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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 1, 257–264

Persistent URL: `http://dml.cz/dmlcz/142055`

Czechoslovak Mathematical Journal, 62 (137) (2012), 257–264

TOPOLOGICAL CALCULUS FOR SEPARATING POINTS FROM CLOSED SETS BY MAPS

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(Received January 7, 2011)

Abstract. Pointfree formulas for three kinds of separating points for closed sets by maps are given. These formulas allow controlling the amount of factors of the target product space so that it does not exceed the weight of the embeddable space. In literature, the question of how many factors of the target product are needed for the embedding has only been considered for specific spaces. Our approach is algebraic in character and can thus be viewed as a contribution to Kuratowski's *topological calculus.*

Keywords: embedding, separating points from closed sets, weak separating points from closed sets, separating family, weight

MSC 2010: 54C25, 54A25

1. Introduction

Some proofs of topological theorems are algebraic in character. These can either by phrased in terms of closure algebras [11] or even in terms of locales [4]. The *topological calculus*—developed by Kuratowski [7]—is, roughly speaking, the part of topology which deals with closure operators (cf. [10]).

The aim of this paper is to enrich Kuratowski's topological calculus with algebraic reformulations of three known concepts of separating points from closed sets (we mean the traditional concept as well as those of [3] and [9]).

Let X and Y_j $(j \in J)$ be topological spaces and $C(X, Y_j)$ the set of all continuous functions from X to Y_j. We recall that a family $F \subseteq \bigcup\limits_{j\in J} C(X, Y_j)$ separates points if, whenever $x \neq y$ in X, there is an $f \in F$ such that $f(x) \neq f(y)$. Further, F

The authors acknowledge financial support from the Ministry of Science and Innovation of Spain under grant MTM2009-12872-C02-02. The first named author also acknowledges financial support from the UPV-EHU under grant GIU07/27.

separates points from closed sets if, whenever $K \subseteq X$ is closed and $x \in X \setminus K$, there is an $f \in F$ with $f(x) \notin \overline{f(K)}$. If F separates both the points and the points from closed sets, then the evaluation map $e\colon X \to \prod_{j \in J} Y_j$—determined by $\pi_j \circ e = f_j$ for all $j \in J$—is an embedding (this is the well-known *diagonal theorem* [1]). In particular, if $Y_j = Y$ for all j, then e embeds X into $Y^{|J|}$. In the literature, the question of how many factors of the latter product are actually needed to perform the embedding has only been considered for specific spaces X and Y. One then usually shows that X embeds into $Y^{w(X)}$ where $w(X) \geqslant \aleph_0$ is the weight of X (cf. proofs of the universal space theorems in [1, 2.3.23, 4.4.9, 6.2.16]). For arbitrary spaces one exception can be found in Mrówka [9] who—by repeating the classical Tychonoff's technique of reducing the amount of factors (cf. [12] or [1])—was able to show the following (cf. [9, 5.3]):

Theorem (Mrówka). *If* $\bigcup_{n \in \mathbb{N}} C(X, Y^n)$ *separates points and points from closed sets, then the evaluation map embeds* X *into* $Y^{w(X)}$.

In this paper we will show that one can reduce the amount of factors already on the level of the general diagonal theorem provided X is T_0, in which case X can always be embedded into $\prod_{f \in F_0} Y_f$ where $F_0 \subseteq F$ is such that $|F_0| \leqslant w(X)$ (see Theorem 2.2). Note that we assume X to be T_0 in order to guarantee that after reducing the family F to a smaller family F_0, the latter will not only keep separating points from closed sets but will also separate points [if X is T_0 and F_0 separates points from closed sets, then it separates points too]. It is remarkable to notice that (when X is T_0) Theorem 2.2 immediately yields the above theorem of Mrówka (see also Remark 2.3).

Removing the redundant factors will be achieved by formulating the concept of separating points from closed sets by maps purely in terms of closed sets without mentioning points (cf. [5]), which allows calculations in which membership of points in sets plays no role as in Kuratowski's topological calculus—hence the title of this article.

Our approach also applies to *weakly separating families* in the sense of Chattopadhyay et al. [3] (Section 3) and *separating families* of Mrówka [9] (Section 4). In both the cases we shall again be able to control the amount of factors of the target product. We will observe that the concept of separating points from closed sets can be formulated purely in terms of open sets and, thus, makes sense in the theory of locales viewed as pointfree topologies (Section 5).

We start with a pointfree version of the concept of separating points from closed sets (cf. [5]). It is not difficult to check that the following holds:

Fact 2.1. A family $F \subseteq \bigcup_{j \in J} C(X, Y_j)$ separates points from closed sets if and only if $K = \bigcap_{f \in F} f^{-1}(\overline{f(K)})$ for all closed $K \subseteq X$.

We note that in the latter formula the inclusion $\subseteq$ always holds. The weight $w(X)$ of a space X is the smallest infinite cardinal such that X has a base of cardinality smaller or equal to $w(X)$. The following is the diagonal theorem for spaces of prescribed weight.

Theorem 2.2. *Let X be T_0 and let F be a family of continuous maps $f\colon X \to Y_f$ which separates points from closed sets. Then there exists a subfamily $F_0 \subseteq F$ with $|F_0| \leqslant w(X)$ and such that the evaluation map embeds X into $\prod_{f \in F_0} Y_f$.*

Proof. Let $\mathscr{H}$ be a closed base of X with $|\mathscr{H}| \leqslant w(X)$. By [1, 1.1.14], for each $H \in \mathscr{H}$ we can find an $F_H \subseteq F$ such that $|F_H| \leqslant w(X)$ and $H = \bigcap_{f \in F_H} f^{-1}(\overline{f(H)})$. Let $F_0 = \bigcup_{H \in \mathscr{H}} F_H$. Clearly, $|F_0| \leqslant w(X)$ (as $w(X)$ is infinite). To check that F_0 still separates points from closed sets, let $K \subseteq X$ be an arbitrary closed set and let $\mathscr{H}_K \subseteq \mathscr{H}$ be such that $K = \bigcap \mathscr{H}_K$. Since $K \subseteq H$ for all $H \in \mathscr{H}_K$, we have

$$K = \bigcap_{H \in \mathscr{H}_K} \bigcap_{f \in F_H} f^{-1}(\overline{f(H)}) \supseteq \bigcap_{f \in F_0} f^{-1}(\overline{f(K)}) \supseteq K.$$

Since F_0 separates points from closed sets and X is T_0, e is well known to be an embedding (cf. [1, Th. 2.3.20]). However, according to the goal of this paper we wish to calculate in order to show that e is an embedding. For this purpose, let $K \subseteq X$ be closed. We use Fact 2.1 for F_0 and *calculate* in order to show that e is a closed map from X onto $e(X)$:

$$\begin{aligned} e(K) &= \bigcap_{f \in F_0} e((\pi_f \circ e)^{-1}(\overline{(\pi_f \circ e)(K)})) \\ &\supseteq \bigcap_{f \in F_0} e(e^{-1}(\pi_f^{-1}(\pi_f(\overline{e(K)})))) \\ &\supseteq e(e^{-1}(\overline{e(K)})) \\ &= \mathrm{Cl}_{e(X)} e(K). \end{aligned}$$

□

Remark 2.3. Regarding the embeding theorem of Mrówka (see Section 1) we note that—by Theorem 2.2—a T_0-space X is embeddable into $Y^{w(X)}$ whenever $C(X,Y)$ separates points and points from closed sets. This is enough for the typical embedding theorems (cf. [1]).

3. Weak separation of points from closed sets

When $F \subseteq C(X,[0,1])$ (with X being a Tychonoff space) separates points from closed sets, then $e\colon X \to [0,1]^{|F|}$ is an embedding. However, the converse need not be true: if e is an embedding, then F need not separate points from closed sets. In order to have an iff criterion Chattopadhyay et al. [3] introduced the concept of a weak separation of points from closed sets and characterized those subsets $F \subseteq C(X,[0,1])$ for which e is an embedding. In this section we formulate their concept without involving points so that we will be able to enrich the main result of [3] with the prescribed amount of factors of the target product space depending on the weight of the embeddable space.

Definition 3.1 ([3, Def. 1.1])**.** A family F of continuous maps $f\colon X \to Y_f$ (where X and Y_f are topological spaces) is said to *weakly separate* points from closed sets if for each closed set $K \subseteq X$ and each $x \in X \setminus K$ there exists a finite cover $\mathscr{C} \subseteq \mathscr{P}(X)$ of K such that for all $A \in \mathscr{C}$ there exists an $f \in F$ satisfying $f(x) \notin \overline{f(A)}$.

Let $\mathrm{Cov}_{\mathrm{fin}}(K)$ be the collection of all finite covers of K.

Lemma 3.2. *A family F of continuous maps $f\colon X \to Y_f$ weakly separates points from closed sets iff*

$$K = \bigcap_{\mathscr{C}\in\mathrm{Cov}_{\mathrm{fin}}(K)} \bigcup_{A\in\mathscr{C}} \bigcap_{f\in F} f^{-1}(\overline{f(A)}) \tag{WS}$$

for each closed $K \subseteq X$.

P r o o f. We first observe that the inclusion $\subseteq$ always holds. Indeed, let $\mathscr{C}$ be a finite cover of K. We have $A \subseteq \bigcap_{f\in F} f^{-1}(\overline{f(A)})$ for each $A \in \mathscr{C}$, so that $K \subseteq \bigcup_{A\in\mathscr{C}} \bigcap_{f\in F} f^{-1}(\overline{f(A)})$. Since $\mathscr{C} \in \mathrm{Cov}_{\mathrm{fin}}(K)$ is arbitrary we get that $K \subseteq \bigcap_{\mathscr{C}\in\mathrm{Cov}_{\mathrm{fin}}(K)} \bigcup_{A\in\mathscr{C}} \bigcap_{f\in F} f^{-1}(\overline{f(A)})$. Now it is easy to see that the inclusion

$$\bigcap_{\mathscr{C}\in\mathrm{Cov}_{\mathrm{fin}}(K)} \bigcup_{A\in\mathscr{C}} \bigcap_{f\in F} f^{-1}(\overline{f(A)}) \subseteq K$$

says precisely that F weakly separates any $x \in X \setminus K$ from K. □

Notation. Given $\mathscr{C} \in \mathrm{Cov}_{\mathrm{fin}}(K)$ and F, we put

$$\Delta(\mathscr{C}, F) = \bigcup_{A \in \mathscr{C}} \bigcap_{f \in F} f^{-1}(\overline{f(A)}).$$

Lemma 3.3. *Let the family F of continuous maps $f\colon X \to Y_f$ weakly separate points from closed sets. Then F has a subfamily $F_0 \subseteq F$ which weakly separates points from closed subsets and is such that $|F_0| \leqslant w(X)$.*

Proof. Let $\mathscr{H}$ be a closed base of X such that $|\mathscr{H}| \leqslant w(X)$. By [1, 1.1.14], we will reduce the cardinalities of the involved families in two steps. First, for each closed $H \in \mathscr{H}$ we select $\mathscr{C}_H \subseteq \mathrm{Cov}_{\mathrm{fin}}(H)$ such that $|\mathscr{C}_H| \leqslant w(X)$ and $H = \bigcap\limits_{\mathscr{C} \in \mathscr{C}_H} \Delta(\mathscr{C}, F)$.

Next, for a given H and for each $A \in \mathscr{C} \in \mathscr{C}_H$ we select $F_A \subseteq F$ such that $|F_A| \leqslant w(X)$ and $\bigcap\limits_{f \in F} f^{-1}(\overline{f(A)}) = \bigcap\limits_{f \in F_A} f^{-1}(\overline{f(A)})$. Take $F_{\mathscr{C}} = \bigcup\limits_{A \in \mathscr{C}} F_A$. Then

$$H \supseteq \bigcap_{\mathscr{C} \in \mathscr{C}_H} \Delta(\mathscr{C}, F_{\mathscr{C}}).$$

Then the family

$$F_0 = \bigcup_{H \in \mathscr{H}} \bigcup_{\mathscr{C} \in \mathscr{C}_H} F_{\mathscr{C}} = \bigcup_{H \in \mathscr{H}} \bigcup_{\mathscr{C} \in \mathscr{C}_H} \bigcup_{A \in \mathscr{C}} F_A$$

is such that $|F_0| \leqslant w(X)$ and weakly separates points from closed sets. Indeed, let K be an arbitrary closed subset of X. Then $K = \bigcap \mathscr{K}$ where $\mathscr{K} \subseteq \mathscr{H}$. For all $H \in \mathscr{K}$ we have $\mathscr{C}_H \subseteq \mathrm{Cov}_{\mathrm{fin}}(H) \subseteq \mathrm{Cov}_{\mathrm{fin}}(K)$ and we can write

$$H \supseteq \bigcap_{\mathscr{C} \in \mathscr{C}_H} \Delta(\mathscr{C}, F_{\mathscr{C}}) \supseteq \bigcap_{\mathscr{C} \in \mathrm{Cov}_{\mathrm{fin}}(K)} \Delta(\mathscr{C}, F_0).$$

Thus $K \supseteq \bigcap\limits_{\mathscr{C} \in \mathrm{Cov}_{\mathrm{fin}}(K)} \Delta(\mathscr{C}, F_0)$. We have already noticed that the opposite inclusion always holds, so the proof is complete. $\square$

After Lemma 3.3 we can again control the amount of factors of the target product according to the weight of the embeddable space. Thanks to the pointfree formula for weak separation the proof is again a *calculation* line by line. Before stating our enrichment of [3, Thm. 1.2], we first check that if F is weakly separating and X is T_0, then F separates points. Indeed, assume $x \neq y$. If K is closed, $x \notin K$ and $y \in K$, say, then there is a finite cover $\mathscr{C}$ of K such that for all $A \in \mathscr{C}$ there is an $f \in F$ with $f(x) \notin \overline{f(A)}$. Let $B \in \mathscr{C}$ be such that $y \in B$. Then $f(x) \notin \overline{f(B)} \ni f(y)$, i.e. $f(x) \neq f(y)$.

Theorem 3.4. *Let F be a family of continuous functions $f\colon X \to Y_f$ where X is a T_0-space and Y_f are arbitrary topological spaces. If F weakly separates points from closed sets, then the evaluation map embeds X into $\prod\limits_{f\in F_0} Y_f$ where $F_0 \subseteq F$ is such that $|F_0| \leqslant w(X)$.*

Proof. By Lemma 3.3, we can find $F_0 \subseteq F$ which satisfies the condition (WS). Since e is injective, we only need to check that the evaluation map e is closed. Let $f \in F_0$ and $A \subseteq X$. We calculate:

$$\begin{aligned} f^{-1}(\overline{f(A)}) &= (\pi_f \circ e)^{-1}(\overline{(\pi_f \circ e)(A)}) \\ &\supseteq e^{-1}(\pi_f^{-1}(\pi_f(\overline{e(A)}))) \\ &\supseteq e^{-1}(\overline{e(A)}), \end{aligned}$$

so that $\bigcap\limits_{f\in F_0} f^{-1}(\overline{f(A)}) \supseteq e^{-1}(\overline{e(A)})$ and hence $K \supseteq \bigcap\limits_{\mathscr{C}\in \mathrm{Cov}_{\mathrm{fin}}(K)} \bigcup\limits_{A\in\mathscr{C}} e^{-1}(\overline{e(A)})$.

We calculate further as follows:

$$\begin{aligned} e(K) &\supseteq \bigcap_{\mathscr{C}\in \mathrm{Cov}_{\mathrm{fin}}(K)} e\Big(\bigcup_{A\in\mathscr{C}} e^{-1}(\overline{e(A)})\Big) \\ &\supseteq e(e^{-1}(\overline{e(K)})) \\ &= \mathrm{Cl}_{e(X)} e(K), \end{aligned}$$

since e maps X onto $e(X)$. □

4. Separating families in the sense of Mrówka

To treat Mrówka's approach (see [9]), we need more notation. If $f\colon X \to Y_f$ is continuous for all $f \in \mathbb{F}$, then the evaluation map associated with $F \subseteq \mathbb{F}$ will be written as $e_F\colon X \to \prod\limits_{f\in F} Y_f$.

Definition 4.1 ([9, Def. 2.2]). The family $\mathbb{F}$ is called separating if for each closed set $K \subseteq X$ and each $x \in X \setminus K$ there exists a finite subfamily $F \subseteq \mathbb{F}$ such that $e_F(x) \notin \overline{e_F(K)}$.

Let $\mathbb{F}_{\mathrm{fin}}$ denote the family of all finite subsets of $\mathbb{F}$. It should already be clear that a family $\mathbb{F}$ is separating iff for all closed $K \subseteq X$ we have

$$K = \bigcap_{F\in\mathbb{F}_{\mathrm{fin}}} e_F^{-1}(\overline{e_F(K)}). \tag{S}$$

Remark 4.2. It follows that $\mathbb{F}$ is separating iff $\{e_F\colon F \in \mathbb{F}_{\mathrm{fin}}\}$ separates points from closed sets. Also, if X is T_0 and $\mathbb{F}$ is separating, then $\mathbb{F}$ separates points. Consequently, if $\mathbb{F}$ is separating and X is T_0, then—by Theorem 2.2—X embeds into $\prod\limits_{F\in\mathbb{F}_{\mathrm{fin}}}\Big(\prod\limits_{f\in F} Y_f\Big)$. In particular, if $Y_f = Y$ for all $f \in \mathbb{F}$, then X embeds into $Y^{w(X)}$ (cf. the theorem of Mrówka stated in Section 1).

The following result improves and extends the just mentioned Mrówka's theorem provided X is T_0. It will be convenient to denote the evaluation map from X into $\prod\limits_{f\in F} Y_f$ by e_F.

Theorem 4.3. *Let $\mathbb{F}$ be a separating family of functions $f\colon\ X \to Y_f$ where X is a T_0-space and Y_f are arbitrary topological spaces for all $f \in \mathbb{F}$. Then the evaluation map embeds X into $\prod\limits_{f\in\mathbb{F}_0} Y_f$ where $\mathbb{F}_0 \subseteq \mathbb{F}$ is such that $|\mathbb{F}_0| \leqslant w(X)$.*

P r o o f. By Lemma 3.3, we can find $\mathbb{F}_0 \subseteq \mathbb{F}$ which satisfies the condition (S). For each $F \subseteq \mathbb{F}_0$ let $\pi_F\colon \prod\limits_{f\in\mathbb{F}_0} Y_f \to \prod\limits_{f\in F} Y_f$ be the (continuous) projection determined by $e_F = \pi_F \circ e_{\mathbb{F}_0}$. We can now continue precisely as in the proof of Theorem 2.2 replacing f and F by F and $\mathbb{F}$, respectively. $\square$

5. Separating points from closed sets in terms of open sets

Let $\mathscr{O}X$ stand for the topology of X. One can go further in dismantling the concept of separating points from closed sets not only from points but also from closed sets and formulate it purely in terms of open sets. Namely, if in $K = \bigcap\limits_{f\in F} f^{-1}(\overline{f(K)})$ we put $K = X \setminus U$ and take complements on both sides, then we obtain

$$U = \bigcup_{f\in F} f^{-1}(Y \setminus \overline{f(X \setminus U)}).$$

Further, $Y \setminus \overline{f(X \setminus U)} = f_*(U)$ where $f_*\colon\ \mathscr{O}X \to \mathscr{O}Y$ is the right adjoint of the inverse image map $f^{-1}\colon\ \mathscr{O}Y \to \mathscr{O}X$, i.e.

$$f_*(U) = \bigcup\{V \in \mathscr{O}Y\colon\ f^{-1}(V) \subseteq U\}$$

(see [8]). Consequently, F separates points from closed sets iff for each open $U \subseteq X$ the following formula holds:

$$U = \bigcup_{f\in F} f^{-1}(f_*(U)).$$

This is an invitation to both localic and many-valued setting (cf. [2] and [6]).

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