the set L. Defining an order in L such that $x > y$ if and only if either $x, y \in P$ and $x > y$ in P or $x, y \in P_-$ and $y > x$ or $x \in P, y \in P_-, x \neq y$ (the symbol $(0, 0)$ denotes also the element of L containing $(0, 0)$). The set L with the order topology will be called the transfinite line. Clearly P and P_- are subspaces of L. Thus the transfinite line was obtained by sticking two copies of the transfinite half-line.

From the definition of the transfinite line and from (10), (11) and (12) it follows at once the following result.

Proposition 2. The transfinite line is a countably compact 1-manifold which is not compact. The closure of every countable subset of L is compact.

4. THE MAIN THEOREM

Let V be a 1-manifold which is not compact and let z be a point of V. Let $>$ be a linear order in V satisfying (6). Put

(13) $I_1 = \{x; x \in V, x \geqq z\}$

(14) $I_2 = \{x; x \in V, x \leqq z\}$

We shall prove that I_i (i $= 1, 2$) is the Euclidean half-line or the transfinite half-line.

Let F be the set of all homeomorphical mappings which transform open sections of the transfinite halb-line P (i. e., either the sets of the form $\{p; p \in P, p < q\}$ or P) onto open sections of I_1. Thus any $f \in F$ is defined on an open section U of P and $f [U]$ is an open section of I_1. For $f, g \in F$ we shall write $f \supset g$, if g is a restriction of f. One can prove at once that any linearly ordered (by $\supset$) subset F_1 of F has an upper bound in F. Thus there exists a maximal element f in F. Let D be the domain of the definition of f. Thus either $D = P$ or D is an open section of P. We shall prove

(15) $f [D] = I_1$

If $D \neq P$, then (12) follows at once from the maximality of f. If $D = P$, then $f [P]$ is an open section of I_1. If $f [D] \neq I_1$, then one can choose a point x in $I_1 — f [D]$ and an arc $A \subset I_1$ with end-points z and x. Clearly $f [D] \subset A$. But it is impossible because $f [P]$ contains no countable dense set and any subset of A contains a countable dense set.

Thus we have proved that I_i is either the Euclidean half-line or the transfinite half-line. Thus we have proved the following main result.

THEOREM 3. *Let V be a 1-manifold which is not compact. Then exactly one of the conditions (1)—(3) is fulfilled. (1) is fulfilled if and only if V is the Euclidean line. (3) is fulfilled if and only if V is the transfinite line. (2) is fulfilled if and only if V is the union of the Euclidean half-line I_1 and the transfinite half-line I_2 such that the intersection of I_1 and I_2 is a one-point set.*

5. TRANSFINITE PLANE

By the transfinite plane we mean the topological product of two transfinite lines. Clearly the transfinite plane is a 2-manifold.

Proposition 3. The transfinite plane L_2 is countably compact but not compact.

Proof. L_2 is not compact because the transfinite line is not. On the other hand one can prove at once that every countable subset of L_2 is contained in a compact set. This follows from Proposition 1.

From Proposition 3 we have that the transfinite plane contains no countable dense set. One can prove that the transfinite plane is a normal space. Also one can prove that every 1-manifold is a normal space. I do not know whether every 2-manifold is a normal space.

Note. Of course, other examples of non-separable 2-manifolds are well-known.

KLASIFIKACE JEDNOROZMĚRNÝCH VARIET

Souhrn

n-rozměrnou varietou se rozumí Hausdorffův souvislý topologický prostor, jehož každý bod má okolí homeomorfní s n-rozměrným Eukleidovským prostorem. V článku je dokázáno, že jednorozměrná varieta je kompaktní tehdy a jen tehdy, je-li topologickou kružnicí. Nekompaktní jednorozměrná varieta je buď homeomorfní s Eukleidovskou přímkou (to nastane právě tehdy, jestliže obsahuje spočetnou hustou množinu, nebo ekvivalentně, je σ-kompaktní), nebo je „transfinitní přímkou "(to nastane právě tehdy, jestliže je spočetně kompaktní) a nebo vznikne slepením Eukleidovské polopřímky a „transfinitní polopřímky" (to nastane právě tehdy, není-li spočetně kompaktní, ale obsahuje spočetně kompaktní množinu, která není kompaktní). Přitom transfinitní polopřímka se dá sestrojit z prostoru všech spočetných ordinálních čísel spojením každých dvou sousedních bodů uzavřenou úsečkou a transfinitní přímka se sestrojí „slepením" dvou transfinitních polopřímek.

Topologický součin dvou transfinitních přímek je příkladem neseparabilní plochy

О КЛАССИФИКАЦИИ МНОГООБРАЗИЙ РАЗМЕРНОСТИ 1.

Резюме

В статье дается полная классификация многообразий размерности 1.

Zdeněk Frolík
Matematicko-fysikální fakulta
Ke Karlovu 3
Praha 2 — Nové Město