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ON THE STRUCTURE OF THE AUGMENTATION QUOTIENT GROUP FOR SOME NONABELIAN 2-GROUPS

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Abstract. Let G be a finite nonabelian group, $\mathbb{Z}G$ its associated integral group ring, and $\Delta(G)$ its augmentation ideal. For the semidihedral group and another nonabelian 2-group the problem of their augmentation ideals and quotient groups $Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G)$ is deal with. An explicit basis for the augmentation ideal is obtained, so that the structure of its quotient groups can be determined.

Keywords: integral group ring, augmentation ideal, augmentation quotient groups, finite 2-group, semidihedral group

MSC 2010: 16S34, 20C05

1. Introduction

Let G be a finite group, $\mathbb{Z}G$ its integral group ring and $\Delta(G)$ the kernel of the augmentation homomorphism $\mathbb{Z}G \to \mathbb{Z}$, $\sum\limits_{g\in G} a_g g \to \sum\limits_{g\in G} a_g$, the augmentation ideal of $\mathbb{Z}G$. It is clear that $\Delta(G)$ is the free abelian group on the elements $[g] = g - 1$ for all $g \in G$ modulo the relation $[1] = 0$. The n-th power ideal $\Delta^n(G) := (\Delta(G))^n$ of the augmentation ideal $\Delta(G)$ is generated as an abelian group by the products $[g_1, \ldots g_n] = [g_1] \ldots [g_n]$, $g_1, \ldots, g_n \in G$. It is well known that if G is a finite group of order r, then $\Delta^n(G)$ is a free $\mathbb{Z}$-module of rank $r - 1$ for any $n \geqslant 1$ [6, p. 122]. The augmentation quotient group is defined as

$$Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G).$$

The problem of determining the structure of augmentation ideals $\Delta^n(G)$ and quotient groups $Q_n(G)$ is an interesting topic in group ring theory. For abelian groups

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much work has been done [1], [2], [4], [5], [6], [7]. In [4], Hales and Passi (see also [1]) proved that for a finite abelian group G there exists a number N such that for all $n \geqslant N$, $Q_n(G)$ is isomorphic to $Q_N(G)$. However, it is usually difficult to write down explicitly a basis of $\Delta^n(G)$ for an arbitrary finite nonabelian group, even for the finite 2-group.

Nonabelian finite 2-groups are one kind of important clases in nonabelian groups. For every positive integer k greater than or equal to 4, there are exactly four isomorphism classes of nonabelian groups of order 2^k which have a cyclic subgroup of index 2. Let $M \cong \mathbb{Z}/2^{k-1}\mathbb{Z}$ be a cyclic maximal subgroup of the nonabelian finite 2-group G, then there are four possibilities:

(i) G is a dihedral group.

(ii) G is a generalized quaternion group.

(iii) G is a semidirect product of M and a cyclic group of order two which acts on M via multiplication by $1 + 2^{k-2}$. Its presentation is

$$\langle a, b \mid a^{2^{k-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+2^{k-2}} \rangle.$$

In Daniel Gorenstein's influential text [3], this group was denoted by $M_k(2)$ and not given a special name.

(iv) G is a semidirect product of M and a cyclic group of order two which acts on M via multiplication by $-1 + 2^{k-2}$. The presentation for this group is

$$\langle a, b \mid a^{2^{k-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1+2^{k-2}} \rangle.$$

In [3], Gorenstein called this group the semidihedral group and we denote it by SD in this paper.

The structure of augmentation quotient groups is well established for two of them: the dihedral group [8], [9] and the generalized quaternion group [10]. The current paper investigates the structure of augmentation ideals and quotient groups of the group $M_k(2)$ and the semidihedral group SD, respectively. We start with some known results.

In [5], M. M. Parmenter proved

Theorem 1.1. *Let $G = \langle g \rangle$ be cyclic of order m. Then the set*

$$B_n(G) = \{(g-1)^n, (g-1)^{n+1}, \ldots, (g-1)^{n+m-2}\}$$

is a $\mathbb{Z}$-basis for $\Delta^n(G)$.

Let G be a finite group, and denote by $G_1 = [G, G]$ the commutator subgroup of G. For $i \geqslant 1$, define $G_i = [G, G_{i-1}]$. Then we have the sequence: $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \ldots$. The next theorem is easy to see:

Theorem 1.2. *$g-1\in\Delta^{i+1}(G)$, if $g\in G_i$.*

2. Structure of augmentation quotients for the group $M_k(2)$

Let $M_k(2)=\langle a,b \mid a^{2^{k-1}}=1, b^2=1, b^{-1}ab=a^{1+2^{k-2}}\rangle$, where $k\geqslant 4$. It is not hard to see that

$$M_k(2)=\{b^ta^u \mid 0\leqslant t\leqslant 1, 0\leqslant u\leqslant 2^{k-1}-1\} \text{ and } b^ta^u\cdot b^ia^j=b^{t+i}a^{u(1+2^{k-2})^i+j}.$$

Consequently, we have

Lemma 2.1. *$M_k(2)_1=[M_k(2),M_k(2)]=\langle a^{2^{k-2}}\rangle$ and $M_k(2)_i=\{1\}$, for $i\geqslant 2$.*

Proof. By the definition of the group $M_k(2)$, $a^{-1}b^{-1}ab=a^{2^{k-2}}$. Moreover, $a^{-2^{k-2}}b^{-1}a^{2^{k-2}}b=a^{-2^{k-2}}(b^{-1}ab)^{2^{k-2}}=a^{-2^{k-2}}a^{2^{k-2}}=1$. □

Now using Theorem 1.2, Lemma 2.1 and the formula

$$a^{2^{k-2}}-1=\binom{2^{k-2}}{1}(a-1)+\binom{2^{k-2}}{2}(a-1)^2+\ldots+(a-1)^{2^{k-2}},$$

we have

Lemma 2.2. $2^{k-2}(a-1)\in\Delta^2(M_k(2))$.

Lemma 2.3. *For $i\geqslant 0$, $l\geqslant 1$, we have $2^i(b-1)=(-1)^i(b-1)^{i+1}$, $(b-1)^{i+l}=(-2)^i(b-1)^l$.*

Proof. We give the proof of the first equality by induction on i. If $i=0$ the equation is trivial. For $i=1$ we have $2(b-1)=-(b-1)^2$ since $b^2=1$. Assume that $i>1$. Then by induction we obtain

$$\begin{aligned}2^i(b-1)=2\cdot 2^{i-1}(b-1)&=2\cdot(-1)^{i-1}(b-1)^i\\&=2(b-1)\cdot(-1)^{i-1}(b-1)^{i-1}\\&=(-1)^i(b-1)^{i+1}.\end{aligned}$$

Furthermore,

$$2^i(b-1)^l=2^i(b-1)\cdot(b-1)^{l-1}=(-1)^i(b-1)^{i+1}(b-1)^{l-1}=(-1)^i(b-1)^{i+l},$$

so the second equality can be derived. □

Recall that the n-th power $\Delta^n(M_k(2))$ is a free $\mathbb{Z}$-module of rank $|M_k(2)|-1$ for any $n \geqslant 1$, and is generated as an abelian group by the products $(x_1-1)\dots(x_n-1)$, $x_1,\dots,x_n \in M_k(2)$, where $k \geqslant 4$,

$$M_k(2)=\{b^t a^u \mid 0 \leqslant t \leqslant 1, 0 \leqslant u \leqslant 2^{k-1}-1\} \text{ and } b^t a^u \cdot b^i a^j = b^{t+i} a^{u(1+2^{k-2})^i+j}.$$

The recurrence relation of $\Delta^n(M_k(2))$ is given as follows:

Lemma 2.4. $\Delta^2(M_k(2)) = (b-1)^2\mathbb{Z} + \Delta(M_k(2))(a-1) + 2^{k-2}(a-1)\mathbb{Z}$.
Moreover, if $n \geqslant 2$,

$$\Delta^{n+1}(M_k(2)) = (b-1)^{n+1}\mathbb{Z} + \Delta^n(M_k(2))(a-1).$$

P r o o f. Since $\Delta^n(M_k(2))$ is generated by the elements $\{(x_1-1)\dots(x_n-1) \mid x_i \in M_k(2)\}$ with

(1) $a^i - 1 = \binom{i}{1}(a-1) + \binom{i}{2}(a-1)^2 + \dots \binom{i}{i-1}(a-1)^{i-1} + (a-1)^i$,

(2) $ba^i - 1 = (b-1)(a^i-1) + (b-1) + (a^i-1)$,

$$(a^i-1)(b-1) = (b-1)(a^{i(1+2^{k-2})}-1) + (a^i-1)(a^{2^{k-2}i}-1) + (a^{2^{k-2}i}-1)$$

for any $i \geqslant 1$, we have

$$\begin{aligned}\Delta^{n+1}(M_k(2)) &\subseteq (b-1)^{n+1}\mathbb{Z} + \Delta^n(M_k(2))(a-1) + \sum_{i+j\geqslant n} 2^{k-2}(b-1)^i(a-1)^j\mathbb{Z} \\ &\subseteq \Delta^{n+1}(M_k(2)).\end{aligned}$$

For $j \geqslant 1$, $2^{k-2}(b-1)^i(a-1)^j = [2^{k-2}(b-1)^i(a-1)^{j-1}](a-1)$:

(i) If $i \neq 0$ or $j > 1$, by Lemma 2.2 and Lemma 2.3 we have

$$2^{k-2}(b-1)^i(a-1)^j \in \Delta^n(M_k(2))(a-1), \text{ for any } n.$$

(ii) If $i=0$ and $j=1$, $2^{k-2}(b-1)^i(a-1)^j = 2^{k-2}(a-1) \in \Delta^2(M_k(2))$, i.e., $n=1$.
For $j=0$, we have

$$\begin{aligned}2^{k-2}(b-1)^i &= (-1)^{k-2}(b-1)^{k-2+i} \\ &= (-1)^{k-2}(-2)^{k-2+i-n-1}(b-1)^{n+1} \in (b-1)^{n+1}\mathbb{Z}.\end{aligned}$$

It follows that

$$\Delta^2(M_k(2)) = (b-1)^2\mathbb{Z} + \Delta(M_k(2))(a-1) + 2^{k-2}(a-1)\mathbb{Z}$$

and

$$\Delta^{n+1}(M_k(2)) = (b-1)^{n+1}\mathbb{Z} + \Delta^n(M_k(2))(a-1)$$

for $n \geqslant 2$. □

Theorem 2.5. *Let* $B_n(M_k(2))$ *be as follows:*

$$\begin{aligned}B_1(M_k(2)) &= \{(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\} \cup \{(b-1)\}\\ &\quad \cup \{(b-1)(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\},\\ B_2(M_k(2)) &= \{(a-1)^i \mid 2 \leqslant i \leqslant 2^{k-1}-1\} \cup \{2^{k-2}(a-1)\}\\ &\quad \cup \{(b-1)(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\} \cup \{(b-1)^2\},\end{aligned}$$

for $3 \leqslant n \leqslant k+1$,

$$\begin{aligned}B_n(M_k(2)) &= \{(a-1)^i \mid n \leqslant i \leqslant 2^{k-1}+n-3\} \cup \{2^{k-2}(a-1)^{n-1}\}\\ &\quad \cup \{2^{n-1-i}(b-1)(a-1)^i \mid 1 \leqslant i \leqslant n-2\}\\ &\quad \cup \{(b-1)(a-1)^i \mid n-1 \leqslant i \leqslant 2^{k-1}-1\} \cup \{(b-1)^n\},\end{aligned}$$

and for any $n \geqslant k+1$,

$$\begin{aligned}B_n(M_k(2)) &= \{(a-1)^i \mid n \leqslant i \leqslant 2^{k-1}+n-3\} \cup \{2^{k-2}(a-1)^{n-1}\}\\ &\quad \cup \{2^{k-i}(b-1)(a-1)^{n-1-(k-i)} \mid 1 \leqslant i \leqslant k-1\}\\ &\quad \cup \{(b-1)(a-1)^i \mid n-1 \leqslant i \leqslant 2^{k-1}-(k+2)+n\} \cup \{(b-1)^n\}.\end{aligned}$$

Then $B_n(M_k(2))$ *is a* $\mathbb{Z}$*-basis for* $\Delta^n(M_k(2))$.

P r o o f. We give the proof by induction on $n \geqslant 1$.

(I) $n=1$. It is obvious by (1), (2) and the definition of the augmentation ideal.

(II) $2 \leqslant n \leqslant k+1$. For $n=2$, by Lemma 2.4 we have

$$\begin{aligned}\Delta^2(M_k(2)) &= (b-1)^2\mathbb{Z} + \Delta(M_k(2))(a-1) + 2^{k-2}(a-1)\mathbb{Z}\\ &= (b-1)^2\mathbb{Z} + \sum_{i=2}^{2^{k-1}}(a-1)^i\mathbb{Z} + \sum_{i=1}^{2^{k-1}}(b-1)(a-1)^i\mathbb{Z} + 2^{k-2}(a-1)\mathbb{Z}.\end{aligned}$$

Since

$$(3)\quad -(a-1)^{2^{k-1}} = 2^{k-1}(a-1) + \binom{2^{k-1}}{2}(a-1)^2 + \ldots + \binom{2^{k-1}}{2^{k-1}-1}(a-1)^{2^{k-1}-1},$$

it follows that

$$\begin{aligned}B_2(M_k(2)) &= \{(a-1)^i \mid 2 \leqslant i \leqslant 2^{k-1}-1\} \cup \{2^{k-2}(a-1)\}\\ &\quad \cup \{(b-1)(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\} \cup \{(b-1)^2\}.\end{aligned}$$

Assume that the result is true for n $(2 \leqslant n \leqslant k)$, then by (3) and Lemma 2.4,

$$\begin{aligned}\Delta^{n+1}(M_k(2)) &= (b-1)^{n+1}\mathbb{Z} + \Delta^n(M_k(2))(a-1)\\ &= \sum_{i=n+1}^{2^{k-1}+n-2}(a-1)^i\mathbb{Z} + 2^{k-2}(a-1)^n\mathbb{Z} + \sum_{i=2}^{n-1} 2^{n-i}(b-1)(a-1)^i\mathbb{Z}\\ &\quad + \sum_{i=n}^{2^{k-1}-1}(b-1)(a-1)^i\mathbb{Z} + (b-1)^n(a-1)\mathbb{Z} + (b-1)^{n+1}\mathbb{Z}.\end{aligned}$$

By Lemma 2.3,

$$(b-1)^n(a-1) = 2^{n-1}(b-1)(a-1)\mathbb{Z},$$

so $B_{n+1}(M_k(2))$ is a set of $\mathbb{Z}$-generators for $\Delta^{n+1}(M_k(2))$. Direct computation shows $|B_n(M_k(2))| = 2^k - 1 = |M_k(2)| - 1$. By Theorem 1.1, $B_n(M_k(2))$ is a linearly independent set. Therefore $B_n(M_k(2))$ is a set of $\mathbb{Z}$-basis for $\Delta^n(M_k(2))$.

(III) $n \geqslant k+1$. The result is true for $n = k+1$ by (II). Suppose it is true for $n \geqslant k+1$. By Lemma 2.4, we have

$$\begin{aligned}\Delta^{n+1}(M_k(2)) &= (b-1)^{n+1}\mathbb{Z} + \Delta^n(M_k(2))(a-1)\\ &= \sum_{i=n+1}^{2^{k-1}+n-2}(a-1)^i\mathbb{Z} + 2^{k-2}(a-1)^n\mathbb{Z} + \sum_{i=1}^{k-1} 2^{k-i}(b-1)(a-1)^{n-(k-i)}\mathbb{Z}\\ &\quad + \sum_{i=n}^{2^{k-1}-(k+2)+n+1}(b-1)(a-1)^i\mathbb{Z} + (b-1)^n(a-1)\mathbb{Z} + (b-1)^{n+1}\mathbb{Z}.\end{aligned}$$

By our inductive hypothesis, Lemma 2.3 and (3), we deduce that

$$\begin{aligned}(b-1)^n(a-1) &= (b-1)[(b-1)^{n-1}(a-1)]\\ &= (b-1)^{n+1}\mathbb{Z} + \sum_{i=n}^{2^{k-1}+n-3}(b-1)(a-1)^i\mathbb{Z} + 2^{k-2}(b-1)(a-1)^{n-1}\mathbb{Z}\\ &\quad + \sum_{i=1}^{k-1} 2^{k-i}(b-1)^2(a-1)^{n-(k+1-i)}\mathbb{Z} + \sum_{i=n-1}^{2^{k-1}-(k+2)+n}(b-1)^2(a-1)^i\mathbb{Z}\\ &= (b-1)^{n+1}\mathbb{Z} + \sum_{i=1}^{k-1} 2^{k-i}(b-1)(a-1)^{n-(k-i)}\mathbb{Z}\\ &\quad + 2^k(b-1)(a-1)^{n-k}\mathbb{Z} + \sum_{i=n}^{2^{k-1}-(k+2)+n}(b-1)(a-1)^i\mathbb{Z}.\end{aligned}$$

But

$$-2(b-1)(a-1)^{2^{k-1}-(k+2)+n+1} = 2^k(b-1)(a-1)^{n-k}$$
$$+ 2\binom{2^{k-1}}{2}(b-1)(a-1)^{n-k+1} + \ldots + 2\binom{2^{k-1}}{2^{k-1}-1}(b-1)(a-1)^{2^{k-1}-(k+2)+n},$$

so $2^k(b-1)(a-1)^{n-k}$ can be generated by

$$\{2^{k-i}(b-1)(a-1)^{n-(k-i)} \mid 1 \leqslant i \leqslant k-1\}$$
$$\cup \{(b-1)(a-1)^i \mid n \leqslant i \leqslant 2^{k-1}-(k+2)+n+1\}.$$

Hence we obtain that $\Delta^{n+1}(M_k(2))$ can be generated by the set

$$B_{n+1}(M_k(2)) = \{(a-1)^i \mid n+1 \leqslant i \leqslant 2^{k-1}+n-2\} \cup \{2^{k-2}(a-1)^n\}$$
$$\cup \{2^{k-i}(b-1)(a-1)^{n-(k-i)} \mid 1 \leqslant i \leqslant k-1\}$$
$$\cup \{(b-1)(a-1)^i \mid n \leqslant i \leqslant 2^{k-1}-(k+2)+n+1\} \cup \{(b-1)^{n+1}\}.$$

Similarly to (II), we know that $B_n(M_k(2))$ is a set of $\mathbb{Z}$-basis for $\Delta^n(M_k(2))$. □

Theorem 2.6. *Let $M_k(2)$ be the nonabelian 2-group of order 2^k $(k \geqslant 4)$ as defined in the previous section. Then*

(i) $Q_1(M_k(2)) \cong \mathbb{Z}_{2^{k-2}} \oplus \mathbb{Z}_2$, *and* $Q_n(M_k(2)) \cong \mathbb{Z}_{2^{k-2}} \oplus \underbrace{\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{n+1}$, *if* $2 \leqslant n \leqslant k$;

(ii) $Q_n(M_k(2)) \cong \mathbb{Z}_{2^{k-2}} \oplus \underbrace{\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{k+2}$, *if* $n \geqslant k+1$.

Proof. (i) It is easy to see that $\{\overline{(b-1)}\} \cup \{\overline{(a-1)}\}$ is a $\mathbb{Z}$-basis for $Q_1(M_k(2))$ and $\{\overline{(a-1)^2}\} \cup \{\overline{2^{k-2}(a-1)}\} \cup \{\overline{(b-1)(a-1)}\} \cup \{\overline{(b-1)^2}\}$ is a $\mathbb{Z}$-basis for $Q_2(M_k(2))$.

For $3 \leqslant n \leqslant k$,

$$\{\overline{(a-1)^n}\} \cup \{\overline{2^{n-1-i}(b-1)(a-1)^i} \mid 1 \leqslant i \leqslant n-2\}$$
$$\cup \{\overline{2^{k-2}(a-1)^{n-1}}\} \cup \{\overline{(b-1)(a-1)^{n-1}}\} \cup \{\overline{(b-1)^n}\}$$

is a set of $\mathbb{Z}$-basis for $Q_n(M_k(2))$.

(ii) For $n \geqslant k+1$,

$$\{\overline{(a-1)^n}\} \cup \{\overline{2^{k-i}(b-1)(a-1)^{n-1-(k-i)}} \mid 1 \leqslant i \leqslant k-1\}$$
$$\cup \{\overline{2^{k-2}(a-1)^{n-1}}\} \cup \{\overline{(b-1)(a-1)^{n-1}}\} \cup \{\overline{(b-1)^n}\}$$

is a set of $\mathbb{Z}$-basis for $Q_n(M_k(2))$.

By Lemma 2.2 and 2.3, we get the conclusion. □

3. Structure of $Q_n(SD)$ for the semidihedral group SD

Let

$$SD = \langle a, b \mid a^{2^{k-1}} = 1, b^2 = 1, b^{-1}ab = a^{-1+2^{k-2}} \rangle$$

be the semidihedral group of order 2^k, where $k \geqslant 4$. It is an easy calculation to show that

$$SD = \{b^t a^u \mid 0 \leqslant t \leqslant 1, 0 \leqslant u \leqslant 2^{k-1} - 1\} \text{ and } b^t a^u \cdot b^i a^j = b^{t+i} a^{u(-1+2^{k-2})^i + j}.$$

Hence we have

Lemma 3.1. $SD_i = \langle a^{2^i} \rangle$ $(i \geqslant 1)$. *Furthermore,* $a^{2^i} - 1 \in \Delta^{i+1}(SD)$.

Proof. Since $a^{-1}b^{-1}ab = a^{-2+2^{k-2}}$, we have

$$SD_1 = [SD, SD] = \langle a^{-2+2^{k-2}} \rangle = \langle a^{2+2^{k-2}} \rangle = \langle a^2 \rangle.$$

In fact, $m(2 + 2^{k-2}) = 2^{k-1}s + 2t$, so

$$(a^{2+2^{k-2}})^m = (a^{2^{k-1}})^s (a^2)^t = (a^2)^t$$

for any integer m, where $s, t \in \mathbb{Z}$. This can be seen as follows:

(i) If m is even, i.e., $m = 2l (l \in \mathbb{Z})$, then

$$m(2 + 2^{k-2}) = 2l(2 + 2^{k-2}) = 2^{k-1}l + 2(2l).$$

(ii) If m is odd, i.e., $m = 2l + 1 (l \in \mathbb{Z})$, then

$$m(2 + 2^{k-2}) = (2l + 1)(2 + 2^{k-2}) = 2^{k-1}l + 2(2l + 1 + 2^{k-3}).$$

Notice that $|\langle a^{2+2^{k-2}} \rangle| = |\langle a^2 \rangle| = 2^{k-2}$, hence $SD_1 = \langle a^2 \rangle$ as claimed. We proceed to the general case. Since

$$a^{-2}b^{-1}a^2b = a^{-2}(b^{-1}ab)^2 = a^{-2}a^{-2+2^{k-1}} = a^{-4},$$

we have

$$SD_2 = [SD, SD_1] = \langle a^{-4} \rangle = \langle a^4 \rangle.$$

Iteratively, we obtain

$$SD_i = [SD, SD_{i-1}] = \langle a^{2^i} \rangle.$$

It follows from Theorem 1.2 that the second assertion can be easily derived. □

Then by Lemma 3.1 and the equality

$$a^{2^i} - 1 = \binom{2^i}{1}(a - 1) + \binom{2^i}{2}(a - 1)^2 + \ldots + (a - 1)^{2^i},$$

we can obtain

Lemma 3.2 (also see the proof in [10])**.** $2^i(a-1)\in\Delta^{i+1}(SD)$ $(i\geqslant 0)$.

Analogously to the proof of Lemma 2.3 in Section 2, we obtain the following results.

Lemma 3.3. $2^i(b-1)=(-1)^i(b-1)^{i+1}$, $(b-1)^{i+l}=(-2)^i(b-1)^l$, *for* $i\geqslant 0$, $l\geqslant 1$.

Note that the n-th power $\Delta^n(SD)$ of $\Delta(SD)$ is a free $\mathbb{Z}$-module of rank $|SD|-1$ for any $n\geqslant 1$, and is generated as an abelian group by the products

$$(x_1-1)\dots(x_n-1), x_1,\dots,x_n\in SD,$$

where $SD=\{b^ta^u\mid 0\leqslant t\leqslant 1, 0\leqslant u\leqslant 2^{k-1}-1\}$ is the semidihedral group of order $2^k(k\geqslant 4)$ and $b^ta^u\cdot b^ia^j=b^{t+i}a^{u(-1+2^{k-2})^i+j}$. Now we give the recurrence relation of $\Delta^n(SD)$.

Lemma 3.4. $\Delta^{n+1}(SD)=(b-1)^{n+1}\mathbb{Z}+\Delta^n(SD)(a-1)+2^n(a-1)\mathbb{Z}$.
Moreover, if $n\geqslant k$,

$$\Delta^{n+1}(SD)=(b-1)^{n+1}\mathbb{Z}+\Delta^n(SD)(a-1).$$

P r o o f. Since $\Delta^n(SD)$ is generated by the elements $\{(x_1-1)\dots(x_n-1)\mid x_i\in SD\}$ with

$$a^i-1=\binom{i}{1}(a-1)+\binom{i}{2}(a-1)^2+\dots\binom{i}{i-1}(a-1)^{n-1}+(a-1)^i, \tag{4}$$

$$\begin{aligned} ba^i-1&=(b-1)(a^i-1)+(b-1)+(a^i-1), \\ (a^i-1)(b-1)&=(b-1)(a^{i(-1+2^{k-2})}-1) \\ &\quad+(a^i-1)(a^{i(-2+2^{k-2})}-1)+(a^{i(-2+2^{k-2})}-1) \end{aligned} \tag{5}$$

for any $i\geqslant 1$, we have

$$\begin{aligned}\Delta^{n+1}(SD)&\subseteq(b-1)^{n+1}\mathbb{Z}+\Delta^n(SD)(a-1)+\sum_{i+j+s\geqslant n+1}2^i(b-1)^j(a-1)^s\mathbb{Z}\\ &\subseteq\Delta^{n+1}(SD).\end{aligned}$$

There are two cases to consider.
(i) $s\geqslant 1$. For $s>1$ or $j\neq 0$, by Lemma 3.2 and Lemma 3.3 we have

$$2^i(b-1)^j(a-1)^s=[2^i(b-1)^j(a-1)^{s-1}](a-1)\subseteq\Delta^n(SD)(a-1).$$

For $s = 1$ and $j = 0$,

$$\sum_{i \geqslant n} 2^i(a-1) = 2^n(a-1)\mathbb{Z}.$$

(ii) $s = 0$. Then

$$2^i(b-1)^j = (-1)^i(b-1)^{i+j} = (-1)^i(-2)^{i+j-n-1}(b-1)^{n+1} \in (b-1)^{n+1}\mathbb{Z}.$$

Hence we have

$$\Delta^{n+1}(SD) = (b-1)^{n+1}\mathbb{Z} + \Delta^n(SD)(a-1) + 2^n(a-1)\mathbb{Z}.$$

For $n \geqslant k-1$, $a^{2^n} = a^{2^{k-1}} = 1$, we have

$$\begin{aligned} -2^n(a-1) &= -(a^{2^n}-1) + \binom{2^n}{2}(a-1)^2 + \binom{2^n}{3}(a-1)^3 + \ldots + (a-1)^{2^n} \\ &= \left[\binom{2^n}{2}(a-1) + \binom{2^n}{3}(a-1)^2 + \ldots + (a-1)^{2^n-1}\right](a-1). \end{aligned}$$

Let $u = 2^m t \geqslant 2$ where $(t,2) = 1$. Then $\binom{2^n}{u} = 2^{n-m}v$, v is an integer, and

$$\binom{2^n}{u}(a-1)^{u-1} = 2^{n-m}v(a-1)^{u-1} \subseteq \Delta^{n-m+1+2^m t-2}(SD) \subseteq \Delta^n(SD).$$

So $2^n(a-1)\mathbb{Z} \subseteq \Delta^n(SD)(a-1)$, and $\Delta^{n+1}(SD) = (b-1)^{n+1}\mathbb{Z} + \Delta^n(SD)(a-1)$ for $n \geqslant k-1$. □

Theorem 3.5. *Let $B_n(SD)$ be as follows:*

$$\begin{aligned} B_1(SD) = \{(b-1)\} &\cup \{(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\} \\ &\cup \{(b-1)(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\}, \\ B_2(SD) = \{(b-1)^2\} &\cup \{2(a-1)\} \cup \{(a-1)^i \mid 2 \leqslant i \leqslant 2^{k-1}-1\} \\ &\cup \{(b-1)(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\}, \end{aligned}$$

for $3 \leqslant n \leqslant k$,

$$\begin{aligned} B_n(SD) = \{(b-1)^n\} &\cup \{2^{n-i}(a-1)^i \mid 1 \leqslant i \leqslant n-1\} \\ &\cup \{(a-1)^i \mid n \leqslant i \leqslant 2^{k-1}-1\} \\ &\cup \{2^{n-1-i}(b-1)(a-1)^i \mid 1 \leqslant i \leqslant n-2\} \\ &\cup \{(b-1)(a-1)^i \mid n-1 \leqslant i \leqslant 2^{k-1}-1\}, \end{aligned}$$

and for $n \geqslant k$,

$$\begin{aligned}B_n(SD) = \{(b-1)^n\} &\cup \{2^{k-i}(a-1)^{n-(k-i)} \mid 1 \leqslant i \leqslant k-1\}\\ &\cup \{(a-1)^i \mid n \leqslant i \leqslant 2^{k-1}-(k+1)+n\}\\ &\cup \{2^{k-1-i}(b-1)(a-1)^{n-(k-i)} \mid 1 \leqslant i \leqslant k-2\}\\ &\cup \{(b-1)(a-1)^i \mid n-1 \leqslant i \leqslant 2^{k-1}-(k+1)+n\}.\end{aligned}$$

Then $B_n(SD)$ *is a* $\mathbb{Z}$*-basis for* $\Delta^n(SD)$.

Proof. We give the proof by induction on n.

(I) $n=1$. It is obvious by (4), (5) and the definition of the augmentation ideal.

(II) $2 \leqslant n \leqslant k$. For $n=2$, by Lemma 3.4 we have

$$\begin{aligned}\Delta^2(SD) &= (b-1)^2\mathbb{Z} + \Delta(SD)(a-1) + 2(a-1)\mathbb{Z}\\ &= (b-1)^2\mathbb{Z} + \sum_{i=2}^{2^{k-1}}(a-1)^i\mathbb{Z}\\ &\quad + \sum_{i=1}^{2^{k-1}}(b-1)(a-1)^i\mathbb{Z} + 2(a-1)\mathbb{Z}.\end{aligned}$$

By (3) we have

$$\begin{aligned}B_2(SD) = \{(b-1)^2\} \cup \{2(a-1)\} &\cup \{(a-1)^i \mid 2 \leqslant i \leqslant 2^{k-1}-1\}\\ &\cup \{(b-1)(a-1)^i \mid 1 \leqslant i \leqslant 2^{k-1}-1\}.\end{aligned}$$

Assume that the result is true for n $(2 \leqslant n \leqslant k-1)$. Then by Lemma 3.4 we have

$$\begin{aligned}\Delta^{n+1}(SD) &= (b-1)^{n+1}\mathbb{Z} + \Delta^n(SD)(a-1) + 2^n(a-1)\mathbb{Z}\\ &= (b-1)^n(a-1)\mathbb{Z} + \sum_{i=2}^{n}2^{n+1-i}(a-1)^i\mathbb{Z} + \sum_{i=n+1}^{2^{k-1}-1}(a-1)^i\mathbb{Z}\\ &\quad + (a-1)^{2^{k-1}}\mathbb{Z} + \sum_{i=2}^{n-1}2^{n-i}(b-1)(a-1)^i\mathbb{Z} + \sum_{i=n}^{2^{k-1}-1}(b-1)(a-1)^i\mathbb{Z}\\ &\quad + (b-1)(a-1)^{2^{k-1}}\mathbb{Z} + (b-1)^{n+1}\mathbb{Z} + 2^n(a-1)\mathbb{Z}.\end{aligned}$$

But

$$
\begin{aligned}
(b-1)^n(a-1) &= (b-1)[(b-1)^{n-1}(a-1)] \\
&= (b-1)^{n+1}\mathbb{Z} + \sum_{i=1}^{n-1} 2^{n-i}(b-1)(a-1)^i\mathbb{Z} + \sum_{i=n}^{2^{k-1}-1} (b-1)(a-1)^i\mathbb{Z} \\
&\quad + \sum_{i=1}^{n-2} 2^{n-1-i}(b-1)^2(a-1)^i\mathbb{Z} + \sum_{i=n-1}^{2^{k-1}-1} (b-1)^2(a-1)^i\mathbb{Z} \\
&= (b-1)^{n+1}\mathbb{Z} + \sum_{i=1}^{n-1} 2^{n-i}(b-1)(a-1)^i\mathbb{Z} + \sum_{i=n}^{2^{k-1}-1} (b-1)(a-1)^i\mathbb{Z} \\
&\quad + \sum_{i=1}^{n-2} 2^{n-i}(b-1)(a-1)^i\mathbb{Z} + \sum_{i=n}^{2^{k-1}-1} (b-1)(a-1)^i\mathbb{Z}
\end{aligned}
$$

by our inductive hypothesis and $(b-1)^2 = -2(b-1)$. So again by equality (3) we obtain

$$
\begin{aligned}
\Delta^{n+1}(SD) = (b-1)^{n+1}\mathbb{Z} &+ \sum_{i=1}^{n} 2^{n+1-i}(a-1)^i\mathbb{Z} + \sum_{i=n+1}^{2^{k-1}-1} (a-1)^i\mathbb{Z} \\
&+ \sum_{i=1}^{n-1} 2^{n-i}(b-1)(a-1)^i\mathbb{Z} + \sum_{i=n}^{2^{k-1}-1} (b-1)(a-1)^i\mathbb{Z}
\end{aligned}
$$

and hence $B_{n+1}(SD)$ is a set of $\mathbb{Z}$-generators for $\Delta^{n+1}(SD)$. Direct computation shows

$$|B_n(SD)| = 2^k - 1 = |SD| - 1.$$

By Theorem 1.1, $B_n(SD)$ is a linearly independent set. Therefore $B_n(SD)$ is a set of $\mathbb{Z}$-basis for $\Delta^n(SD)$.

(III) $n \geqslant k$. The result is true for $n = k$ by (II). Suppose it is true for n. By Lemma 3.4, we have

$$
\begin{aligned}
\Delta^{n+1}(SD) &= (b-1)^{n+1}\mathbb{Z} + \Delta^n(SD)(a-1) \\
&= (b-1)^{n+1}\mathbb{Z} + (b-1)^n(a-1)\mathbb{Z} \\
&\quad + \sum_{i=1}^{k-1} 2^{k-i}(a-1)^{n+1-(k-i)}\mathbb{Z} + \sum_{i=n+1}^{m} (a-1)^i\mathbb{Z} \\
&\quad + \sum_{i=1}^{k-2} 2^{k-1-i}(b-1)(a-1)^{n+1-(k-i)}\mathbb{Z} + \sum_{i=n}^{m}(b-1)(a-1)^i\mathbb{Z},
\end{aligned}
$$

where $m = 2^{k-1} - (k+1) + n + 1$. By induction we deduce that

$$\begin{aligned}
&(b-1)^n(a-1) = (b-1)[(b-1)^{n-1}(a-1)] \\
&= (b-1)^{n+1}\mathbb{Z} + \sum_{i=1}^{k-1} 2^{k-i}(b-1)(a-1)^{n-(k-i)}\mathbb{Z} + \sum_{i=n}^{2^{k-1}-(k+1)+n} (b-1)(a-1)^i\mathbb{Z} \\
&\quad + \sum_{i=1}^{k-2} 2^{k-1-i}(b-1)^2(a-1)^{n-(k-i)}\mathbb{Z} + \sum_{i=n-1}^{2^{k-1}-(k+1)+n} (b-1)^2(a-1)^i\mathbb{Z} \\
&= (b-1)^{n+1}\mathbb{Z} + 2^{k-1}(b-1)(a-1)^{n-(k-1)}\mathbb{Z} \\
&\quad + \sum_{i=1}^{k-2} 2^{k-1-i}(b-1)(a-1)^{n+1-(k-i)}\mathbb{Z} + \sum_{i=n}^{2^{k-1}-(k+1)+n} (b-1)(a-1)^i\mathbb{Z}.
\end{aligned}$$

Since

$$\begin{aligned}
-(b-1)(a-1)^{2^{k-1}-(k+1)+n+1} &= 2^{k-1}(b-1)(a-1)^{n-(k-1)} \\
&\quad + 2^{k-2}(b-1)(a-1)^{n-(k-2)}\mathbb{Z} + \sum_{l=3}^{2^{k-1}-1} \binom{2^{k-1}}{l}(b-1)(a-1)^{l+n-k},
\end{aligned}$$

it follows that $2^{k-1}(b-1)(a-1)^{n-(k-1)}$ can be generated by

$$\begin{aligned}
&\{2^{k-1-i}(b-1)(a-1)^{n+1-(k-i)} \mid 1 \leqslant i \leqslant k-2\} \\
&\quad \cup \{(b-1)(a-1)^i \mid n \leqslant i \leqslant 2^{k-1} - (k+1) + n + 1\}.
\end{aligned}$$

Therefore $\Delta^{n+1}(SD)$ can be generated by the set

$$\begin{aligned}
B_{n+1}(SD) = \{(b-1)^{n+1}\} &\cup \{2^{k-i}(a-1)^{n+1-(k-i)} \mid 1 \leqslant i \leqslant k-1\} \\
&\cup \{(a-1)^i \mid n+1 \leqslant i \leqslant 2^{k-1} - (k+1) + n + 1\} \\
&\cup \{2^{k-1-i}(b-1)(a-1)^{n+1-(k-i)} \mid 1 \leqslant i \leqslant k-2\} \\
&\cup \{(b-1)(a-1)^i \mid n \leqslant i \leqslant 2^{k-1} - (k+1) + n + 1\}.
\end{aligned}$$

Similarly to (II), we know that $B_n(SD)$ is a set of $\mathbb{Z}$-basis for $\Delta^n(SD)$. □

Theorem 3.6. *Let SD be the semidihedral group of order 2^k $(k \geqslant 4)$. Then*
(i) $Q_n(SD) \cong \underbrace{\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{2n}$, *if* $1 \leqslant n \leqslant k-1$;
(ii) $Q_n(SD) \cong \underbrace{\mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2}_{2k}$, *if* $n \geqslant k$.

P r o o f. (i) It is easy to see that $\{\overline{(b-1)}\} \cup \{\overline{(a-1)}\}$ is a $\mathbb{Z}$-basis for $Q_1(SD)$.

For $2 \leqslant n \leqslant k-1$,

$$\{\overline{(b-1)^n}\} \cup \{\overline{2^{n-i}(a-1)^i} \mid 1 \leqslant i \leqslant n-1\} \cup \{\overline{(a-1)^n}\} \\ \cup \{\overline{2^{n-1-i}(b-1)(a-1)^i} \mid 1 \leqslant i \leqslant n-2\} \cup \{\overline{(b-1)(a-1)^{n-1}}\}$$

is a set of $\mathbb{Z}$-basis for $Q_n(SD)$.

(ii) For $n \geqslant k$,

$$\{\overline{(b-1)^n}\} \cup \{\overline{2^{k-i}(a-1)^{n-(k-i)}} \mid 1 \leqslant i \leqslant k-1\} \cup \{\overline{(a-1)^n}\} \\ \cup \{\overline{2^{k-1-i}(b-1)(a-1)^{n-(k-i)}} \mid 1 \leqslant i \leqslant k-2\} \cup \{\overline{(b-1)(a-1)^{n-1}}\}$$

is a set of $\mathbb{Z}$-basis for $Q_n(SD)$.

The result follows immediately from Lemma 3.2 and 3.3. □

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