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Homogeneous variational problems: a minicourse

David J. Saunders

Abstract. A Finsler geometry may be understood as a homogeneous variational problem, where the Finsler function is the Lagrangian. The extremals in Finsler geometry are curves, but in more general variational problems we might consider extremal submanifolds of dimension m . In this minicourse we discuss these problems from a geometric point of view.

1 Introduction

This paper is a written-up version of the major part of a minicourse given at the sixth Bilateral Workshop on Differential Geometry and its Applications, held in Ostrava in May 2011. Much of the discussion at these workshops is on Finsler geometry, where the interest is in variational problems defined on tangent manifolds by a ‘Finsler function’, a smooth function defined on the slit tangent manifold (excluding the zero section) and satisfying certain homogeneity and nondegeneracy properties. The extremals of such problems are geometric curves in the original (base) manifold, without any particular parametrization but with an orientation.

For this particular workshop it was felt that it might be worthwhile to describe slightly more general problems, looking at variational problems where the extremals were submanifolds of dimension m , but where the action function still depended upon no more than the first derivatives of the submanifold [2], [4]; for example, minimal surface problems would be included in this description. This minicourse introduces a version of the geometric background needed to express such problems, in terms of velocity manifolds. There is an alternative approach to such problems involving manifolds of contact elements (quotients of velocity manifolds); we refer to this only briefly, when we consider the action of the jet group.

Although we consider only first order variational problems, we nevertheless need to use second order velocities: for instance, the Euler-Lagrange equations for first order variational problems are second-order differential equations. We do this in

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a slightly unusual way, looking at a particular submanifold of the double velocity manifold. Having done this, we look at some geometrical and cohomological constructions, before obtaining a version of the first variation formula for variational problems with fixed boundary conditions. The final part of the minicourse, which considered various concepts of regularity, has been omitted from this paper for reasons of space; the concepts described may be found in a recent paper [1]. We give only a few other references: [3] provides extensive background material on various types of jet manifold and the actions of the jet groups; [5] introduces in a more general context the type of cohomological approach we use these types of variational problem; and [6], with a philosophy similar to that of the present paper, compares these problems with those defined on jets of sections of fibrations.

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2 Velocities

In this section we see how to construct manifolds of first order and second order velocities, and also how certain groups, the jet groups, act on these manifolds.

2.1 First order velocities

Let E be a connected, paracompact, Hausdorff manifold of class C^∞ and of finite dimension n ; let $O \subset \mathbb{R}^m$ (with $m < n$) be open and connected, with $0 \in O$. A map $\gamma : O \rightarrow E$ will be called an m -curve in E . The 1-jet $j_0^1\gamma$ of γ at zero will be called a *velocity* (or m -velocity), and the set $T_m E = \{j_0^1\gamma\}$ of velocities of all m -curves in E will be called the *velocity (or m -velocity) manifold of E* . We map $T_m E$ to E by

$$\tau_{mE} : T_m E \rightarrow E, \quad \tau_{mE}(j_0^1\gamma) = \gamma(0).$$

We shall show that $T_m E$ really is a manifold (and is connected, paracompact and Hausdorff, and indeed is a vector bundle over E) by identifying it with the Whitney sum over E of m copies of the tangent manifold TE .

Lemma 1. *There is a canonical identification $T_m E \cong \bigoplus^m TE$.*

Proof. Let $i_k : \mathbb{R} \rightarrow \mathbb{R}^m$ be the inclusion $i_k(s) = (0, \dots, 0, s, 0, \dots, 0)$. Then each $\gamma \circ i_k$ is a curve in E , and the map

$$j_0^1\gamma \mapsto (j_0^1(\gamma \circ i_1), \dots, j_0^1(\gamma \circ i_m))$$

is a bijection $T_m E \rightarrow \bigoplus^m TE$ preserving the fibration over E . □

Corollary 1. *Let $\{dt^i\}$ be the canonical basis of \mathbb{R}^{m*} ; then*

$$T_m E \rightarrow TE \otimes \mathbb{R}^{m*}, \quad (\xi_1, \dots, \xi_m) \mapsto \xi_i \otimes dt^i$$

is a vector bundle isomorphism. □

If $(U; u^a)$ is a chart on E then $(U^1; u^a, u_i^a)$ is a chart on $T_m E$, where

$$U^1 = \tau_{mE}^{-1}(U), \quad u_i^a(j_0^1 \gamma) = D_i \gamma^a(0) = D_i(u^a \circ \gamma)(0).$$

If $j_0^1 \gamma = (\xi_1, \dots, \xi_m)$ then it is clear that $u_i^a(j_0^1 \gamma) = \dot{u}^a(\xi_i)$. The rule for changing coordinates on $T_m E$ is therefore

$$v_i^b(j_0^1 \gamma) = \left. \frac{\partial v^b}{\partial u^a} \right|_{\gamma(0)} u_i^a(j_0^1 \gamma).$$

We can see from this that the superscript a labeling the coordinate function u_i^a depends on the original choice of chart u^a on E , whereas the subscript i is independent of this choice and so is the index of a component of the velocity (namely, the tangent vector ξ_i). We call indices of this latter type *counting indices* rather than *coordinate indices*.

We shall be particularly interested in the subsets of $T_m E$ containing those velocities $j_0^1 \gamma$ where the m -curve γ has certain properties. Write $\overset{\circ}{T}_m E$ for the subset

$$\{j_0^1 \gamma \in T_m E : \gamma \text{ is an immersion near zero}\};$$

if $j_0^1 \gamma = (\xi_1, \dots, \xi_m)$ and $j_0^1 \gamma \in \overset{\circ}{T}_m E$ then $\{\xi_1, \dots, \xi_m\}$ will be linearly independent. An element of $\overset{\circ}{T}_m E \subset T_m E$ will be called a *regular velocity*.

Proposition 1. *The regular velocities form an open-dense submanifold.*

Proof. To show that $\overset{\circ}{T}_m E$ is open in $T_m E$, define the map $\wedge : T_m E \rightarrow \wedge^m TE$ by $(\xi_1, \dots, \xi_m) \mapsto \xi_1 \wedge \dots \wedge \xi_m$. Then

- The map \wedge is fibred over the identity on E and is continuous (it is polynomial in the fibre coordinates u_i^a);
- $j_0^1 \gamma \in \overset{\circ}{T}_m E$ exactly when $\wedge(j_0^1 \gamma) \neq 0$;
- the zero section of $\wedge^m TE$ is closed.

To show that $\overset{\circ}{T}_m E$ is dense in $T_m E$, define the map $f : U^1 \rightarrow \mathbb{R}$ by $f(j_0^1 \gamma) = \det(u_i^j(j_0^1 \gamma))$, where (u_i^j) is the $m \times m$ submatrix containing the first m rows of the $n \times m$ matrix u_i^a . If $j_0^1 \gamma \in O \subset U^1$ where O is open and $O \cap \overset{\circ}{T}_m E = \emptyset$ then f vanishes on O . But

$$\left. \frac{\partial^m f}{\partial u_1^1 \partial u_2^2 \dots \partial u_m^m} \right|_{j_0^1 \gamma} = 1. \quad \square$$

2.2 Second order velocities

We define a *second-order m -velocity* in the same way as a 2-jet at zero of an m -curve, and write

$$T_m^2 E = \{j_0^2 \gamma\}, \quad \overset{\circ}{T}_m^2 E = \{j_0^2 \gamma : \gamma \text{ is an immersion near zero}\}.$$

We also let $\tau_{mE}^2 : T_m^2 E \rightarrow E$, $\tau_{mE}^{2,1} : T_m^2 E \rightarrow T_m E$ be the projections

$$\tau_{mE}^2(j_0^2\gamma) = \gamma(0), \quad \tau_{mE}^{2,1}(j_0^2\gamma) = j_0^1\gamma.$$

We take charts on $T_m^2 E$ to be $(U^2; u^a, u_i^a, u_{ij}^a)$ where $U^2 = (\tau_{mE}^2)^{-1}(U)$ and

$$u_i^a(j_0^2\gamma) = D_i\gamma^a(0), \quad u_{ij}^a(j_0^2\gamma) = D_i D_j \gamma^a(0)$$

so that $u_{ij}^a = u_{ji}^a$ (this constraint will cause complications in certain coordinate formulæ). These charts form an atlas such that $T_m^2 E$ becomes a manifold with the standard properties. We shall not demonstrate this directly; we shall show instead that it may be identified with a closed submanifold of a larger manifold, the manifold of double velocities.

2.3 Double velocities

We know that $T_m E$ is a manifold, so it has its own velocity manifold

$$T_{m'} T_m E = \{j_0^1 \tilde{\gamma}\}$$

where $\tilde{\gamma}$ is an m' -curve in $T_m E$. This is the (m', m) double velocity manifold. Charts on $T_{m'} T_m E$ are therefore

$$((U^1)^1; u^a, u_i^a, u_{i,j}^a, u_{i,i}^a),$$

where $1 \leq i \leq m$ and $1 \leq j \leq m'$, corresponding to the charts $(U^1; u^a, u_i^a)$ on $T_m E$. In most applications we have either $m' = m$ or $m' = 1$. We shall be interested in a particular submanifold of double velocities, known as holonomic double velocities.

2.4 Holonomic double velocities

If γ is an m -curve in E then its *prolongation* is the m -curve $\bar{j}^1\gamma$ in $T_m E$ where

$$\bar{j}^1\gamma(t) = j_0^1(\gamma \circ \tau_t)$$

and $\tau_t : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the translation map $\tau_t(s) = t + s$. Thus $j_0^1 \bar{j}^1\gamma \in T_m T_m E$. We use the notation $\bar{j}^1\gamma$ rather than $j^1\gamma$; the latter would be a map satisfying $j^1\gamma(t) = j_t^1\gamma$ whose codomain would be a set containing jets at arbitrary points of \mathbb{R}^m rather than just at zero.

Proposition 2. *The map*

$$\iota : T_m^2 E \rightarrow T_m T_m E, \quad \iota(j_0^2\gamma) = j_0^1 \bar{j}^1\gamma$$

is an injection. Its image is the submanifold described in coordinates by

$$u_i^a = u_{i,i}^a, \quad u_{i,j}^a = u_{j,i}^a.$$

The image of the chart $(U^2; u^a, u_i^a, u_{ij}^a)$ under the injection is the restriction of the chart $((U^1)^1; u^a, u_i^a, u_{i,j}^a, u_{i,i}^a)$ to the submanifold.

Proof. Suppose γ_1, γ_2 are two m -curves in E such that $j_0^1 \bar{j}^1 \gamma_1 = j_0^1 \bar{j}^1 \gamma_2$. Then for γ_1

$$\begin{aligned} u^a(j_0^2 \gamma_1) &= u^a(\gamma_1(0)) = u^a(\bar{j}^1 \gamma_1(0)) = u^a(j_0^1 \bar{j}^1 \gamma_1); \\ u_i^a(j_0^2 \gamma_1) &= D_i(u^a \circ \gamma_1)(0) = u_i^a(\bar{j}^1 \gamma_1(0)) = u_i^a(j_0^1 \bar{j}^1 \gamma_1); \\ u_{ij}^a(j_0^2 \gamma_1) &= D_i D_j(u^a \circ \gamma_1)(0) = D_i(u_{ij}^a \circ \bar{j}^1 \gamma_1)(0) = u_{ij}^a(j_0^1 \bar{j}^1 \gamma_1) \end{aligned}$$

and similarly for γ_2 , so that $j_0^2 \gamma_1 = j_0^2 \gamma_2$ and the map is an injection.

For any m -curve γ in E

$$u_{ij}^a(j_0^1 \bar{j}^1 \gamma) = D_i(u^a \circ \bar{j}^1 \gamma)(0) = D_i(u^a \circ \gamma)(0) = u_i^a(\bar{j}^1 \gamma(0)) = u_i^a(j_0^1 \bar{j}^1 \gamma)$$

and

$$u_{j_i i}^a(j_0^1 \bar{j}^1 \gamma) = D_i(u_{j_i i}^a \circ \bar{j}^1 \gamma)(0) = D_i(D_j(u^a \circ \gamma))(0)$$

so that $u_i^a = u_{i,i}^a$ and $u_{i,j}^a = u_{j,i}^a$ when restricted to the image of the injection.

Furthermore, if $\tilde{\gamma}$ is an m -curve in $T_m E$ satisfying

$$u_i^a(j_0^1 \tilde{\gamma}) = u_{i,i}^a(j_0^1 \tilde{\gamma}), \quad u_{ij}^a(j_0^1 \tilde{\gamma}) = u_{j,i}^a(j_0^1 \tilde{\gamma})$$

then the m -curve γ in E given in coordinates near $\tau_{mE}(\tilde{\gamma}(0))$ by

$$\gamma^a(t) = u^a(j_0^1 \tilde{\gamma}) + u_i^a(j_0^1 \tilde{\gamma}) t^i + \frac{1}{2} u_{ij}^a(j_0^1 \tilde{\gamma}) t^i t^j$$

so that $j_0^1 \bar{j}^1 \gamma = j_0^1 \tilde{\gamma}$; thus the image of the injection is described locally by the equations $u_i^a = u_{i,i}^a$, $u_{i,j}^a = u_{j,i}^a$ and is therefore a submanifold of $T_m T_m E$.

The relationship between the charts $(U^2; u^a, u_i^a, u_{ij}^a)$ and $((U^1)^1; u^a, u_i^a, u_{ij}^a, u_{i,j}^a)$ is immediate. \square

The image of $T_m^2 E$ in $T_m T_m E$ is called the submanifold of *holonomic* double velocities. There is no canonical projection $T_m T_m E \rightarrow T_m^2 E$; we may, however, consider a tubular neighbourhood $\nu : N \rightarrow T_m^2 E$ of $T_m^2 E$ in $T_m T_m E$, and then the condition $\nu \circ \iota = \text{id}_{T_m^2 E}$ (where $\iota : T_m^2 E \rightarrow T_m T_m E$ is the injection) gives rise to the constraints

$$\begin{aligned} \frac{\partial \nu^a}{\partial u^c} &= \delta_c^a, & \frac{\partial \nu^a}{\partial u_p^c} + \frac{\partial \nu^a}{\partial u_{;p}^c} &= 0, & \frac{\partial \nu^a}{\partial u_{p;q}^c} + \frac{\partial \nu^a}{\partial u_{q;p}^c} &= 0, \\ \frac{\partial \nu_i^a}{\partial u^c} &= 0, & \frac{\partial \nu_i^a}{\partial u_p^c} + \frac{\partial \nu_i^a}{\partial u_{;p}^c} &= \delta_c^a \delta_i^p, & \frac{\partial \nu_i^a}{\partial u_{p;q}^c} + \frac{\partial \nu_i^a}{\partial u_{q;p}^c} &= 0, \\ \frac{\partial \nu_{ij}^a}{\partial u^c} &= 0, & \frac{\partial \nu_{ij}^a}{\partial u_p^c} + \frac{\partial \nu_{ij}^a}{\partial u_{;p}^c} &= 0, & \frac{\partial \nu_{ij}^a}{\partial u_{p;q}^c} + \frac{\partial \nu_{ij}^a}{\partial u_{q;p}^c} &= \delta_c^a (\delta_i^p \delta_j^q + \delta_j^p \delta_i^q). \end{aligned}$$

for the coordinates of ν , and hence to the conditions

$$\begin{aligned} d\nu^a &= du^a + \frac{\partial \nu^a}{\partial u_p^c} (du_p^c - du_{;p}^c) + \frac{1}{2} \frac{\partial \nu^a}{\partial u_{p;q}^c} (du_{p;q}^c - du_{q;p}^c) \\ d\nu_i^a &= \frac{1}{2} (du_i^a + du_{;i}^a) + \frac{1}{2} \left(\frac{\partial \nu_i^a}{\partial u_p^c} - \frac{\partial \nu_i^a}{\partial u_{;p}^c} \right) (du_p^c - du_{;p}^c) + \frac{1}{2} \frac{\partial \nu_i^a}{\partial u_{p;q}^c} (du_{p;q}^c - du_{q;p}^c) \\ d\nu_{ij}^a &= \frac{1}{2} (du_{i;j}^a + du_{j;i}^a) + \frac{\partial \nu_{ij}^a}{\partial u_p^c} (du_p^c - du_{;p}^c) + \frac{1}{2} \frac{\partial \nu_{ij}^a}{\partial u_{p;q}^c} (du_{p;q}^c - du_{q;p}^c). \end{aligned}$$

We shall use these conditions later on.

2.5 The exchange map

There is another way of describing the submanifold of holonomic velocities.

A map $\psi : O' \times O \rightarrow E$, where $O \subset \mathbb{R}^m$, $O' \subset \mathbb{R}^{m'}$ are open and connected, and where $0_{\mathbb{R}^m} \in O$ and $0_{\mathbb{R}^{m'}} \in O'$, is called a *double* (m', m) -*curve*. For each $s \in O'$

$$\psi_s : O \rightarrow E, \quad \psi_s(t) = \psi(s, t)$$

is then an m -curve in E , so that $j_0^1 \psi_s \in T_m E$. Thus

$$j_0^1 (s \mapsto j_0^1 \psi_s) \in T_{m'} T_m E.$$

Lemma 2. *The exchange map $e : T_{m'} T_m E \rightarrow T_m T_{m'} E$ is well-defined by $\psi \mapsto \hat{\psi}$ where $\hat{\psi}(t, s) = \psi(s, t)$ and is a smooth bijection.*

Proof. The element of $T_m T_m E$ defined by ψ satisfies

$$\begin{aligned} u^a(j_0^1(s \mapsto j_0^1 \psi_s)) &= u^a(j_0^1 \psi_0) = \psi_0^a(0) = \psi^a(0, 0), \\ u_i^a(j_0^1(s \mapsto j_0^1 \psi_s)) &= u_i^a(j_0^1 \psi_0) = D_i(u^a \circ \psi_0)(0) = D_{2;i} \psi^a(0, 0), \\ u_{;j}^a(j_0^1(s \mapsto j_0^1 \psi_s)) &= D_j(u^a \circ (s \mapsto j_0^1 \psi_s))(0) = D_j(s \mapsto \psi_s^a)(0) = D_{1;j} \psi^a(0, 0), \\ u_{i;j}^a(j_0^1(s \mapsto j_0^1 \psi_s)) &= D_j(u_i^a \circ (s \mapsto j_0^1 \psi_s))(0) \\ &= D_j(s \mapsto D_i \psi_s^a(0))(0) = D_{1;j} D_{2;i} \psi^a(0, 0), \end{aligned}$$

and carrying out the same calculation for $\hat{\psi}$ shows that e is a well-defined injection. It is clearly an involution, and hence is a bijection. The coordinate formulæ

$$u^a \circ e = u^a, \quad u_i^a \circ e = u_{;i}^a, \quad u_{;j}^a \circ e = u_j^a, \quad u_{i;j}^a \circ e = u_{j;i}^a$$

show that it is smooth. □

Proposition 3. *The holonomic submanifold of $T_m T_m E$ is the fixed point set of the exchange map.*

Proof. This is immediate from the coordinate formulæ for e . □

2.6 Jet groups

If we consider m -curves in \mathbb{R}^m rather than in some other manifold, then we have the possibility of composing two such m -curves. If we insist that the origin must map to itself then the composition will always exist, although possibly with a smaller domain than the domains of the two original m -curves. We shall want the jets of these m -curves to have inverses, so that the curves themselves will need to be immersions near zero; it is convenient to assume that they are, in fact, diffeomorphisms onto their images.

So let $O \subset \mathbb{R}^m$ be open and connected with $0 \in O$, and let $\phi : O \rightarrow \phi(O) \subset \mathbb{R}^m$ be a diffeomorphism with $\phi(0) = 0$. The *first and second order jet groups* are

$$L_m^1 = \{j_0^1 \phi\}, \quad L_m^2 = \{j_0^2 \phi\}.$$

The products for L_m^1 and L_m^2 are given by

$$j_0^1 \phi_1 \cdot j_0^1 \phi_2 = j_0^1(\phi_1 \circ \phi_2), \quad j_0^2 \phi_1 \cdot j_0^2 \phi_2 = j_0^2(\phi_1 \circ \phi_2).$$

Lemma 3. *The product rules define group structures on L_m^1 and L_m^2 .*

Proof. The products are well-defined because the first (or second) derivatives of a composite depend only upon the first (or second) derivatives of the individual maps, by the first (or second) order chain rule; associativity of the products is inherited from that of composition. The diffeomorphism $\text{id}_{\mathbb{R}^m}$ satisfies

$$j_0^1(\text{id}_{\mathbb{R}^m}) \cdot j_0^1 \phi = j_0^1(\text{id}_{\mathbb{R}^m} \circ \phi) = j_0^1 \phi;$$

the map $\bar{\phi} : \phi(O) \rightarrow O$ given by $\bar{\phi} = \phi^{-1}$ satisfies $\bar{\phi}(0) = 0$, and

$$j_0^1 \bar{\phi} \cdot j_0^1 \phi = j_0^1(\bar{\phi} \circ \phi) = j_0^1(\text{id}_O) = j_0^1(\text{id}_{\mathbb{R}^m}).$$

Similar formulæ hold for second-order jets. □

The map $L_m^1 \rightarrow \mathbb{R}^{m^2}$, $j_0^1 \phi \mapsto (D_j \phi^i(0))$ defines global coordinates on L_m^1 , and identifies it with $\text{GL}(m, \mathbb{R})$. The map $L_m^2 \rightarrow \mathbb{R}^{m^2(m+3)/2}$,

$$j_0^2 \phi \mapsto (D_j \phi^i(0), D_j D_k \phi^i(0))$$

defines global coordinates on L_m^2 . Writing

$$A_j^i = D_j \phi^i(0), \quad B_{jk}^i = D_j D_k \phi^i(0)$$

where $\det A_j^i \neq 0$ because ϕ is a diffeomorphism, the product rule in L_m^1 is

$$(A\hat{A})_j^i = A_h^i \hat{A}_j^h$$

and the product rule in L_m^2 is

$$\begin{aligned} ((A, B)(\hat{A}, \hat{B}))_j^i &= A_h^i \hat{A}_j^h, \\ ((A, B)(\hat{A}, \hat{B}))_{jk}^i &= A_l^i \hat{B}_{jk}^l + B_{hl}^i \hat{A}_j^h \hat{A}_k^l, \end{aligned}$$

the latter formula arising from the second order chain rule

$$\begin{aligned} D_j D_k (\phi \hat{\phi})^i(0) &= D_j (D_l \phi^i \circ \hat{\phi}) D_k \hat{\phi}^l(0) \\ &= D_l \phi^i(0) D_j D_k \hat{\phi}^l(0) + D_h D_l \phi^i(0) D_j \hat{\phi}^h(0) D_k \hat{\phi}^l(0) \end{aligned}$$

using $\phi(0) = \hat{\phi}(0) = 0$.

Corollary 2. *The groups L_m^1 and L_m^2 are Lie groups.* □

Lemma 4. *The oriented subgroups L_m^{1+} and L_m^{2+} , where ϕ preserves orientation, are connected.*

Proof. As L_m^1 may be identified with $\text{GL}(m, \mathbb{R})$, the subgroup L_m^1 where ϕ preserves orientation may be identified with $\text{GL}^+(m, \mathbb{R})$, the subgroup of matrices satisfying $\det A_j^i > 0$, which is connected.

The map $L_m^1 \rightarrow L_m^2$ given by $j_0^1 \phi \mapsto j_0^2 \hat{\phi}$, where $\hat{\phi}$ is the linear map $\hat{\phi}^i(t) = A_j^i t^j$ with (A_j^i) being the matrix corresponding to $j_0^1 \phi$, is continuous; the coordinates of the image are $(A_j^i, 0)$. The image of the subgroup L_m^{1+} under this map is therefore connected. But every element of L_m^{2+} may be joined to an element of this image by a path given in coordinates by

$$s \mapsto (A_j^i, s B_{jk}^i), \quad s \in [0, 1] \quad \square$$

2.7 Group actions

The jet groups L_m^1 and L_m^2 act on the velocity manifolds $T_m E$ and $T_m^2 E$ by

$$(j_0^1 \phi, j_0^1 \gamma) \mapsto j_0^1(\gamma \circ \phi), \quad (j_0^2 \phi, j_0^2 \gamma) \mapsto j_0^2(\gamma \circ \phi).$$

These are right actions, and in coordinates they are

$$\begin{aligned} u^a &\mapsto u^a \\ u_i^a &\mapsto u_h^a A_i^h \\ u_{ij}^a &\mapsto u_{hk}^a A_i^h A_j^k + u_h^a B_{jk}^h \end{aligned}$$

where A_j^i and B_{jk}^i are the global coordinates of $j_0^2 \phi$.

Lemma 5. *The action of L_m^1 on $T_m E$ restricts to $\hat{T}_m E$, and the restricted action is free. The action of L_m^2 on $T_m^2 E$ restricts to $\hat{T}_m^2 E$, and the restricted action is free.*

Proof. The map ϕ is a diffeomorphism onto its image, so if γ is an immersion near zero then so is $\gamma \circ \phi$.

We use coordinates to show that the restricted actions are free. Suppose first that $j_0^1(\gamma \circ \phi) = j_0^1 \gamma$, so that

$$u_j^a(j_0^1 \gamma) = u_i^a(j_0^1 \gamma) A_j^i;$$

as γ is an immersion near zero and $u_i^a(j_0^1\gamma) = D_i\gamma^a(0)$, it follows that the $m \times n$ matrix $u_i^a(j_0^1\gamma)$ must have rank m , so that $A_j^i = \delta_j^i$ and hence $j_0^1\phi = j_0^1(\text{id}_{\mathbb{R}^m})$.

Now suppose that $j_0^2(\gamma \circ \phi) = j_0^2\gamma$, so that $u_j^a(j_0^2\gamma) = u_i^a(j_0^2\gamma)A_j^i$ and now also

$$u_{hk}^a(j_0^2\gamma) = u_{ij}^a(j_0^2\gamma)A_h^iA_k^j + u_i^a(j_0^2\gamma)B_{hk}^i.$$

As before we see that $A_j^i = \delta_j^i$, so that

$$u_{hk}^a(j_0^2\gamma) = u_{hk}^a(j_0^2\gamma) + u_i^a(j_0^2\gamma)B_{hk}^i$$

and therefore that $u_i^a(j_0^2\gamma)B_{hk}^i = 0$; the rank condition on $u_i^a(j_0^2\gamma)$ now tells us that $B_{hk}^i = 0$. \square

2.8 Infinitesimal actions

Let (a_j^i) be an element of the Lie algebra of L_m^1 ; the identification of the group with $\text{GL}(m, r)$ means that its Lie algebra may be identified with $\mathfrak{gl}(m, \mathbb{R})$ so that (a_j^i) is an arbitrary $m \times m$ matrix.

Lemma 6. *The vector field on T_mE corresponding to (a_j^i) is*

$$a_j^i u_i^a \frac{\partial}{\partial u_j^a}.$$

Proof. The map $\sigma : (-\varepsilon, \varepsilon) \rightarrow \text{GL}(m, \mathbb{R})$, defined for sufficiently small ε by $\sigma(s) = (\delta_j^i + sa_j^i)$, is a curve in $\text{GL}(m, \mathbb{R})$ whose tangent vector at the identity is (a_j^i) . If $j_0^1\gamma \in T_mE$ then the corresponding curve through $j_0^1\gamma$ is given in coordinates by

$$s \mapsto (u^b(j_0^1\gamma), (\delta_j^i + sa_j^i)u_i^b(j_0^1\gamma)).$$

The resulting tangent vector $\xi \in T_{j_0^1\gamma}T_mE$ satisfies

$$\dot{u}^b(\xi) = 0, \quad \dot{u}_j^b(\xi) = a_j^i u_i^b(j_0^1\gamma)$$

so that the vector field on T_mE defined by the Lie algebra element (a_j^i) is

$$a_j^i u_i^b \frac{\partial}{\partial u_j^b}. \quad \square$$

We write d_j^i for the Lie derivative operation of the basis vector field $\Delta_i^j = u_i^a \partial / \partial u_j^a$.

2.9 Second order infinitesimal actions

There is a similar result for the action of the Lie algebra of L_m^2 .

Lemma 7. *Let (a_j^i, b_{jk}^i) be an element of the Lie algebra of L_m^2 . The corresponding vector field on T_m^2E is*

$$a_j^i u_i^a \frac{\partial}{\partial u_j^a} + \frac{1}{\#(jk)} (2a_j^i u_{ik}^a + b_{jk}^i u_i^a) \frac{\partial}{\partial u_{jk}^a}.$$

where $\#(jk)$ equals 1 if $j = k$ and equals 2 otherwise.

Proof. Let γ be the curve in L_m^2 through the identity $j_0^2(\text{id})$ given in coordinates by

$$s \mapsto (\delta_j^i + sa_j^i, sb_{jk}^i).$$

If $j_0^2\gamma \in T_m^2E$ then the corresponding curve through $j_0^2\gamma$ is given in coordinates by

$$s \mapsto (u^a(j_0^2\gamma), u_i^a(j_0^2\gamma)(\delta_j^i + sa_j^i), u_{hi}^a(j_0^2\gamma)(\delta_k^h + sa_j^h)(\delta_k^i + sa_k^i) + su_i^a(j_0^2\gamma)b_{jk}^i).$$

The resulting tangent vector $\xi \in T_{j_0^2\gamma}T_m^2E$ satisfies

$$\begin{aligned} \dot{u}^a(\xi) &= 0 \\ \dot{u}_j^a(\xi) &= a_j^i u_i^a(j_0^2\gamma) \\ \dot{u}_{jk}^a(\xi) &= a_k^i u_{ij}^a(j_0^2\gamma) + a_j^i u_{ik}^a(j_0^2\gamma) + b_{jk}^i u_i^a(j_0^2\gamma) \end{aligned}$$

so that the vector field on T_m^2E defined by the Lie algebra element corresponding to (a_j^i, b_{jk}^i) is

$$a_j^i u_i^a \frac{\partial}{\partial u_j^a} + \frac{1}{\#(jk)} (2a_j^i u_{ik}^a + u_{jk}^i u_i^a) \frac{\partial}{\partial u_{jk}^a}. \quad \square$$

We write d_i^j and d_i^{jk} for the Lie derivative operation of the basis vector fields

$$\Delta_i^j = u_i^a \frac{\partial}{\partial u_j^a} + \frac{2}{\#(jk)} u_{ik}^a \frac{\partial}{\partial u_{jk}^a}, \quad \Delta_i^{jk} = \frac{1}{\#(jk)} u_i^a \frac{\partial}{\partial u_{jk}^a}.$$

Note the use of the symbol $\#(jk)$ to compensate for the fact that the coordinate functions u_{jk}^a and u_{kj}^a are equal, so that summing over j and k could result in double-counting.

3 Geometric structures

The special structure of velocity manifolds manifests itself in the existence of certain differential operators ('total derivatives') and differential forms ('contact forms') which capture certain aspects of the structure. The total derivatives and contact forms may also be used to identify those maps between velocity manifolds, and vector fields on velocity manifolds, which have been constructed by a process known as prolongation. Finally, there is an algebraic method of lifting tangent vectors from a manifold to its velocity manifold called the vertical lift, and this gives rise to vertical endomorphisms.

3.1 Total derivatives

The identity map $T_m E \rightarrow T_m E$ defines a section of the pull-back bundle

$$\tau_{mE}^* T_m E \rightarrow T_m E.$$

Its components d_i are the *total derivatives*, vector fields along τ_{mE} . At a point $j_0^1\gamma$, the identification $T_m E \cong \bigoplus^m TE$ from Lemma 1 gives the k -th component of $j_0^1\gamma$ as

$$d_k|_{j_0^1\gamma} = j_0^1(\gamma \circ i_k) = T\gamma(j_0^1 i_k) = T\gamma\left(\left.\frac{\partial}{\partial t^k}\right|_0\right).$$

Note that the subscript k is a counting index, not a coordinate index. In coordinates, if f is a function on E then

$$\begin{aligned} d_k f|_{j_0^1 \gamma} &= d_k|_{j_0^1 \gamma} f = T\gamma \left(\frac{\partial}{\partial t^k} \Big|_0 \right) f = \frac{\partial(f \circ \gamma)}{\partial t^k} \Big|_0 \\ &= \frac{\partial f}{\partial u^a} \Big|_{\gamma(0)} D_k \gamma^a(0) = u_k^a(j_0^1 \gamma) \frac{\partial f}{\partial u^a} \Big|_{\gamma(0)} \end{aligned}$$

so that

$$d_k = u_k^a \frac{\partial}{\partial u^a}.$$

It is clear from this coordinate formula that the image of (d_1, \dots, d_m) , a subspace of $T_{\gamma(0)}E$ corresponding to each point $j_0^1 \gamma \in T_m E$, does not have constant rank on $T_m E$. But its restriction to $\hat{T}_m E$, where the $m \times n$ matrix u_i^a has maximal rank, does have constant rank m .

3.2 Second order total derivatives

We take a similar approach to second order total derivatives. The inclusion map $T_m^2 E \rightarrow T_m T_m E$ defines a section of the pull-back bundle

$$\tau_{mE}^{2,1*} T_m T_m E \rightarrow T_m^2 E;$$

its components d_i are the *second order total derivatives*, vector fields along $\tau_{mE}^{2,1}$. At a point $j_0^2 \gamma$,

$$d_k|_{j_0^2 \gamma} = T(j^1 \gamma) \left(\frac{\partial}{\partial t^k} \Big|_0 \right);$$

in coordinates

$$d_k = u_k^a \frac{\partial}{\partial u^a} + u_{kj}^a \frac{\partial}{\partial u_j^a}.$$

Once again the image of (d_1, \dots, d_m) , a subspace of $T_{j_0^1 \gamma} T_m E$ corresponding to each point $j_0^2 \gamma \in T_m^2 E$, does not have constant rank on $T_m^2 E$, but its restriction to $\hat{T}_m^2 E$ does have constant rank m .

3.3 Contact 1-forms

Contact 1-forms on $T_m E$ or on $T_m^2 E$ are the horizontal 1-forms which annihilate total derivatives, so that θ is a contact 1-form exactly when

$$\langle \theta, d_k \rangle = 0.$$

Here, ‘horizontal’ means horizontal over E for a 1-form on $T_m E$, and it means horizontal over $T_m E$ for a 1-form on $T_m^2 E$, so that it makes sense to evaluate such forms on total derivatives; indeed, the modules of such horizontal 1-forms are dual to the modules of vector fields along $T_m E \rightarrow E$ or along $T_m^2 E \rightarrow T_m E$.

In fact we shall consider contact 1-forms, not on the whole of $T_m E$ or $T_m^2 E$, but on the submanifolds of regular velocities $\hat{T}_m E$ and $\hat{T}_m^2 E$. The reason is that, as mentioned previously, the image of the map (d_1, \dots, d_m) has constant rank m

only on the regular submanifolds; it is, for example, zero on the zero section of $T_m E$, and so every horizontal cotangent vector on that zero section is annihilated by all the total derivatives. If we were to include non-regular velocities then there would be ‘contact’ cotangent vectors which were not the values of any (smooth, and hence continuous) contact 1-form.

The important property of contact 1-forms is that they always pull back to zero under prolongations.

Lemma 8. *If θ is a contact 1-form on $\hat{T}_m E$ then $(\bar{j}^1 \gamma)^* \theta = 0$. If it is a contact 1-form on $\hat{T}_m^2 E$ then $(\bar{j}^2 \gamma)^* \theta = 0$, where the prolonged m -curve $\bar{j}^2 \gamma$ is defined by $\bar{j}^2 \gamma(t) = j_0^2(\gamma \circ \mathbb{T}_t)$.*

Proof. If θ is a contact 1-form on $\hat{T}_m E$ then

$$\begin{aligned} \left\langle (\bar{j}^1 \gamma)^* \theta \Big|_t, \frac{\partial}{\partial t^k} \Big|_t \right\rangle &= \left\langle (j^1(\gamma \circ \mathbb{T}_t))^* \theta \Big|_t, \frac{\partial}{\partial t^k} \Big|_t \right\rangle \\ &= \left\langle (j^1 \gamma)^* \theta \Big|_0, \frac{\partial}{\partial t^k} \Big|_0 \right\rangle \\ &= \left\langle \theta \Big|_{j_0^1 \gamma}, T\gamma \left(\frac{\partial}{\partial t^k} \Big|_0 \right) \right\rangle \\ &= \langle \theta \Big|_{j_0^1 \gamma}, d_k \Big|_{j_0^1 \gamma} \rangle = 0. \end{aligned}$$

The proof for a contact 1-form on $\hat{T}_m^2 E$ is similar. □

Proposition 4. *If θ is a 1-form on $\hat{T}_m E$ satisfying $(\bar{j}^1 \gamma)^* \theta = 0$ for every prolonged m -curve $\bar{j}^1 \gamma$ in $\hat{T}_m E$ then θ is horizontal over E , and is a contact 1-form. A similar result holds for contact 1-forms on $\hat{T}_m^2 E$.*

Proof. We show first that θ is horizontal over E , by showing that it is horizontal at each point $j_0^1 \gamma \in \hat{T}_m E$. Write θ in coordinates around such a point as

$$\theta = \theta_a du^a + \theta_a^i du_i^a;$$

then if γ is a representative m -curve for the velocity $j_0^1 \gamma$ we have

$$(\bar{j}^1 \gamma)^* \theta = (\theta_a \circ \bar{j}^1 \gamma) ((\bar{j}^1 \gamma)^* du^a) + (\theta_a^i \circ \bar{j}^1 \gamma) ((\bar{j}^1 \gamma)^* du_i^a).$$

But

$$\begin{aligned} (\bar{j}^1 \gamma)^* du^a \Big|_0 &= d(u^a \circ \bar{j}^1 \gamma) \Big|_0 = d\gamma^a \Big|_0 = \frac{\partial \gamma^a}{\partial t^j} \Big|_0 dt^j \Big|_0 \\ (\bar{j}^1 \gamma)^* du_i^a \Big|_0 &= d(u_i^a \circ \bar{j}^1 \gamma) \Big|_0 = d \left(\frac{\partial \gamma^a}{\partial t^i} \right) \Big|_0 = \frac{\partial \gamma^a}{\partial t^i} \frac{\partial t^j}{\partial t^i} \Big|_0 dt^j \Big|_0 \end{aligned}$$

so that

$$0 = (\bar{j}^1\gamma)^*\theta|_0 = \left((\theta_a \circ \bar{j}^1\gamma)(0) \frac{\partial\gamma^a}{\partial t^j} \Big|_0 + (\theta_a^i \circ \bar{j}^1\gamma)(0) \frac{\partial\gamma^a}{\partial t^i \partial t^j} \Big|_0 \right) dt^j|_0$$

and hence

$$\theta_a(j_0^1\gamma) \frac{\partial\gamma^a}{\partial t^j} \Big|_0 + \theta_a^i(j_0^1\gamma) \frac{\partial\gamma^a}{\partial t^i \partial t^j} \Big|_0 = 0.$$

Choosing a different representative m -curve $\hat{\gamma}$ of $j_0^1\gamma$ which differs in its second derivatives from γ (although necessarily having the same first derivatives) allows us to conclude that $\theta_a^i(j_0^1\gamma) = 0$, so that θ is horizontal at $j_0^1\gamma$ and hence is a horizontal 1-form. We also see from this argument that

$$\theta_a(j_0^1\gamma) \frac{\partial\gamma^a}{\partial t^j} \Big|_0 = 0.$$

Finally we observe that

$$\langle \theta, d_k \rangle = \left\langle \theta_a du^a, u_k^b \frac{\partial}{\partial u^b} \right\rangle = \theta_a u_k^a$$

so that

$$\langle \theta, d_k \rangle|_{j_0^1\gamma} = \theta_a(j_0^1\gamma) \frac{\partial\gamma^a}{\partial t^k} \Big|_0 = 0$$

for each point $j_0^1\gamma \in T_m E$, showing that $\langle \theta, d_k \rangle = 0$ and hence that θ is a contact 1-form.

The proof for forms on $\hat{T}_m^2 E$ is similar in principle but involves more complicated calculations. □

The coordinate expressions for contact 1-forms on velocity manifolds are quite different from those on jet manifolds, and involve determinants: indeed, contact 1-forms on $\hat{T}_m E$ are sums of scalar multiples of $(m+1) \times (m+1)$ determinants

$$\theta^{a_1 a_2 \dots a_{m+1}} = \begin{vmatrix} u_1^{a_1} & u_1^{a_2} & \dots & u_1^{a_{m+1}} \\ u_2^{a_1} & u_2^{a_2} & \dots & u_2^{a_{m+1}} \\ \vdots & \vdots & & \vdots \\ u_m^{a_1} & u_m^{a_2} & \dots & u_m^{a_{m+1}} \\ du^{a_1} & du^{a_2} & \dots & du^{a_{m+1}} \end{vmatrix}.$$

To see that such a determinant is indeed a contact 1-form, evaluate it on the total derivative $d_k = u_k^b \partial / \partial u^b$ to give

$$\langle \theta^{a_1 a_2 \dots a_{m+1}}, d_k \rangle = \begin{vmatrix} u_1^{a_1} & u_1^{a_2} & \dots & u_1^{a_{m+1}} \\ u_2^{a_1} & u_2^{a_2} & \dots & u_2^{a_{m+1}} \\ \vdots & \vdots & & \vdots \\ u_m^{a_1} & u_m^{a_2} & \dots & u_m^{a_{m+1}} \\ u_k^{a_1} & u_k^{a_2} & \dots & u_k^{a_{m+1}} \end{vmatrix} = 0.$$

To show that these forms span the local contact 1-forms, we show that their values at each point span the contact cotangent vectors at that point. Let the coordinate functions on the fibres of $T^*\mathring{T}_m E$ corresponding to the coordinates (u^a, u_i^a) on $\mathring{T}_m E$ be (p_a, p_a^i) ; then horizontal cotangent vectors satisfy the equations $p_a^i = 0$, and we have seen that the condition $\langle \theta, d_k \rangle = 0$ corresponds to a coordinate condition which may now be written as $u_k^a p_a = 0$.

Now observe that at each point $j_0^1 \gamma$ there is at least one set of m coordinates $(u_1^{a_1}, u_2^{a_2}, \dots, u_m^{a_m})$ such that the determinant $\det u_j^{a_i}$ does not vanish at $j_0^1 \gamma$; suppose, without loss of generality, that this set is $(u_1^1, u_2^2, \dots, u_m^m)$, for we may always rearrange the order of the base coordinates u^a if necessary. It is clear that the cotangent vectors

$$\theta_{j_0^1 \gamma}^{12 \dots m, m+1}, \theta_{j_0^1 \gamma}^{12 \dots m, m+2}, \dots, \theta_{j_0^1 \gamma}^{12 \dots m, n}$$

are linearly independent, so that the subspace of the space of contact cotangent vectors at $j_0^1 \gamma$ spanned by them has dimension $n - m$. But $\dim \tau_{mE}^*(T_{j_0^1 \gamma}^* E) = n$ and the m equations $u_k^a p_a$ characterising contact 1-forms are linearly independent for regular velocities, so that the dimension of the space of contact cotangent vectors at $j_0^1 \gamma$ is $n - m$.

3.4 Contact r -forms

We define contact r -forms using the pull-back condition, so that an r -form ω on $\mathring{T}_m E$ is a contact r -form if $(\mathring{j}^1 \gamma)^* \omega = 0$, and an r -form ω on $\mathring{T}_m^2 E$ is a contact r -form if $(\mathring{j}^2 \gamma)^* \omega = 0$. Note that contact r -forms need not be horizontal if $r > 1$.

We now see another important difference between contact forms on velocity manifolds and contact forms on jet manifolds. In the latter context, the contact r -forms are generated by the contact 1-forms and their exterior derivatives; but this is not the case on velocity manifolds. For example, on $\mathring{T}_2 \mathbb{R}^3$ the contact 1-forms are generated by the single 1-form

$$\theta = \begin{vmatrix} u_1^1 & u_1^2 & u_1^3 \\ u_2^1 & u_2^2 & u_2^3 \\ du^1 & du^2 & du^3 \end{vmatrix};$$

but $(u_1^1 du^2 - u_1^2 du^1) \wedge du_2^3 - (u_2^1 du^2 - u_2^2 du^1) \wedge du_1^3$ is a contact 2-form which cannot be written in terms of θ and $d\theta$.

3.5 Prolongations of maps

Let E_1, E_2 be manifolds, and let $f : E_1 \rightarrow E_2$ a map. The *prolongation* of f to $T_m E_1$ is the map

$$T_m f : T_m E_1 \rightarrow T_m E_2$$

defined by

$$T_m f(j_0^1 \gamma) = j_0^1(f \circ \gamma).$$

It is immediate from this definition that $T_m(f \circ g) = T_m f \circ T_m g$ and that $T_m(\text{id}_E) = \text{id}_{T_m E}$, so that T_m is a covariant functor. In coordinates,

$$u^a \circ T_m f = f^a, \quad u_i^a \circ T_m f = d_i f^a.$$

It is important to note that $T_m f$ might not restrict to a map $\overset{\circ}{T}_m E_1 \rightarrow \overset{\circ}{T}_m E_2$, because $f \circ \gamma$ might not be an immersion, even though γ is an immersion.

3.6 Prolongations and the exchange map

As a particular example, the prolongation of the vector bundle projection $\tau_{mE} : T_m E \rightarrow E$ to $T_m T_m E$ is

$$T_{m'} \tau_{mE} : T_{m'} T_m E \rightarrow T_{m'} E.$$

Lemma 9. *The exchange map $e : T_{m'} T_m E \rightarrow T_m T_{m'} E$ satisfies*

$$T_{m'} \tau_{mE} \circ e = \tau_{m(T_{m'} E)}.$$

Proof. From Lemma 2, e may be expressed in coordinates as

$$u^a \circ e = u^a, \quad u_i^a \circ e = u_{;i}^a, \quad u_{;j}^a \circ e = u_j^a, \quad u_{i;j}^a \circ e = u_{j;i}^a.$$

Thus

$$u^a \circ \tau_{m(T_{m'} E)} = u^a, \quad u_i^a \circ \tau_{m(T_{m'} E)} = u_i^a$$

whereas

$$u^a \circ T_{m'} \tau_{mE} \circ e = u^a \circ e = u^a, \quad u_i^a \circ T_{m'} \tau_{mE} \circ e = u_{;i}^a \circ e = u_i^a. \quad \square$$

In other words, the exchange map interchanges these two diagrams.

$$\begin{array}{ccc}
 T_{m'} T_m E & \xrightarrow{T_{m'} \tau_{mE}} & T_{m'} E \\
 \tau_{m'(T_m E)} \downarrow & & \downarrow \tau_{m' E} \\
 T_m E & \xrightarrow{\tau_{mE}} & E
 \end{array}
 \quad \xleftarrow{e} \quad
 \begin{array}{ccc}
 T_m T_{m'} E & \xrightarrow{\tau_{m(T_{m'} E)}} & T_{m'} E \\
 T_m \tau_{m' E} \downarrow & & \downarrow \tau_{m' E} \\
 T_m E & \xrightarrow{\tau_{mE}} & E
 \end{array}$$

3.7 Prolongations of vector fields

A vector field X on E is a map $E \rightarrow TE$, and so its prolongation (as a map) is $T_m X : T_m E \rightarrow T_m TE$.

Lemma 10. *The composition $X_m^1 = e \circ T_m X$, where $e : T_m TE \rightarrow TT_m E$ is the exchange map, is a vector field on $T_m E$*

Proof. From Lemma 9,

$$\begin{aligned}\tau_{T_mE} \circ e \circ T_m X &= T_m \tau_E \circ T_m X \\ &= T_m(\tau_E \circ X) \\ &= T_m(\text{id}_E) \\ &= \text{id}_{T_mE} .\end{aligned}$$

□

The vector field X_m^1 is called the *prolongation of X to T_mE* .

Proposition 5. *If ψ_s is the flow of X then $T_m\psi_s$ is the flow of X_m^1 .*

Proof. We first compute a coordinate formula for the vector field whose flow is $T_m\psi_s$.

Choose a point $j_0^1\gamma \in T_mE$ and let φ be the flow of X in a neighbourhood of $\gamma(0)$. Let (U, y) be a chart around $\gamma(0)$ so that, if

$$X = X^a \frac{\partial}{\partial u^a} ,$$

φ satisfies

$$\left. \frac{\partial \varphi^a}{\partial s} \right|_{(0, \cdot)} = X^a .$$

Let $\tilde{\varphi}$ denote the map $(s, q) \mapsto T_m\varphi_s(q)$, so that

$$\tilde{\varphi}^a = \varphi^a , \quad \tilde{\varphi}_i^a = d_i\varphi^a$$

where we define $(d_i\varphi^a)(s, q) = (d_i\varphi_s^a)(q)$. Then

$$\begin{aligned}\left. \frac{\partial \tilde{\varphi}_i^a}{\partial s} \right|_{(0, \cdot)} &= \left. \frac{\partial (d_i\varphi^a)}{\partial s} \right|_{(0, \cdot)} = \left. \frac{\partial}{\partial s} \right|_{(0, \cdot)} \left(u_i^b \frac{\partial \varphi^a}{\partial u^b} \right) \\ &= u_i^b \left. \frac{\partial \varphi^a}{\partial u^b \partial s} \right|_{(0, \cdot)} = d_i \left(\left. \frac{\partial \varphi^a}{\partial s} (0, \cdot) \right) \right|_{(0, \cdot)} = d_i X^a ,\end{aligned}$$

so that, in coordinates, the vector field whose flow is $T_m\psi_s$ is

$$X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u_i^a} .$$

On the other hand, regarding X as a map $E \rightarrow TE$, and writing \dot{u}^a as u_1^a ,

$$u^a \circ X = \dot{u}^a , \quad u_1^a \circ X = X^a$$

so that

$$\begin{aligned}u^a \circ T_m X &= \dot{u}^a , & u_1^a \circ T_m X &= X^a , \\ u_i^a \circ T_m X &= \dot{u}_i^a , & u_{1i}^a \circ T_m X &= d_i X^a ;\end{aligned}$$

thus

$$\begin{aligned} u^a \circ e \circ T_m X &= u^a, & u_{;1}^a \circ e \circ T_m X &= X^a, \\ u_i^a \circ e \circ T_m X &= u_i^a, & u_{i1}^a \circ e \circ T_m X &= d_i X^a \end{aligned}$$

so that

$$X_m^1 = e \circ T_m X = X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u_i^a}. \quad \square$$

Unlike prolongations of maps, prolongations of vector fields *do* restrict to $\mathring{T}_m E$.

3.8 Second prolongations

By extending the first order approach, maps $f : E_1 \rightarrow E_2$ may be prolonged to maps $T_m^2 f : T_m^2 E_1 \rightarrow T_m^2 E_2$, and vector fields X on E may be prolonged to vector fields X_m^2 on $T_m^2 E$. In coordinates,

$$u^a \circ T_m^2 f = f^a, \quad u_i^a \circ T_m^2 f = d_i f^a, \quad u_{ij}^a \circ T_m^2 f = d_i d_j f^a$$

and if $X = X^a \partial / \partial u^a$ then

$$X_m^2 = X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u_i^a} + \frac{1}{\#(ij)} (d_i d_j X^a) \frac{\partial}{\partial u_{ij}^a}.$$

The calculations are similar in principle to those given for the first order case, but more complicated in detail. Again $T_m^2 f$ might not restrict to a map $\mathring{T}_m^2 E_1 \rightarrow \mathring{T}_m^2 E_2$, whereas X_m^2 does restrict to $\mathring{T}_m^2 E$.

3.9 Prolongations, contact forms, and total derivatives

Let $f : E_1 \rightarrow E_2$ be a map. If θ is a contact form on $\mathring{T}_m E_2$ and if $T_m f$ restricts to $\mathring{T}_m E_1$ then $(T_m f)^* \theta$ is a contact form on $\mathring{T}_m E_1$, because

$$(\bar{j}^1 \gamma)^* (T_m f)^* \theta = (T_m f \circ \bar{j}^1 \gamma)^* \theta = (\bar{j}^1 (f \circ \gamma))^* \theta = 0.$$

If X is a vector field on E and θ is a contact form on $\mathring{T}_m E$ then the Lie derivative $\mathcal{L}_{X_m^1} \theta$ by the prolongation of X is also a contact form, because the flow of X_m^1 is the prolongation of the flow of X . These results, using the characterisation of a contact form by vanishing pullback, apply to both 1-forms and to r -forms with $r > 1$. They also hold for contact forms on $\mathring{T}_m^2 E$.

The corresponding result for total derivatives is more complicated, as these operators are vector fields along a map rather than on a manifold.

Lemma 11. *Prolongations and basis total derivatives commute, so that*

$$d_i \circ \mathcal{L}_X = \mathcal{L}_{X_m^1} \circ d_i, \quad d_i \circ \mathcal{L}_{X_m^1} = \mathcal{L}_{X_m^2} \circ d_i.$$

Proof. We check this using coordinates. In the first order case, if f is a function on E then

$$d_i(\mathcal{L}_X f) = d_i\left(X^a \frac{\partial f}{\partial u^a}\right) = u_i^b \left(\frac{\partial X^a}{\partial u^b} \frac{\partial f}{\partial u^a} + X^a \frac{\partial f}{\partial u^b \partial u^a}\right)$$

whereas

$$\mathcal{L}_{X_m^1}(d_i f) = \mathcal{L}_{X_m^1}\left(u_i^b \frac{\partial f}{\partial u^b}\right) = (d_i X^b) \frac{\partial f}{\partial u^b} + u_i^b X^a \frac{\partial f}{\partial u^b \partial u^a}.$$

A similar but slightly more lengthy calculation is used in the second order case. \square

3.10 Vertical endomorphisms

We have seen that $T_m E \rightarrow E$ is a vector bundle and so, as with every vector bundle, it has a canonical vertical lift operator. Denote the vertical lift to $(\eta_i) \in \bigoplus^m T E \cong T_m E$ by

$$T_{m|\tau_m(\eta_i)} E \rightarrow T_{(\eta_i)} T_m E, \quad (\xi_k) \mapsto (\xi_k)^{\uparrow(\eta_i)};$$

in coordinates this is

$$(\xi_k)^{\uparrow(\eta_i)} = u_j^a(\xi_k) \left. \frac{\partial}{\partial u_j^a} \right|_{(\eta_i)}.$$

For each vector $\zeta \in T_{(\eta_i)} T_m E$ define the vector $S^j \zeta \in T_{(\eta_i)} T_m E$ by

$$S^j \zeta = (0, \dots, 0, T\tau_m(\zeta), 0, \dots, 0)^{\uparrow(\eta_i)}$$

where the non-zero vector $T\tau_m(\zeta)$ is in the j -th position. It is evident that S^j is a vector bundle map $TT_m E \rightarrow TT_m E$, or alternatively a type $(1,1)$ tensor field on $T_m E$, called a *vertical endomorphism*. Note that the superscript j is a counting index, not a coordinate index. In coordinates

$$S^j = du^a \otimes \frac{\partial}{\partial u_j^a}.$$

There is a close relationship between vertical endomorphisms and total derivatives.

Lemma 12. *If ω is an r -form on E then*

$$S^j d_k \omega = r \delta_k^j (\tau_{mE}^* \omega).$$

Proof. Suppose first that θ is a 1-form; we shall give a proof in coordinates, omitting explicit mention of the pullback map. If $\theta = \theta_a du^a$ then

$$S^j d_k \theta = S^j ((d_k \theta_a) du^a + \theta_a du_k^a) = \delta_k^j \theta_a du^a = \delta_k^j \theta.$$

We now use induction on r . Suppose ω is an r -form and that $S^j d_k \omega = r \delta_k^j (\tau_{mE}^* \omega)$; then

$$\begin{aligned} S^j d_k (\theta \wedge \omega) &= S^j (d_k \theta \wedge \tau_{mE}^* \omega + \tau_{mE}^* \theta \wedge d_k \omega) \\ &= S^j d_k \theta \wedge \tau_{mE}^* \omega + \tau_{mE}^* \theta \wedge S^j d_k \omega \\ &= \delta_k^j (\tau_{mE}^* \theta \wedge \tau_{mE}^* \omega) + r \delta_k^j (\tau_{mE}^* \theta \wedge \tau_{mE}^* \omega) \\ &= (r+1) \delta_k^j \tau_{mE}^* (\theta \wedge \omega) \end{aligned}$$

using the fact that $\tau_{mE}^* \theta$ and $\tau_{mE}^* \omega$ are horizontal over E . The result now follows by linearity. \square

3.11 Second order vertical endomorphisms

There is also a version of the vertical endomorphism defined on second order velocity manifolds. This cannot be constructed in the same way as the first order vertical endomorphism, as $T_m^2 E \rightarrow T_m E$ is not a vector bundle but is instead an affine sub-bundle of $T_m T_m E \rightarrow T_m E$. We shall establish our construction by modifying the first-order vertical endomorphism on $T_m T_m E$. There is an alternative method, based on the construction of vertical lifts using double $(1, m)$ -curves, which may be used in both first and second order cases, but we shall not describe that here.

So let $\nu : T_m^2 E \rightarrow T_m E$ be some tubular neighbourhood of $T_m^2 E$ in $T_m T_m E$, and let $\iota : T_m^2 E \rightarrow T_m T_m E$ be the inclusion from Proposition 2. As before, let $e : T_m T_m E \rightarrow T_m T_m E$ be the exchange map.

Proposition 6. *Let θ be a 1-form on $T_m^2 E$; then the operation*

$$\theta \mapsto \iota^* (S^k \lrcorner (\nu^* \theta + e^* \nu^* \theta)),$$

where S^k is the vertical endomorphism on $T_m T_m E$, does not depend on the choice of tubular neighbourhood map ν and hence defines a vertical endomorphism on $T_m^2 E$.

Proof. We use coordinates to show that the result is independent of ν . Let $\theta = \theta_a du^a + \theta_a^i du_i^a + \theta_a^{ij} du_{ij}^a$; then

$$\begin{aligned} \nu^* \theta &= (\nu^* \theta_a) d\nu^a + (\nu^* \theta_a^i) d\nu_i^a + (\nu^* \theta_a^{ij}) d\nu_{ij}^a \\ &= (\nu^* \theta_a) \left(du^a + \frac{\partial \nu^a}{\partial u_p^c} (du_p^c - du_{;p}^c) + \frac{1}{2} \frac{\partial \nu^a}{\partial u_{p;q}^c} (du_{p;q}^c - du_{q;p}^c) \right) \\ &\quad + (\nu^* \theta_a^i) \left(\frac{1}{2} (du_i^a + du_{;i}^a) + \frac{1}{2} \left(\frac{\partial \nu_i^a}{\partial u_p^c} - \frac{\partial \nu_i^a}{\partial u_{;p}^c} \right) (du_p^c - du_{;p}^c) \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial \nu_i^a}{\partial u_{p;q}^c} (du_{p;q}^c - du_{q;p}^c) \right) \\ &\quad + (\nu^* \theta_a^{ij}) \left(\frac{1}{2} (du_{i;j}^a + du_{j;i}^a) + \frac{\partial \nu_{ij}^a}{\partial u_p^c} (du_p^c - du_{;p}^c) + \frac{1}{2} \frac{\partial \nu_{ij}^a}{\partial u_{p;q}^c} (du_{p;q}^c - du_{q;p}^c) \right) \end{aligned}$$

using the coordinate formulæ for the tubular neighbourhood map given in Section 2. Thus

$$\begin{aligned}
S^k \lrcorner \nu^* \theta &= (\nu^* \theta_a) \left(-\frac{\partial \nu^a}{\partial u_k^c} du^c + \frac{1}{2} \left(\frac{\partial \nu^a}{\partial u_{p;k}^c} - \frac{\partial \nu^a}{\partial u_{k;p}^c} \right) du_p^c \right) \\
&+ (\nu^* \theta_a^i) \left(\delta_i^k \frac{1}{2} du^a - \frac{1}{2} \left(\frac{\partial \nu_i^a}{\partial u_k^c} - \frac{\partial \nu_i^a}{\partial u_{i;k}^c} \right) du^c \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{\partial \nu_i^a}{\partial u_{p;k}^c} - \frac{\partial \nu_i^a}{\partial u_{k;p}^c} \right) du_p^c \right) \\
&+ (\nu^* \theta_a^{ij}) \left(\frac{1}{2} (\delta_j^k du_i^a + \delta_i^k du_j^a) - \frac{\partial \nu_{ij}^a}{\partial u_k^c} du^c \right. \\
&\quad \left. + \frac{1}{2} \left(\frac{\partial \nu_{ij}^a}{\partial u_{p;k}^c} - \frac{\partial \nu_{ij}^a}{\partial u_{k;p}^c} \right) du_p^c \right)
\end{aligned}$$

so that

$$\begin{aligned}
\iota^*(S^k \lrcorner \nu^* \theta) &= \theta_a \left(-\iota^* \left(\frac{\partial \nu^a}{\partial u_k^c} \right) du^c + \frac{1}{2} \iota^* \left(\frac{\partial \nu^a}{\partial u_{p;k}^c} - \frac{\partial \nu^a}{\partial u_{k;p}^c} \right) du_p^c \right) \\
&+ \theta_a^i \left(\delta_i^k \frac{1}{2} du^a - \frac{1}{2} \iota^* \left(\frac{\partial \nu_i^a}{\partial u_k^c} - \frac{\partial \nu_i^a}{\partial u_{i;k}^c} \right) du^c \right. \\
&\quad \left. + \frac{1}{2} \iota^* \left(\frac{\partial \nu_i^a}{\partial u_{p;k}^c} - \frac{\partial \nu_i^a}{\partial u_{k;p}^c} \right) du_p^c \right) \\
&+ \theta_a^{ij} \left(\frac{1}{2} (\delta_j^k du_i^a + \delta_i^k du_j^a) - \iota^* \left(\frac{\partial \nu_{ij}^a}{\partial u_k^c} \right) du^c \right. \\
&\quad \left. + \frac{1}{2} \iota^* \left(\frac{\partial \nu_{ij}^a}{\partial u_{p;k}^c} - \frac{\partial \nu_{ij}^a}{\partial u_{k;p}^c} \right) du_p^c \right);
\end{aligned}$$

and similarly

$$\begin{aligned}
S^k \lrcorner e^* \nu^* \theta &= (e^* \nu^* \theta_a) \left(e^* \left(\frac{\partial \nu^a}{\partial u_k^c} \right) du^c + \frac{1}{2} e^* \left(\frac{\partial \nu^a}{\partial u_{k;p}^c} - \frac{\partial \nu^a}{\partial u_{p;k}^c} \right) du_p^c \right) \\
&+ (e^* \nu^* \theta_a^i) \left(\frac{1}{2} \delta_i^k du^a + \frac{1}{2} e^* \left(\frac{\partial \nu_i^a}{\partial u_k^c} - \frac{\partial \nu_i^a}{\partial u_{i;k}^c} \right) du^c \right. \\
&\quad \left. + \frac{1}{2} e^* \left(\frac{\partial \nu_i^a}{\partial u_{k;p}^c} - \frac{\partial \nu_i^a}{\partial u_{p;k}^c} \right) du_p^c \right) \\
&+ (e^* \nu^* \theta_a^{ij}) \left(\frac{1}{2} (\delta_j^k du_i^a + \delta_i^k du_j^a) + e^* \left(\frac{\partial \nu_{ij}^a}{\partial u_k^c} \right) du^c \right. \\
&\quad \left. + \frac{1}{2} e^* \left(\frac{\partial \nu_{ij}^a}{\partial u_{k;p}^c} - \frac{\partial \nu_{ij}^a}{\partial u_{p;k}^c} \right) du_p^c \right)
\end{aligned}$$

so that, using $\iota^* e^* = \iota^*$,

$$\begin{aligned} \iota^*(S^k \lrcorner e^* \nu^* \theta) &= \theta_a \left(\iota^* \left(\frac{\partial \nu^a}{\partial u_k^c} \right) du^c + \frac{1}{2} \iota^* \left(\frac{\partial \nu^a}{\partial u_{k;p}^c} - \frac{\partial \nu^a}{\partial u_{p;k}^c} \right) du_p^c \right) \\ &\quad + \theta_a^i \left(\frac{1}{2} \delta_i^k du^a + \frac{1}{2} \iota^* \left(\frac{\partial \nu_i^a}{\partial u_k^c} - \frac{\partial \nu_i^a}{\partial u_{;k}^c} \right) du^c \right) \\ &\quad + \frac{1}{2} \iota^* \left(\frac{\partial \nu_i^a}{\partial u_{k;p}^c} - \frac{\partial \nu_i^a}{\partial u_{p;k}^c} \right) du_p^c \\ &\quad + \theta_a^{ij} \left(\frac{1}{2} (\delta_j^k du_i^a + \delta_i^k du_j^a) + \iota^* \left(\frac{\partial \nu_{ij}^a}{\partial u_k^c} \right) du^c \right) \\ &\quad + \frac{1}{2} \iota^* \left(\frac{\partial \nu_{ij}^a}{\partial u_{k;p}^c} - \frac{\partial \nu_{ij}^a}{\partial u_{p;k}^c} \right) du_p^c. \end{aligned}$$

Thus, adding, we obtain

$$\iota^*(S^k \lrcorner (\nu^* \theta + e^* \nu^* \theta)) = \theta_a^k du^a + 2\theta_a^{ik} du_i^a$$

using $\theta_a^{ki} = \theta_a^{ik}$. □

In coordinates, therefore, the second order vertical endomorphisms may be written as tensor fields

$$S^k = du^a \otimes \frac{\partial}{\partial u_k^a} + \frac{2}{\#(ik)} du_i^a \otimes \frac{\partial}{\partial u_{ik}^a};$$

the factor $1/\#(ik)$ arises here because the contraction of $\partial/\partial u_{ik}^a$ with du_{pq}^c equals $\frac{1}{2} \#(ik) \delta_a^c (\delta_p^i \delta_q^k + \delta_q^i \delta_p^k)$, so that

$$\frac{\partial}{\partial u_{ik}^a} \lrcorner (\theta_c^{pq} du_{pq}^c) = \frac{\#(ik)}{2} \delta_a^c (\delta_p^i \delta_q^k + \delta_q^i \delta_p^k) \theta_c^{pq} = \#(ik) \theta_a^{ik}.$$

The relationship given in Lemma 12 between vertical endomorphisms and total derivatives may now be extended to a kind of homotopy formula.

Lemma 13. *If ω is an r -form on $T_m E$ then*

$$S^j d_k \omega - d_k S^j \omega = r \delta_k^j (\tau_{mE}^{2,1} \omega).$$

Proof. Suppose first that θ is a 1-form; we shall give a proof in coordinates, omitting explicit mention of the pullback map. If $\theta = \theta_a du^a + \theta_a^i du_i^a$ then

$$d_k \theta = (d_k \theta_a) du^a + \theta_a du_k^a + (d_k \theta_a^i) du_i^a + \theta_a^i du_{ik}^a$$

so that

$$S^j d_k \theta = (\delta_k^j \theta_a + (d_k \theta_a^j)) du^a + \delta_k^j \theta_a^i du_i^a + \theta_a^j du_k^a.$$

On the other hand, $S^j \theta = \theta_a^j du^a$, so that

$$d_k S^j \theta = (d_k \theta_a^j) du^a + \theta_a^j du_k^a$$

and hence

$$S^j d_k \theta - d_k S^j \theta = \delta_k^j \theta_a du^a + \delta_k^j \theta_a^i du_i^a = \delta_k^j \theta.$$

We now use induction on r . Suppose ω is an r -form and that $S^j d_k \omega - d_k S^j \omega = r \delta_k^j (\tau_{mE}^{2,1*} \omega)$; then, as both S^j and d_k are derivations of degree zero, their commutator is a derivation of degree zero, and so

$$\begin{aligned} (S^j d_k - d_k S^j)(\theta \wedge \omega) &= (S^j d_k - d_k S^j)\theta \wedge \tau_{mE}^{2,1*} \omega + \tau_{mE}^{2,1*} \theta \wedge (S^j d_k - d_k S^j)\omega \\ &= r \delta_k^j \tau_{mE}^{2,1*} \theta \wedge \tau_{mE}^{2,1*} \omega + \delta_k^j \tau_{mE}^{2,1*} \theta \wedge \tau_{mE}^{2,1*} \omega \\ &= (r+1) \delta_k^j \tau_{mE}^{2,1*} (\theta \wedge \omega). \end{aligned}$$

The result now follows by linearity. \square

4 Vector forms

We often use *vectors* of operators, tensors, forms, and so on. For instance, we have defined the total derivatives d_k and the vertical endomorphisms S^j , where j and k are counting indices rather than coordinate indices. These operators fit into a framework of *vector forms*, to which we can associate a cohomology theory. Although the full cohomology theory requires the use of higher-order velocity manifolds, we can see some aspects of the theory in the first and second order cases.

4.1 Vector forms

We consider differential forms on E , $\hat{T}_m E$ and $\hat{T}_m^2 E$ taking values in the vector space \mathbb{R}^{m*} and its exterior powers. Write $\hat{T}_m^k E$ with $k = 0, 1, 2$ and put

$$\Omega_k^{r,s} = \left(\Omega^r \hat{T}_m^k E \right) \otimes (\wedge^s \mathbb{R}^{m*}).$$

Then a typical element of $\Omega_k^{r,s}$ is

$$\Xi = \chi_{i_1 \dots i_s} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_s} \in \Omega_k^{r,s}$$

where the scalar forms $\chi_{i_1 \dots i_s}$ are skew-symmetric in their indices, and where, as in Corollary 1, $\{dt^i\}$ is the canonical basis of \mathbb{R}^{m*} . It is clear that $\Omega_k^{r,s}$ is a module over the algebra of functions on $\hat{T}_m^k E$.

4.2 Operations on vector forms

Define the operators d and d_T on the modules of vector forms by their actions on decomposable forms,

$$\begin{aligned} d : \Omega_k^{r,s} &\rightarrow \Omega_k^{r+1,s}, & d(\chi \otimes \omega) &= d\chi \otimes \omega \\ d_T : \Omega_k^{r,s} &\rightarrow \Omega_{k+1}^{r,s+1}, & d_T(\chi \otimes \omega) &= d_i \chi \otimes (dt^i \wedge \omega), \end{aligned}$$

so that

$$\begin{aligned} dd_T(\chi \otimes \omega) &= d(d_i \chi \otimes (dt^i \wedge \omega)) = dd_i \chi \otimes (dt^i \wedge \omega) \\ &= d_i d\chi \otimes (dt^i \wedge \omega) = d_T(d\chi \otimes \omega) = d_T d(\chi \otimes \omega) \end{aligned}$$

and

$$d_{\mathbb{T}}^2(\chi \otimes \omega) = d_j d_i \chi \otimes (dt^j \wedge dt^i \wedge \omega) = 0,$$

showing that $dd_{\mathbb{T}} = d_{\mathbb{T}}d$ and $d_{\mathbb{T}}^2 = 0$. We say that $d\Xi$ is the *differential* of the vector form Ξ , and that $d_{\mathbb{T}}\Xi$ is its *total derivative*.

The total derivative of a vector form is a type of Lie derivative, and so we can also define the corresponding contraction operation. Put

$$i_{\mathbb{T}} : \Omega_k^{r,s} \rightarrow \Omega_{k+1}^{r-1,s+1}, \quad i_{\mathbb{T}}(\chi \otimes \omega) = (d_i \lrcorner \chi) \otimes dt^i \wedge \omega$$

where $d_i \lrcorner \chi$ denotes the contraction of the ‘vector field along a map’ d_i with the scalar form χ , so that

$$d_{\mathbb{T}} = di_{\mathbb{T}} + i_{\mathbb{T}}d.$$

4.3 Equivariant vector forms

Let $\alpha_{j_0^1 \phi} : \mathring{T}_m E \rightarrow \mathring{T}_m E$ denote the right action of $j_0^1 \phi \in L_m^{1+}$ on $\mathring{T}_m E$ by

$$\alpha_{j_0^1 \phi}(j_0^1 \gamma) = j_0^1(\gamma \circ \phi);$$

also, let $A_{j_0^1 \phi} : \mathbb{R}^{m*} \rightarrow \mathbb{R}^{m*}$ denote the linear map

$$A_{j_0^1 \phi}(dt^i) = (D_j \phi^i(0)) dt^j,$$

and extend this by multilinearity to $A_{j_0^1 \phi} : \bigwedge^s \mathbb{R}^{m*} \rightarrow \bigwedge^s \mathbb{R}^{m*}$. The vector form $\chi_{i_1 \dots i_s} \otimes (dt^{i_1} \wedge \dots \wedge dt^{i_s}) \in \Omega_1^{r,s}$ is said to be *equivariant* if, for every $j_0^1 \phi$,

$$\alpha_{j_0^1 \phi}^*(\chi_{i_1 \dots i_s}) \otimes (dt^{i_1} \wedge \dots \wedge dt^{i_s}) = \chi_{i_1 \dots i_s} \otimes A_{j_0^1 \phi}(dt^{i_1} \wedge \dots \wedge dt^{i_s}).$$

Thus an equivariant form, regarded as a map from objects defined on a velocity manifold to elements of a vector space, commutes with the action of the jet group on the manifold and the vector space. We use the *oriented* jet group in our definition, as our application will be to problems in the calculus of variations where we need to integrate the forms.

We shall be particularly interested in equivariant elements of $\Omega_1^{0,m}$, namely 0-forms (functions) taking their values in the one-dimensional vector space $\bigwedge^m \mathbb{R}^{m*}$. Then

$$A_{j_0^1 \phi}(dt^1 \wedge \dots \wedge dt^m) = \mathcal{J}\phi(0)(dt^1 \wedge \dots \wedge dt^m)$$

where $\mathcal{J}\phi = \det(D_j \phi^i)$ is the Jacobian of ϕ , and so, writing $d^m t$ for $dt^1 \wedge \dots \wedge dt^m$, an element $\Lambda = L d^m t$ is equivariant when

$$(L \circ \alpha_{j_0^1 \phi})d^m t = \det(D_j \phi^i(0))L d^m t.$$

Thus, writing an element of $T_m E \cong \bigoplus^m TE$ as (ξ_1, \dots, ξ_m) , Λ is equivariant when for each matrix $A \in \text{GL}^+(m, \mathbb{R})$,

$$L(\xi_i A_j^i) = (\det A)L(\xi_j).$$

As the oriented jet group L_m^{1+} is connected, there is an infinitesimal condition for equivariance. For a vector form $\chi_{i_1 \dots i_s} \otimes (dt^{i_1} \wedge \dots \wedge dt^{i_s}) \in \Omega_1^{r,s}$, we require

$$d_i^j(\chi_{i_1 \dots i_s}) \otimes (dt^{i_1} \wedge \dots \wedge dt^{i_s}) = \chi_{i_1 \dots i_s} \otimes \mathcal{L}_{t^j \partial / \partial t^i} (dt^{i_1} \wedge \dots \wedge dt^{i_s})$$

In the particular case where $s = m$ we have $\mathcal{L}_{t^j \partial / \partial t^i} d^m t = \delta_i^j d^m t$, so the condition simplifies to

$$d_i^j \chi = \delta_i^j \chi.$$

4.4 The bicomplex

It is clear that for $-1 \leq s \leq m-2$ we can use the operators d and d_T to construct a bicomplex:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overline{\Omega}_0^{0,s} & \xrightarrow{d} & \Omega_0^{1,s} & \xrightarrow{d} & \Omega_0^{2,s} & \xrightarrow{d} & \Omega_0^{3,s} & \longrightarrow \\ & & d_T \downarrow & & d_T \downarrow & & d_T \downarrow & & d_T \downarrow & \\ 0 & \longrightarrow & \overline{\Omega}_1^{0,s+1} & \xrightarrow{d} & \Omega_1^{1,s+1} & \xrightarrow{d} & \Omega_1^{2,s+1} & \xrightarrow{d} & \Omega_1^{3,s+1} & \longrightarrow \\ & & d_T \downarrow & & d_T \downarrow & & d_T \downarrow & & d_T \downarrow & \\ 0 & \longrightarrow & \overline{\Omega}_2^{0,s+2} & \xrightarrow{d} & \Omega_2^{1,s+2} & \xrightarrow{d} & \Omega_2^{2,s+2} & \xrightarrow{d} & \Omega_2^{3,s+2} & \longrightarrow \end{array}$$

where if $s = -1$ then $\Omega_*^{*,s} = 0$. In this bicomplex $\overline{\Omega}_*^{0,*}$ means ‘modulo constant functions’, and is used instead of the usual beginning $0 \rightarrow \mathbb{R} \rightarrow \Omega^0 \rightarrow \dots$ of the de Rham sequence.

An important property of the bicomplex is that all columns (apart from the first) are globally exact, we show this by obtaining a homotopy formula for d_T . Strictly speaking the homotopy formula involves *third order* forms which are horizontal over E , because the operator P_2 defined in the statement of the theorem involves applying a total derivative to (scalar) second-order forms which are horizontal over E ; but if $d_T \Xi = 0$ then the operator P_2 is not involved and the formula is genuinely second order. We feel, nevertheless, that it is worthwhile giving the more general statement, on the understanding that the definition of the total derivative of a second order form, and the consequent generalisation of Lemma 13, follow exactly the same pattern as before. We also use the operator P_2 when studying equivalents of first-order Lagrangians, although in that context the image of P_2 is always second-order rather than third-order.

Theorem 1. *If $\Xi \in \Omega_1^{r,s+1}$ with $r > 0$ then, to within a pullback,*

$$P_2 d_T \Xi + d_T P_1 \Xi = \Xi,$$

where

$$P_1(\chi_{i_1 \dots i_{s+1}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}}) = \frac{1}{r(m-s)} S^j \chi_{i_1 \dots i_{s+1}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \right)$$

for first-order r -forms $\chi_{i_1 \dots i_{s+1}}$, and

$$\begin{aligned} & P_2(\eta_{i_1 \dots i_{s+2}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{s+2}}) \\ &= \left(\frac{1}{r(m-s-1)} S^j \eta_{i_1 \dots i_{s+2}} - \frac{1}{r^2(m-s)(m-s-1)} d_l S^l S^j \eta_{i_1 \dots i_{s+2}} \right) \otimes \\ & \quad \otimes \left(\frac{\partial}{\partial t^j} \lrcorner dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \right) \end{aligned}$$

for second-order r -forms $\eta_{i_1 \dots i_{s+2}}$.

Proof. This is a consequence of Lemma 13. Put

$$\begin{aligned} P_1^j &= \frac{1}{r(m-s)} S^j \\ P_2^j &= \frac{1}{r(m-s-1)} S^j - \frac{1}{r^2(m-s)(m-s-1)} d_l S^l S^j; \end{aligned}$$

then

$$\begin{aligned} P_2 d_T \Xi &= P_2(d_k \chi_{i_1 \dots i_{s+1}} \otimes dt^k \wedge dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}}) \\ &= P_2^j d_k \chi_{i_1 \dots i_{s+1}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner dt^k \wedge dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \right) \\ &= P_2^j d_k \chi_{i_1 \dots i_{s+1}} \otimes (\delta_j^k dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}}) \\ & \quad - P_2^j d_k \chi_{i_1 \dots i_{s+1}} \otimes dt^k \wedge \left(\frac{\partial}{\partial t^j} \lrcorner dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \right) \\ &= P_2^j d_k \chi_{i_1 \dots i_{s+1}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \\ & \quad - (s+1) P_2^j d_{i_1} \chi_{j i_2 \dots i_{s+1}} \otimes dt^{i_1} \wedge dt^{i_2} \wedge \dots \wedge dt^{i_{s+1}} \end{aligned}$$

whereas

$$\begin{aligned} d_T P_1 \Xi &= d_T \left(P_1^j \chi_{i_1 \dots i_{s+1}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \right) \right) \\ &= d_k P_1^j \chi_{i_1 \dots i_{s+1}} \otimes dt^k \wedge \left(\frac{\partial}{\partial t^j} \lrcorner dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \right) \\ &= (s+1) d_{i_1} P_1^j \chi_{j i_2 \dots i_{s+1}} \otimes dt^{i_1} \wedge dt^{i_2} \wedge \dots \wedge dt^{i_{s+1}} \end{aligned}$$

so that

$$\begin{aligned} P_2 d_T \Xi + d_T P_1 \Xi &= P_2^k d_k \chi_{i_1 \dots i_{s+1}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \\ & \quad - (s+1) P_2^j d_{i_1} \chi_{j i_2 \dots i_{s+1}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}} \\ & \quad + (s+1) d_{i_1} P_1^j \chi_{j i_2 \dots i_{s+1}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{s+1}}. \end{aligned}$$

But, using Lemma 13, the operators acting on $\chi_{j i_2 \dots i_{s+1}}$ satisfy

$$\begin{aligned} \delta_{i_1}^j P_2^k d_k &= \frac{1}{r(m-s-1)} \delta_{i_1}^j S^k d_k - \frac{1}{r^2(m-s)(m-s-1)} \delta_{i_1}^j d_l S^l S^k d_k \\ &= \frac{1}{r(m-s-1)} \delta_{i_1}^j (d_k S^k + mr) \\ &\quad - \frac{1}{r^2(m-s)(m-s-1)} \delta_{i_1}^j (d_l d_k S^l S^k + (m+1) r d_l S^l) \\ &= \frac{m}{m-s-1} \delta_{i_1}^j - \frac{s+1}{r(m-s)(m-s-1)} \delta_{i_1}^j d_k S^k, \end{aligned}$$

using the fact that $S^l S^k \chi_{i_1 \dots i_{s+1}} = 0$ because the $\chi_{i_1 \dots i_{s+1}}$ are first-order forms. Similarly

$$\begin{aligned} -(s+1) P_2^j d_{i_1} &= -\frac{s+1}{r(m-s-1)} S^j d_{i_1} + \frac{s+1}{r^2(m-s)(m-s-1)} d_l S^l S^j d_{i_1} \\ &= -\frac{s+1}{r(m-s-1)} (d_{i_1} S^j + r \delta_{i_1}^j) \\ &\quad + \frac{s+1}{r^2(m-s)(m-s-1)} (d_l d_{i_1} S^l S^j + r d_{i_1} S^j + r \delta_{i_1}^j d_l S^l) \\ &= -\frac{s+1}{(m-s-1)} \delta_{i_1}^j - \frac{(s+1)}{r(m-s)} d_{i_1} S^j \\ &\quad + \frac{s+1}{r(m-s)(m-s-1)} \delta_{i_1}^j d_k S^k \end{aligned}$$

and

$$(s+1) d_{i_1} P_1^j = \frac{s+1}{r(m-s)} d_{i_1} S^j,$$

from which we see that

$$\delta_{i_1}^j P_2^k d_k - (s+1) P_2^j d_{i_1} + (s+1) d_{i_1} P_1^j = \delta_{i_1}^j$$

and the result follows. \square

4.5 The bottom left corner

The part of the bicomplex which holds the major interest for the calculus of variations is in the bottom left-hand corner; we shall repeat it, with a pull-back map shown explicitly where appropriate.

$$\begin{array}{ccccc} & & & & \Omega_1^{1,m-1} \\ & & & \nearrow S & \downarrow d_T \\ \overline{\Omega}_1^{0,m} & \xrightarrow{d} & \Omega_1^{1,m} & \xrightarrow{\tau_{mE}^{2,1*}} & \Omega_2^{1,m} \end{array}$$

Take $[\Lambda] \in \overline{\Omega}_1^{0,m}$, so that, for some function L on $\mathring{T}_m E$, we have for any representative

$$\Lambda = L dt^1 \wedge \cdots \wedge dt^m = L d^m t.$$

Here, L will play the role of a (first order) Lagrangian function in the calculus of variations, and the vector-valued function Λ will have the capability of being integrated along m -curves in $\mathring{T}_m E$ (and, in particular, along prolongations to $\mathring{T}_m E$ of m -curves in E). So, given the equivalence class $[\Lambda]$, define

$$\Theta_1 = Sd\Lambda, \quad \mathcal{E}_0 = \tau_{mE}^{2,1*} d\Lambda - d_T \Theta_1,$$

where the choice of representative in the equivalence class is immaterial as we consider only $d\Lambda$ in the definition. We may compute Θ_1 and \mathcal{E}_0 in coordinates; they are

$$\begin{aligned} \Theta_1 &= S^j \left(\frac{\partial L}{\partial u^a} du^a + \frac{\partial L}{\partial u_i^a} du_i^a \right) \otimes \left(\frac{\partial}{\partial t^j} \lrcorner (dt^1 \wedge \cdots \wedge dt^m) \right) \\ &= \left(\frac{\partial L}{\partial u_j^a} du^a \right) \otimes \left(\frac{\partial}{\partial t^j} \lrcorner (dt^1 \wedge \cdots \wedge dt^m) \right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_0 &= \left(\frac{\partial L}{\partial u^a} du^a + \frac{\partial L}{\partial u_i^a} du_i^a \right) \otimes (dt^1 \wedge \cdots \wedge dt^m) \\ &\quad - d_k \left(\frac{\partial L}{\partial u_j^a} du^a \right) \otimes dt^k \wedge \left(\frac{\partial}{\partial t^j} \lrcorner (dt^1 \wedge \cdots \wedge dt^m) \right) \\ &= \left(\frac{\partial L}{\partial u^a} - d_k \left(\frac{\partial L}{\partial u_k^a} \right) \right) du^a \otimes (dt^1 \wedge \cdots \wedge dt^m). \end{aligned}$$

5 Variational problems

Our main application of the theory of vector forms, and their associated cohomology, will be to problems in the calculus of variations. These will be *parametric problems*: that is, problems where the solutions are submanifolds without a given parametrization (although with a particular orientation). In the one-dimensional case, as exemplified by Finsler geometry, all the vector forms are essentially scalar forms, and so this theory only provides further insight in the case where the submanifolds have dimension two or more.

5.1 Homogeneous variational problems

We now study m -dimensional variational problems on E , with fixed boundary conditions. As before, a vector function $\Lambda = L d^m t \in \overline{\Omega}_1^{0,m}$ will be called a *Lagrangian* for a variational problem. It will be called *homogeneous* if it is equivariant with respect to the action of the oriented jet group L_m^{1+} . Thus Λ is homogeneous when the scalar function L satisfies the infinitesimal condition

$$d_j^i L = \delta_j^i L$$

or, equivalently, the finite condition

$$L \circ \alpha_{j_0^1 \phi} = (\det D_j \phi^i(0))L$$

for every every $j_0^1 \phi \in L_m^{1+}$.

We now consider submanifolds of E of the form $\gamma(C)$ where $\gamma : \mathbb{R}^m \rightarrow E$ is an immersion and $C \subset \mathbb{R}^m$ is a connected compact m -dimensional submanifold with boundary ∂C . The *fixed-boundary variational problem* defined by Λ is the search for extremal submanifolds $\gamma(C) \subset E$ satisfying

$$\int_C ((\bar{j}^1 \gamma)^* \mathcal{L}_{X_m^1} L) d^m t = 0$$

for every variation field X on E satisfying $X|_{\gamma(\partial C)} = 0$.

Theorem 2. *If Λ is homogeneous and $\gamma(C)$ is an extremal submanifold then $\gamma \circ \phi$ is also an extremal submanifold, for any orientation-preserving reparametrization ϕ whose image contains C .*

Proof. We shall show that if Λ is homogeneous then, for any immersion γ ,

$$\int_{\phi^{-1}(C)} (L \circ \bar{j}^1(\gamma \circ \phi)) d^m t = \int_C (L \circ \bar{j}^1 \gamma) d^m t$$

so that the integral itself is invariant under reparametrization; hence extremals will be invariant under reparametrization. As

$$\begin{aligned} \int_{\phi^{-1}(C)} (L \circ \bar{j}^1(\gamma \circ \phi)) d^m t &= \int_C (\phi^{-1})^* ((L \circ \bar{j}^1(\gamma \circ \phi)) d^m t) \\ &= \int_C (L \circ \bar{j}^1(\gamma \circ \phi) \circ \phi^{-1}) (\phi^{-1})^* d^m t, \end{aligned}$$

it will be sufficient to show that

$$(L \circ \bar{j}^1(\gamma \circ \phi) \circ \phi^{-1}) (\phi^{-1})^* d^m t = (L \circ \bar{j}^1 \gamma) d^m t.$$

Now for any $s \in \mathbb{R}^m$

$$d^m t|_s = (\mathcal{J} \phi \circ \phi^{-1})(s) (\phi^{-1})^* d^m t|_s,$$

and so it will be sufficient to show that, for each s ,

$$(L \circ \bar{j}^1(\gamma \circ \phi) \circ \phi^{-1})(s) = (L \circ \bar{j}^1 \gamma)(s) (\mathcal{J} \phi \circ \phi^{-1})(s).$$

Note that we do not require the diffeomorphism ϕ to satisfy the condition $\phi(0) = 0$.

To see how this can be obtained from the homogeneity condition, write the latter as

$$L \circ \alpha_{j_0^1 \phi} = (\mathcal{J} \phi)(0)L$$

where φ is a diffeomorphism which *does* satisfy $\varphi(0) = 0$; then, for any immersion $\gamma : \mathbb{R}^m \rightarrow E$,

$$\begin{aligned} (\mathcal{J}\varphi)(0)L(j_0^1(\gamma \circ T_s)) &= L(\alpha_{j_0^1\varphi}(j_0^1(\gamma \circ T_s))) \\ &= L(j_0^1(\gamma \circ T_s \circ \varphi)). \end{aligned}$$

Now put $\varphi = T_{-s} \circ \phi \circ T_{\phi^{-1}(s)}$, and note that $\varphi(0) = 0$; also

$$(\gamma \circ T_s) \circ \varphi = \gamma \circ \phi \circ T_{\phi^{-1}(s)}$$

and

$$(\mathcal{J}\varphi)(0) = (\mathcal{J}\phi)(\phi^{-1}(s)),$$

so that

$$(\mathcal{J}\phi)(\phi^{-1}(s))L(j_0^1(\gamma \circ T_s)) = L(j_0^1(\gamma \circ \phi \circ T_{\phi^{-1}(s)}))$$

and hence

$$(\mathcal{J}\phi)(\phi^{-1}(s))L(\bar{j}^1\gamma(s)) = L(\bar{j}^1(\gamma \circ \phi) \circ \phi^{-1}(s)). \quad \square$$

5.2 Equivalents of Lagrangians

Let $\Lambda \in \Omega_1^{0,m}$ be a homogeneous Lagrangian. Any scalar m -form $\Theta_m \in \Omega_1^{m,0}$ which is horizontal over E will be called an *integral equivalent* of Λ if

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!} \right) i_T^m \Theta_m;$$

any vector r -form $\Theta_r \in \Omega_1^{r,m-r}$ which is horizontal over E will be called an *intermediate equivalent* if

$$\Lambda = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!} i_T^r \Theta_r \quad 0 \leq r \leq m-1.$$

Lemma 14. *If Θ_{r+1} is an equivalent of Λ then*

$$\Theta_r = \frac{(-1)^r}{m-r} i_T \Theta_{r+1}$$

is also an equivalent.

Proof. If Θ_{r+1} is an equivalent of Λ then by definition

$$\Lambda = \frac{(-1)^{r(r+1)/2}(m-r-1)!}{m!} i_T^{r+1} \Theta_{r+1},$$

so that

$$\frac{(-1)^{r(r-1)/2}(m-r)!}{m!} i_T^r \Theta_r = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!} i_T^r \left(\frac{(-1)^r}{m-r} i_T \Theta_{r+1} \right) = \Lambda. \quad \square$$

In the case $r = m$ we use the term ‘integral equivalent’ for the following reason.

Lemma 15. *If γ is an m -curve in E then $(\bar{j}^1\gamma)^*\Lambda = (\bar{j}^1\gamma)^*\Theta_m$, so that*

$$\int_C (\bar{j}^1\gamma)^*\Lambda = \int_C (\bar{j}^1\gamma)^*\Theta_m.$$

It follows that $\Lambda = \Theta_0$ and Θ_m have the same extremals.

Proof. Suppose $\Theta \in \Omega^{r,m-r}$ may be written in coordinates in the particular form

$$\Theta = \Theta_{a_1 \dots a_m} u_{k_{r+1}}^{a_{r+1}} \dots u_{k_m}^{a_m} du^{a_1} \wedge \dots \wedge du^{a_r} \otimes dt^{k_{r+1}} \wedge \dots \wedge dt^{k_m}$$

where the functions $\Theta_{a_1 \dots a_m}$ are skew-symmetric in their indices; then

$$\begin{aligned} i_T \Theta &= \Theta_{a_1 \dots a_m} u_{k_{r+1}}^{a_{r+1}} \dots u_{k_m}^{a_m} \left(u_{k_r}^b \frac{\partial}{\partial u^b} \lrcorner du^{a_1} \wedge \dots \wedge du^{a_r} \right) \otimes \\ &\quad \otimes dt^{k_r} \wedge dt^{k_{r+1}} \wedge \dots \wedge dt^{k_m} \\ &= \sum_{p=1}^r (-1)^{p-1} \Theta_{a_1 \dots a_m} u_{k_{r+1}}^{a_{r+1}} \dots u_{k_m}^{a_m} \left(u_{k_r}^{a_p} du^{a_1} \wedge \dots \wedge \widehat{du^{a_p}} \dots \wedge du^{a_r} \right) \otimes \\ &\quad \otimes dt^{k_r} \wedge dt^{k_{r+1}} \wedge \dots \wedge dt^{k_m} \\ &= r(-1)^{r-1} \Theta_{a_1 \dots a_m} u_{k_r}^{a_r} u_{k_{r+1}}^{a_{r+1}} \dots u_{k_m}^{a_m} du^{a_1} \wedge \dots \wedge du^{a_{r-1}} \otimes \\ &\quad \otimes dt^{k_r} \wedge dt^{k_{r+1}} \wedge \dots \wedge dt^{k_m}. \end{aligned}$$

Thus if $\Theta \in \Omega^{m,0}$ we see that

$$\begin{aligned} i_T^m \Theta &= m!(-1)^{m(m-1)/2} \Theta_{a_1 \dots a_m} u_{k_1}^{a_1} \dots u_{k_m}^{a_m} dt^{k_1} \wedge \dots \wedge dt^{k_m} \\ &= m!(-1)^{m(m-1)/2} \Theta_{a_1 \dots a_m} \det(u_{k_j}^{a_i}) dt^1 \wedge \dots \wedge dt^m \end{aligned}$$

so that

$$(\bar{j}^1\gamma)^* i_T^m \Theta = m!(-1)^{m(m-1)/2} (\Theta_{a_1 \dots a_m} \circ \bar{j}^1\gamma) \det\left(\frac{\partial \gamma^{a_i}}{\partial t^{k_j}}\right) dt^1 \wedge \dots \wedge dt^m.$$

On the other hand,

$$\begin{aligned} (\bar{j}^1\gamma)^*\Theta &= (\Theta_{a_1 \dots a_m} \circ \bar{j}^1\gamma) (\bar{j}^1\gamma)^*(du^{a_1} \wedge \dots \wedge du^{a_m}) \\ &= (\Theta_{a_1 \dots a_m} \circ \bar{j}^1\gamma) \det\left(\frac{\partial \gamma^{a_i}}{\partial t^{k_j}}\right) dt^1 \wedge \dots \wedge dt^m. \quad \square \end{aligned}$$

5.3 Euler forms

Let Θ_m be an integral equivalent of Λ . Define the scalar $(m+1)$ -form $\mathcal{E}_m \in \Omega_1^{m+1,0}$ by

$$\mathcal{E}_m = d\Theta_m$$

and the vector forms $\mathcal{E}_r \in \Omega_2^{r+1,m-r}$ by

$$\mathcal{E}_r = \tau_{mE}^{2,1*} d\Theta_r - (-1)^r d_T \Theta_{r+1} \quad 0 \leq r \leq m-1.$$

The forms \mathcal{E}_r are called the *Euler forms* of Θ_m .

Lemma 16. *The Euler forms satisfy the recurrence relation*

$$\mathcal{E}_r = \frac{(-1)^{r+1}}{m-r} i_{\mathbb{T}} \mathcal{E}_{r+1} \quad 0 \leq r \leq m-1;$$

consequently if $d\Theta_m = \mathcal{E}_m = 0$ then $\mathcal{E} = 0$.

Proof. This follows from the definition and Lemma 14. We have, omitting the pull-back maps,

$$\begin{aligned} i_{\mathbb{T}} \mathcal{E}_{r+1} &= i_{\mathbb{T}} d\Theta_{r+1} (-1)^{r+1} i_{\mathbb{T}} d_{\mathbb{T}} \Theta_{r+2} \\ &= d_{\mathbb{T}} \Theta_{r+1} - di_{\mathbb{T}} \Theta_{r+1} - (-1)^r d_{\mathbb{T}} i_{\mathbb{T}} \Theta_{r+2} \\ &= d_{\mathbb{T}} \Theta_{r+1} - (-1)^r d\Theta_r + (m-r-1) d_{\mathbb{T}} \Theta_{r+1} \\ &= (m-r) (d_{\mathbb{T}} \Theta_{r+1} - (-1)^r d\Theta_r) \end{aligned}$$

when $r+1 < m$, so that

$$\frac{(-1)^{r+1}}{m-r} i_{\mathbb{T}} \mathcal{E}_{r+1} = (-1)^{r+1} d_{\mathbb{T}} \Theta_{r+1} + d\Theta^r = \mathcal{E}_r.$$

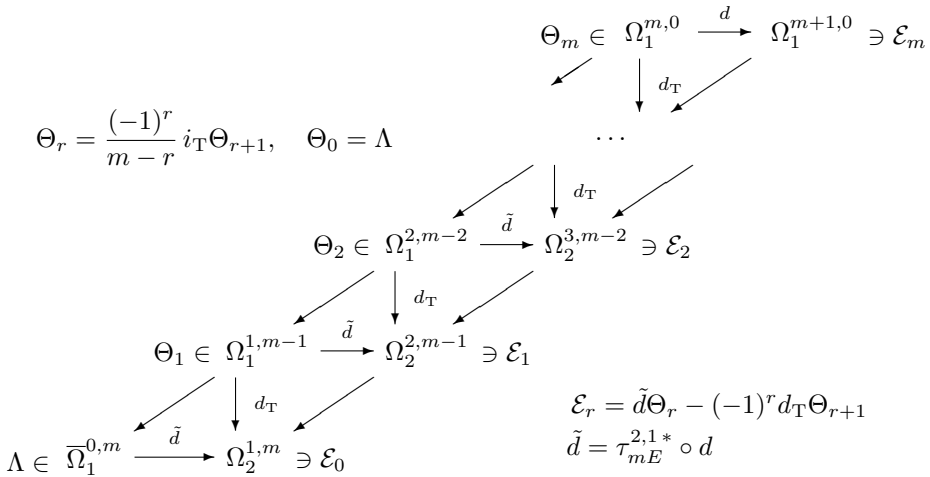
Similarly,

$$\begin{aligned} i_{\mathbb{T}} \mathcal{E}_m &= i_{\mathbb{T}} d\Theta_m \\ &= d_{\mathbb{T}} \Theta_m - di_{\mathbb{T}} \Theta_m \\ &= d_{\mathbb{T}} \Theta_m - (-1)^{m-1} d\Theta_{m-1} \end{aligned}$$

so that

$$(-1)^m i_{\mathbb{T}} \mathcal{E}_m = (-1)^m d_{\mathbb{T}} \Theta_m + d\Theta_{m-1} = \mathcal{E}_{m-1}. \quad \square$$

The different spaces containing the various equivalents and Euler forms may be seen in this diagonal part of the bicomplex.



5.4 Lepagean forms

Let Λ be a homogeneous Lagrangian, and let Θ_r be an equivalent of Λ ($1 \leq r \leq m$). We shall say that Θ_r is *Lepagean* if the corresponding Euler form $\mathcal{E}_0 \in \Omega_2^{1,m}$ satisfies

$$S\mathcal{E}_0 = 0,$$

so that \mathcal{E}_0 is horizontal over E .

Theorem 3. *The vector 1-form*

$$\Theta_1 = Sd\Lambda$$

is an integral equivalent of Λ ($m = 1$) or an intermediate equivalent ($m \geq 2$), and is Lepagean. It is called the Hilbert equivalent of $\Lambda = L d^m t$.

Proof. From the definition of S ,

$$S\Xi = S^j \chi \otimes d^{m-1} t_j,$$

so that

$$\begin{aligned} i_T Sd\Lambda &= i_T S(dL \otimes d^m t) \\ &= i_T (S^j dL \otimes d^{m-1} t_j) \\ &= i_k S^j dL \otimes dt^k \wedge d^{m-1} t_j \\ &= i_j S^j dL \otimes d^m t. \end{aligned}$$

But for any 1-form θ on $\hat{T}_m E$, if in coordinates $\theta = \theta_a du^a + \theta_a^i du_i^a$ then

$$i_j S^j \theta = i_j (\theta_a^j du^a) = u_j^a \theta_a^j,$$

so that

$$i_j S^j dL = u_j^a \frac{\partial L}{\partial u_j^a} = d_j^j L = mL$$

using the homogeneity of the Lagrangian.

To show that Θ_1 is Lepagean, note that

$$\begin{aligned} Sd_T \Theta_1 &= Sd_T Sd\Lambda \\ &= Sd_T (S^j dL \otimes d^{m-1} t_j) \\ &= S(d_i S^j dL \otimes (dt^i \wedge d^{m-1} t_j)) \\ &= S(d_j S^j dL \otimes d^m t) \\ &= S^i d_j S^j dL \otimes d^{m-1} t_i \\ &= (d_j S^i + \delta_j^i) S^j dL \otimes d^{m-1} t_i \\ &= S^i dL \otimes d^{m-1} t_i \\ &= S(dL \otimes d^m t) = Sd\Lambda \end{aligned}$$

using Lemma 13 and the fact that L is defined on $\hat{T}_m E$ so that $S^i S^j dL = 0$; thus $S\mathcal{E}_0 = 0$, as required. \square

Theorem 4. *If $\tilde{\Theta}_1$ is another Lepagean vector 1-form equivalent to Λ , with corresponding Euler form $\tilde{\mathcal{E}}_0$, then*

$$\tilde{\mathcal{E}}_0 = \mathcal{E}_0, \quad \tilde{\Theta}_1 - \Theta_1 = d_T \Phi \quad (\Phi \in \Omega_0^{1, m-2}).$$

Proof. It follows straightforwardly from the Lepagean condition $S\tilde{\mathcal{E}}_0 = 0$ that $P_2\tilde{\mathcal{E}}_0 = 0$, so that we may use the homotopy condition of Theorem 1 to see that

$$0 = P_2\tilde{\mathcal{E}}_0 = P_2(d\Lambda - d_T\tilde{\Theta}_1) = \Theta_1 - P_2d_T\tilde{\Theta}_1 = \Theta_1 - (1 - d_T P_1)\tilde{\Theta}_1,$$

giving $\tilde{\Theta}_1 - \Theta_1 = d_T P_1 \tilde{\Theta}_1$ (or $\tilde{\Theta}_1 = \Theta_1$ if $m = 1$). Thus

$$\tilde{\mathcal{E}}_0 - \mathcal{E}_0 = (d\Lambda - d_T\tilde{\Theta}_1) - (d\Lambda - d_T\Theta_1) = -d_T^2 P_1 \tilde{\Theta}_1 = 0.$$

(Note that, as $d\Lambda$ is a first-order vector 1-form, $P_2d\Lambda = Sd\Lambda = \Theta_1$.) □

5.5 The First Variation Formula

Theorem 5. *Let C be a compact connected m -dimensional submanifold of \mathbb{R}^m with boundary ∂C , let γ be an m -curve in E whose domain contains C , and let X be a variation field on E vanishing on $\gamma(\partial C)$ with prolongation X_m^1 on $\dot{T}_m E$. Then*

$$\int_C (\bar{j}^1 \gamma)^* \mathcal{L}_{X_m^1} \Lambda = \int_C (\bar{j}^2 \gamma)^* i_X \mathcal{E}_0;$$

consequently γ is an extremal of Λ precisely when \mathcal{E}_0 vanishes along the image of $\bar{j}^2 \gamma$.

Proof. We note first that

$$\begin{aligned} \int_C (\bar{j}^1 \gamma)^* \mathcal{L}_{X_m^1} \Lambda &= \int_C (\bar{j}^1 \gamma)^* i_{X_m^1} d\Lambda \\ &= \int_C (\bar{j}^2 \gamma)^* i_{X_m^2} \tau_{mE}^{2,1} d\Lambda \\ &= \int_C (\bar{j}^2 \gamma)^* i_{X_m^2} \mathcal{E}_0 + \int_C (\bar{j}^2 \gamma)^* i_{X_m^2} d_T \Theta_1, \end{aligned}$$

using the definition of the Euler form \mathcal{E}_0 . But prolongations commute with basis total derivatives and Θ_1 is horizontal over E , so that

$$\int_C (\bar{j}^2 \gamma)^* i_{X_m^2} d_T \Theta_1 = \int_C (\bar{j}^2 \gamma)^* d_T i_{X_m^1} \Theta_1 = \int_C d(\bar{j}^1 \gamma)^* i_X \Theta_1 = 0$$

and we see that the second integral vanishes; thus

$$\int_C (\bar{j}^1 \gamma)^* \mathcal{L}_{X_m^1} \Lambda = \int_C (\bar{j}^2 \gamma)^* i_{X_m^2} \mathcal{E}_0 = \int_C (\bar{j}^2 \gamma)^* i_X \mathcal{E}_0$$

because \mathcal{E}_0 is horizontal over E .

Now let γ be an immersion. If $\mathcal{E}_0 = 0$ at every point in the image of $\bar{j}^2 \gamma$, then for any vector field X on E and any $t \in C$ we will have $(\bar{j}^2 \gamma)^* i_X \mathcal{E}_0|_t = 0$, so that the integral over C will vanish and γ will be an extremal.

If, instead, $q = j_0^2(\gamma \circ \tau_t)$ is some point in the image of $j^2\gamma$ where $\mathcal{E}_0|_q$ is non-zero, then there must be a vector field X on E such that the vector-valued function $i_X\mathcal{E}_0$ gives a strictly positive multiple of $d^m t$ when evaluated at q , and hence when evaluated in some neighbourhood U of q . Let b be a positive bump function on E whose support lies in the interior of U and which satisfies $b(q) = 1$. Then

$$\int_C (j^1\gamma)^* \mathcal{L}_{(bX)_m^1} \Lambda = \int_C (j^2\gamma)^* i_{bX} \mathcal{E}_0 > 0,$$

so that γ cannot be an extremal. □

5.6 Integral equivalents for $m \geq 2$

Let $\Lambda = L d^m t$ be a homogeneous Lagrangian with $m \geq 2$, and write its Hilbert equivalent Θ_1 as

$$\Theta_1 = \vartheta^i \otimes d^{m-1} t_i;$$

the scalar 1-forms ϑ_i are called the *Hilbert forms* of Λ . If Λ never vanishes, define the *Carathéodory equivalent* $\tilde{\Theta}_m \in \Omega_1^{m,0}$ by

$$\tilde{\Theta}_m = \frac{1}{L^{m-1}} \bigwedge_{i=1}^m \vartheta^i.$$

Theorem 6. *The Carathéodory equivalent $\tilde{\Theta}_m$ is an integral equivalent of Λ .*

Proof. We must show that $i_T^m \Theta_m = (-1)^{m(m-1)/2} m! \Lambda$, so rewrite Θ_m as

$$\Theta_m = \frac{1}{m! L^{m-1}} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m)},$$

where \mathfrak{S}_m is the permutation group, and use induction. The calculation uses $d_j \lrcorner \vartheta^i = \delta_j^i L$, the proof of which is similar to that used to show that $i_T \Theta_1 = m\Lambda$; we also define $\tau_{r,s} \in \mathfrak{S}_m$ by

$$\tau_{r,s}(i) = \begin{cases} m-s & (i=r) \\ i-1 & (r+1 \leq i \leq m-s) \\ i & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned}
& i_{\mathbb{T}} \left(\sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s)} \otimes dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \right) \\
&= \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma d_j \lrcorner (\vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s)}) \otimes dt^j \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \\
&= \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (-1)^{r-1} (\vartheta^{\sigma(1)} \wedge \dots \wedge (d_j \lrcorner \vartheta^{\sigma(r)}) \wedge \dots \wedge \vartheta^{\sigma(m-s)}) \otimes \\
&\quad \otimes dt^j \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \\
&= L \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (-1)^{r-1} (\vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(r-1)} \wedge \vartheta^{\sigma(r+1)} \wedge \dots \wedge \vartheta^{\sigma(m-s)}) \otimes \\
&\quad \otimes dt^{\sigma(r)} \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \\
&= L \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (-1)^{r-1} (-1)^{m-r-s} \left\{ \right. \\
&\quad \left. (\vartheta^{\sigma\tau_{r,s}(1)} \wedge \dots \wedge \vartheta^{\sigma\tau_{r,s}(r-1)} \wedge \vartheta^{\sigma\tau_{r,s}(r+1)} \wedge \dots \wedge \vartheta^{\sigma\tau_{r,s}(m-s)}) \otimes \right. \\
&\quad \left. \otimes dt^{\sigma\tau_{r,s}(r)} \wedge dt^{\sigma\tau_{r,s}(m-s+1)} \wedge \dots \wedge dt^{\sigma\tau_{r,s}(m)} \right\} \\
&= (-1)^{m-s-1} L \sum_{r=1}^{m-s} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (\vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s-1)}) \otimes \\
&\quad \otimes dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \\
&= (-1)^{m-s-1} (m-s)L \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (\vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s-1)}) \otimes \\
&\quad \otimes dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)},
\end{aligned}$$

so if

$$\begin{aligned}
i_{\mathbb{T}}^s \Theta_m &= \frac{(-1)^{s(2m-s-1)/2}}{(m-s)! L^{m-s-1}} \left\{ \right. \\
&\quad \left. \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s)} \otimes dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \right\}
\end{aligned}$$

then

$$\begin{aligned}
i_{\mathbb{T}}^{s+1} \Theta_m &= \frac{(-1)^{s(2m-s-1)/2}}{(m-s)! L^{m-s-1}} \left\{ \right. \\
&\quad (-1)^{m-s-1} (m-s)L \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (\vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s-1)}) \otimes \\
&\quad \left. \otimes dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)} \right\}
\end{aligned}$$

$$= \frac{(-1)^{(s+1)(2m-s-2)/2}}{(m-s-1)!L^{m-s-2}} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma (\vartheta^{\sigma(1)} \wedge \dots \wedge \vartheta^{\sigma(m-s-1)}) \otimes \\ \otimes dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \dots \wedge dt^{\sigma(m)}$$

as required. Hence

$$\begin{aligned} i_{\mathbb{T}}^m \Theta_m &= \frac{(-1)^{m(m-1)/2}}{L^{-1}} \sum_{\sigma \in \mathfrak{S}_m} (-1)^\sigma dt^{\sigma(1)} \wedge \dots \wedge dt^{\sigma(m)} \\ &= (-1)^{m(m-1)/2} m! L dt^1 \wedge \dots \wedge dt^m \\ &= (-1)^{m(m-1)/2} m! \Lambda. \end{aligned} \quad \square$$

We see also from the induction formula that

$$i_{\mathbb{T}}^{m-1} \Theta_m = (-1)^{m(m-1)/2} (m-1)! \Theta_1$$

where Θ_1 is the Hilbert equivalent; consequently Θ_m is Lepagean. Then, as $d\Theta_m = \mathcal{E}_m$,

$$\begin{aligned} \int_C (j^1 \gamma)^* \mathcal{L}_{X_m^1} \Theta_m &= \int_C (j^1 \gamma)^* i_{X_m^1} \mathcal{E}_m \\ &= (-1)^{m(m-1)/2} m! \int_C (j^2 \gamma)^* i_X \mathcal{E}_0 \end{aligned}$$

for any vector field X on E vanishing on $\gamma(\partial C)$, because contractions by vector fields anticommute, so that $i_{\mathbb{T}}^m i_{X_m^1} \mathcal{E}_m = (-1)^m i_{X_m^1} i_{\mathbb{T}}^m \mathcal{E}_m$.

5.7 Another integral equivalent

When $m = 1$ then the only Lepagean integral equivalent of a Lagrangian is the Hilbert equivalent. But when $m > 1$ there may be other integral equivalents. Put

$$\Theta_{r+1} = \frac{(-1)^r}{(r+1)^2} S d\Theta_r \quad (1 \leq r < m)$$

where, as usual, $\Theta_0 = \Lambda$.

Lemma 17. *Each Θ_r is a first-order vector form, an element of $\Omega_1^{r, m-r}$, horizontal over E .*

Proof. Each Θ_r is first-order because neither S nor d increases the order of a vector form. By definition Θ_0 is horizontal over E , and if Θ_r is horizontal over E then the contraction of $d\Theta_r$ with any vector field on $\overset{\circ}{T}_m E$ vertical over E will again be horizontal over E ; thus Θ_{r+1} will also be horizontal over E . \square

Theorem 7. *The scalar m -form Θ_m is a Lepagean integral equivalent of Λ called the fundamental equivalent of Λ .*

Proof. We first show that, in coordinates,

$$\Theta_r = \frac{1}{(r!)^2} \frac{\partial^r L}{\partial u_{i_1}^{a_1} \dots \partial u_{i_r}^{a_r}} du^{a_1} \wedge \dots \wedge du^{a_r} \otimes \left(\frac{\partial}{\partial t^{i_r}} \lrcorner \dots \lrcorner \frac{\partial}{\partial t^1} \lrcorner d^m t \right).$$

This formula clearly holds for $r = 1$ (and, indeed, for $r = 0$); so suppose that it holds for a given value of r . Then

$$\begin{aligned} \Theta_{r+1} &= \frac{(-1)^r}{(r+1)^2} Sd\Theta_r \\ &= \frac{(-1)^r}{(r+1)^2} \frac{1}{(r!)^2} S^j \left(\frac{\partial^{r+1} L}{\partial u_{i_1}^{a_1} \dots \partial u_{i_r}^{a_r} \partial u_{i_{r+1}}^{a_{r+1}}} du_{i_{r+1}}^{a_{r+1}} + \dots \right) \wedge \\ &\quad \wedge du^{a_1} \wedge \dots \wedge du^{a_r} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner \frac{\partial}{\partial t^{i_r}} \lrcorner \dots \lrcorner \frac{\partial}{\partial t^1} \lrcorner d^m t \right) \\ &= \frac{1}{((r+1)!)^2} \frac{\partial^{r+1} L}{\partial u_{i_1}^{a_1} \dots \partial u_{i_{r+1}}^{a_{r+1}}} du^{a_1} \wedge \dots \wedge du^{a_{r+1}} \otimes \\ &\quad \otimes \left(\frac{\partial}{\partial t^{i_{r+1}}} \lrcorner \dots \lrcorner \frac{\partial}{\partial t^1} \lrcorner d^m t \right) \end{aligned}$$

so that the formula also holds for the case $r + 1$. In particular, therefore, we have

$$\begin{aligned} \Theta_m &= \frac{1}{(m!)^2} \frac{\partial^m L}{\partial u_{i_1}^{a_1} \dots \partial u_{i_m}^{a_m}} du^{a_1} \wedge \dots \wedge du^{a_m} \times \left(\frac{\partial}{\partial t^{i_m}} \lrcorner \dots \lrcorner \frac{\partial}{\partial t^1} \lrcorner d^m t \right) \\ &= \frac{1}{(m!)^2} \frac{\partial^m L}{\partial u_{i_1}^{a_1} \dots \partial u_{i_m}^{a_m}} du^{a_1} \wedge \dots \wedge du^{a_m} \times \begin{vmatrix} \delta_{i_1}^1 & \dots & \delta_{i_m}^1 \\ \vdots & & \vdots \\ \delta_{i_1}^m & \dots & \delta_{i_m}^m \end{vmatrix} \\ &= \frac{1}{m!} \frac{\partial^m L}{\partial u_1^{a_1} \dots \partial u_m^{a_m}} du^{a_1} \wedge \dots \wedge du^{a_m}. \end{aligned}$$

Thus, using the calculation in the proof of Lemma 15,

$$\begin{aligned} i_{\mathbb{T}}^m \Theta_m &= m! (-1)^{m(m-1)/2} \left(\frac{1}{m!} \frac{\partial^m L}{\partial u_1^{a_1} \dots \partial u_m^{a_m}} \det(u_{k_j}^{a_i}) \right) dt^1 \wedge \dots \wedge dt^m \\ &= (-1)^{m(m-1)/2} \frac{\partial^m L}{\partial u_1^{a_1} \dots \partial u_m^{a_m}} \det(u_{k_j}^{a_i}) dt^1 \wedge \dots \wedge dt^m \\ &= (-1)^{m(m-1)/2} m! L dt^1 \wedge \dots \wedge dt^m \\ &= (-1)^{m(m-1)/2} m! \Lambda. \quad \square \end{aligned}$$

Theorem 8. *The fundamental equivalent Θ_m of a homogeneous Lagrangian Λ has the property that $d\Theta_m = \mathcal{E}_m = 0$ if, and only if, $\mathcal{E}_0 = 0$.*

Proof. If $\mathcal{E}_m = 0$ then $\mathcal{E}_0 = 0$ by the recurrence relation of Lemma 16. So show the converse, we use the definition

$$\Theta_{r+1} = \frac{(-1)^r}{(r+1)^2} Sd\Theta_r$$

and the fact that $d\Theta_r \in \Omega_1^{r+1, m-r}$ to see that the homotopy operator P_1 from Theorem 1 takes the form

$$P_1(\chi_{i_1 \dots i_{m-r}} \otimes dt^{i_1} \wedge \dots \wedge dt^{i_{m-r}}) = \frac{1}{(r+1)^2} S^j \chi_{i_1 \dots i_{m-r}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner dt^{i_1} \wedge \dots \wedge dt^{i_{m-r}} \right)$$

(the formula in the proof of Theorem 1 was for an element of $\Omega_1^{r, s+1}$); thus we may rewrite the definition of Θ_{r+1} as

$$\Theta_{r+1} = (-1)^r P d\Theta_r.$$

Now from

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathbb{T}} \Theta_{r+1}$$

we obtain

$$P_2 d\mathcal{E}_r = -(-1)^r P_2 d_{\mathbb{T}} d\Theta_{r+1} = (-1)^r (d_{\mathbb{T}} P_1 d\Theta_{r+1} - d\Theta_{r+1})$$

so that

$$(-1)^{r+1} P_2 d\mathcal{E}_r = d\Theta_{r+1} - d_{\mathbb{T}} P_1 d\Theta_{r+1}$$

using the homotopy formula of Theorem 1; but

$$\mathcal{E}_{r+1} = d\Theta_{r+1} - (-1)^{r+1} d_{\mathbb{T}} \Theta_{r+2} = d\Theta_{r+1} - d_{\mathbb{T}} P_1 d\Theta_{r+1}$$

so that

$$\mathcal{E}_{r+1} = (-1)^{r+1} P_2 d\mathcal{E}_r.$$

Similarly,

$$P_2 d\mathcal{E}_{m-1} = -(-1)^{m-1} P_2 d_{\mathbb{T}} d\Theta_m = (-1)^m d\Theta_m = (-1)^m \mathcal{E}_m.$$

It follows that if $\mathcal{E}_0 = 0$ then $\mathcal{E}_m = 0$. □

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