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A NOTE ON TOPOLOGICAL GROUPS AND THEIR REMAINDERS

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Abstract. In this note we first give a summary that on property of a remainder of a non-locally compact topological group G in a compactification bG makes the remainder and the topological group G all separable and metrizable.

If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ of G belongs to $\mathscr{P}$, then G and $bG \setminus G$ are separable and metrizable, where $\mathscr{P}$ is a class of spaces which satisfies the following conditions:
(1) if $X \in \mathscr{P}$, then every compact subset of the space X is a G_δ-set of X;
(2) if $X \in \mathscr{P}$ and X is not locally compact, then X is not locally countably compact;
(3) if $X \in \mathscr{P}$ and X is a Lindelöf p-space, then X is metrizable.

Some known conclusions on topological groups and their remainders can be obtained from this conclusion. As a corollary, we have that if a non-locally compact topological group G has a compactification bG such that compact subsets of $bG \setminus G$ are G_δ-sets in a uniform way (i.e., $bG\setminus G$ is CSS), then G and $bG\setminus G$ are separable and metrizable spaces.

In the last part of this note, we prove that if a non-locally compact topological group G has a compactification bG such that the remainder $bG\setminus G$ has a point-countable weak base and has a dense subset D such that every point of the set D has countable pseudo-character in the remainder $bG\setminus G$ (or the subspace D has countable π-character), then G and $bG\setminus G$ are both separable and metrizable.

Keywords: topological group, remainder, compactification, metrizable space, weak base

MSC 2010: 54A25, 54B05

1. Introduction

All spaces in this note are Tychonoff spaces unless stated otherwise, a "compactification" is a "Hausdorff compactification". A *remainder* of a space X is the subspace $bX \setminus X$ of a compactification bX of X.

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In 1958, M. Henriksen and J. R. Isbell [15] showed that a space X is of countable type if and only if the remainder in any (or in some) compactification of X is Lindelöf. In recent years, there are many results on topological groups and their remainders. In 2005, A. V. Arhangel'skii [2] showed that if a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ has a G_δ-diagonal, then G is metrizable. In 2007, A. V. Arhangel'skii [3] obtained that both G and $bG \setminus G$ are separable and metrizable if G is a non-locally compact topological group and has a compactification bG such that the remainder $bG \setminus G$ has a G_δ-diagonal. Some other results on a topological group and its remainder can be found in [4], [5], [6], [7], and [18].

Most of the known results on topological groups and their remainders study the relationship between properties of topological groups and their remainders. In this note, we give a summary on what property of a remainder of a non-locally compact topological group G in a compactification bG makes the remainder $bG \setminus G$ and G all separable and metrizable. The following is a result on it.

If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ belongs to $\mathscr{P}$, then G and $bG \setminus G$ are separable and metrizable, where $\mathscr{P}$ is a class of spaces which satisfies the following conditions:

(1) if $X \in \mathscr{P}$, then every compact subset of the space X is a G_δ-set of X;

(2) if $X \in \mathscr{P}$ and X is not locally compact, then X is not locally countably compact;

(3) if $X \in \mathscr{P}$ and X is a Lindelöf p-space, then X is metrizable.

Some known conclusions on topological groups and their remainders can be obtained from this conclusion. As a corollary, we have that if a non-locally compact topological group G has a compactification bG such that compact sets of $bG \setminus G$ are G_δ-sets in a uniform way (i.e., $bG \setminus G$ is CSS), then G and $bG \setminus G$ are separable and metrizable spaces.

In [7] Arhangel'skii showed that if G is a non-locally compact topological group, and the remainder of G in a compactification bG is the union of a finite collection of hereditarily D-spaces each of which is first countable (of countable π-character) at a dense set of points, then G is metrizable. In [21] Peng proved that a space with a point-countable weak base is a D-space. So we will study the property of a non-locally compact topological group G which has a compactification bG such that the remainder $bG \setminus G$ has countable tightness and is the union of a finite collection of spaces with point-countable weak bases. The following question appears in [19].

Let G be a non-locally compact topological group, if the remainder $Y = bG \setminus G$ of G in a compactification bG of G has a point-countable weak base, are G and bG separable and metrizable ([19, Question 5.2])?

In the last part of this note, we prove that if a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset D such that every point of the set D has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace D has countable π-character), then G and $bG \setminus G$ are both separable and metrizable; if a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ has countable tightness and is the union of a finite collection $\{X_i\colon\ i \leqslant n\}$ of spaces such that X_i has a point-countable weak base and has a dense subspace D_i which has countable π-character for each $i \leqslant n$, then G is metrizable.

The set of all positive integers is denoted by $\mathbb{N}$, and ω is $\mathbb{N} \cup \{0\}$. In notions and terminology we will follow [11], [13], and [26].

2. On remainders of metrizable spaces

Recall that a space X is of *countable type* if every compact subset P of X is contained in a compact subset $F \subset X$ that has a countable base of open neighborhoods in X [1]. All metrizable spaces, and all locally compact Hausdorff spaces, as well as all Čech-complete spaces are of countable type [1].

Recall that a space X is a *p-space* [1], if in any (or in some) compactification bX of X there exists a countable family $\{\mathscr{U}_n\}_{n\in\mathbb{N}}$ of families $\mathscr{U}_n$ of open subsets of bX such that $x \in \bigcap\limits_{n\in\mathbb{N}} \operatorname{st}(x, \mathscr{U}_n) \subset X$ for each $x \in X$. It was shown in [1] that every p-space is of countable type, and that every metrizable space is of countable type. A. V. Arhangel'skii [1] proved that a paracompact p-space is a preimage of a metrizable space under a perfect mapping. A *Lindelöf p-space* is a preimage of a separable and metrizable space under a perfect mapping. A mapping is said to be *perfect* if it is continuous, closed and all fibers are compact.

Lemma 2.1 ([15]). *A space X is of countable type if and only if the remainder in any (or in some) compactification of X is Lindelöf.*

Recall that a space X is said to have a *G_δ-diagonal* if the diagonal $\Delta_X = \{(x, x)\colon x \in X\}$ is the intersection of countably many of open subsets of $X \times X$. A countably compact space X with a G_δ-diagonal is metrizable [9].

Lemma 2.2 ([13]). *A Lindelöf p-space with a G_δ-diagonal is separable and metrizable.*

Proposition 2.3. *Let X be a locally separable meta-Lindelöf space, then $X = \bigoplus\limits_{\alpha\in\Lambda} X_\alpha$, where $\{X_\alpha\colon\ \alpha \in \Lambda\}$ is a discrete family of open separable subspaces of X.*

Lemma 2.4 ([2, Theorem 2.1]). *If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

By the proof of the last part of Theorem 5 in [3], we can get Theorem 2.5. To assist the reader, we give a proof.

Theorem 2.5. *If a nowhere locally compact locally separable metrizable space X has a compactification bX such that every compact subset of the remainder $bX \setminus X$ is a G_δ-set of $bX \setminus X$ and every Lindelöf p-subspace of the remainder $bX \setminus X$ is metrizable, then X and $bX \setminus X$ are separable and metrizable.*

P r o o f. Since X is a locally separable metrizable space, $X = \bigoplus_{\alpha\in\Lambda} X_\alpha$ by Proposition 2.3, where X_α is separable and metrizable for each $\alpha \in \Lambda$. If F is the set of all accumulation points for the family $\{X_\alpha\colon \alpha \in \Lambda\}$ in bX, then the set F is a closed subset of bX and $F \subset bX \setminus X$. Thus F is a compact subset of bX.

Since every compact subset of the remainder $bX \setminus X$ is a G_δ-set of $bX \setminus X$ and every Lindelöf p-subspace of the remainder $bX \setminus X$ is metrizable, the subspace F is a G_δ-set of $bX \setminus X$ and is separable and metrizable.

Put $F = \bigcap\{O_n\colon n \in \mathbb{N}\}$, where O_n is an open subset of $bX \setminus X$ for each $n \in \mathbb{N}$. Denote $M = (bX \setminus X) \setminus F = \bigcup\{A_n\colon n \in \mathbb{N}\}$, where $A_n = (bX \setminus X) \setminus O_n$ for each $n \in \mathbb{N}$. Thus the set A_n is a closed subset of $bX \setminus X$. X is a metrizable space, hence X is of countable type. So $bX \setminus X$ is Lindelöf by Lemma 2.1. Thus the subspace A_n is Lindelöf for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ and for each $y \in A_n$, there exists an open subset U_y of bX such that $y \in U_y$ and $\overline{U_y} \cap F = \emptyset$. So there exists $m_y \in \mathbb{N}$ such that $U_y \cap X = \bigcup\{U_y \cap X_{\alpha_i}\colon i \leqslant m_y, \alpha_i \in \Lambda\}$. If we let $P = \bigcup\{X_{\alpha_i}\colon i \leqslant m_y, \alpha_i \in \Lambda\}$, then $U_y \cap X \subset P$. Since $\overline{U_y \cap X} = \overline{U_y}$ and $\overline{U_y \cap X} \subset \overline{P}$, $\overline{U_y} \subset \overline{P}$.

The set P is a separable and metrizable subspace of X, hence $\overline{P} \setminus P$ is a Lindelöf p-space by Lemma 2.4. Thus $\overline{P} \setminus P$ is a separable and metrizable space, and so is the set $U_y \cap (bX \setminus X)$. Thus the subspace A_n is Lindelöf and every point of A_n has a neighborhood which has a countable base, hence the subspace A_n has a countable network for each $n \in \mathbb{N}$. The subspace $(bX \setminus X) \setminus F$ has a countable network and the subspace F has a countable network, so $bX \setminus X$ has a countable network. Thus $bX \setminus X$ is separable, hence the Souslin number of $bX \setminus X$ is countable. So the Souslin numbers of bX and X are both countable. Thus X is separable and metrizable. So $bX \setminus X$ is a Lindelöf p-space by Lemma 2.4. In addition, $bX \setminus X$ has a countable network, so it has a G_δ-diagonal. Thus $bX \setminus X$ is separable and metrizable by Lemma 2.2. □

Lemma 2.6. *If X is a regular space and X has a G_δ-diagonal, then every compact subset of X is a G_δ-set of X.*

Proof. X has a G_δ-diagonal, thus there is a sequence $\{\mathscr{U}_n\colon n\in\mathbb{N}\}$ of open covers of X such that for any distinct points x and y of X there is $n\in\mathbb{N}$ such that $x\notin \operatorname{st}(y,\mathscr{U}_n)$, where $\operatorname{st}(y,\mathscr{U}_n)=\bigcup\{U\colon y\in U \text{ and } U\in\mathscr{U}_n\}$. Let C be any compact subset of X. In what follows we show that the set C is a G_δ-set of X.

For each $m\in\mathbb{N}$ there are $n_m\in\mathbb{N}$ and an open subset $V(m,i)$ of X for each $i\leqslant n_m$ such that $C\subset\bigcup\{V(m,i)\colon i\leqslant n_m\}$, $V(m,i)\cap C\neq\emptyset$, and there are $U(m,i)\in\mathscr{U}_m$ and some $j\leqslant n_{m-1}$ such that $V(m,i)\subset U(m,i)$ and $\overline{V(m,i)}\subset V(m-1,j)$.

Suppose there is a point $y\in\bigcap\{\bigcup\{V(m,i)\colon i\leqslant n_m\}\colon m\in\mathbb{N}\}\setminus C$. For each $m\in\mathbb{N}$ and for each $j\leqslant m$ there is $i_j^m\leqslant n_j$ such that $y\in V(j,i_j^m)\subset\overline{V(j,i_j^m)}\subset V(j-1,i_{j-1}^m)$. Since $\{V(m,i)\colon i\leqslant n_m\}$ is a finite family for each $m\in\mathbb{N}$, there is $i_m\leqslant n_m$ such that $y\in V(m,i_m)\subset\overline{V(m,i_m)}\subset V(m-1,i_{m-1})$ for each $m\in\mathbb{N}$ by König's Lemma. Thus $\bigcap\{V(m,i_m)\colon m\in\mathbb{N}\}\cap C=\bigcap\{\overline{V(m,i_m)}\colon m\in\mathbb{N}\}\cap C\neq\emptyset$. Let $x\in\bigcap\{V(m,i_m)\colon m\in\mathbb{N}\}\cap C$. Since the point $y\in V(m,i_m)$ and $V(m,i_m)\subset U(m,i_m)$ for each $m\in\mathbb{N}$, $y\in\operatorname{st}(x,\mathscr{U}_m)$ for each $m\in\mathbb{N}$. This contradicts $y\notin\bigcap\{\operatorname{st}(x,\mathscr{U}_m)\colon m\in\mathbb{N}\}$.

So $\bigcap\{\bigcup\{V(m,i)\colon i\leqslant n_m\},m\in\mathbb{N}\}=C$, hence C is a G_δ-set of X. □

By Lemma 2.2, Theorem 2.5, and Lemma 2.6, we get a corollary.

Corollary 2.7. *Let X be a nowhere locally compact locally separable metrizable space. If X has a compactification bX such that the remainder $bX\setminus X$ has a G_δ-diagonal, then both X and $bX\setminus X$ are separable and metrizable.*

Lemma 2.8 ([3, Proposition 4]). *Let X be a nowhere locally separable metrizable space and let bX be a compactification of X. If $\mathscr{B}=\bigcup\{\mathscr{B}_n\colon n\in\omega\}$ is a base of X such that each family $\mathscr{B}_n$ is discrete in X, then $Z=\bigcup\{F_n\colon n\in\omega\}$ is dense in $Y=bX\setminus X$ and F_n is compact for each n, where F_n is the set of all accumulation points for $\mathscr{B}_n$ in bX for each n.*

Let us recall that a topological space X is *homogeneous* if for any two points $a,b\in X$ there exists a homeomorphism $f\colon X\to X$ such that $f(a)=b$.

Theorem 2.9. *Let X be a nowhere locally compact homogeneous metrizable space and let bX be a compactification of X such that every compact subset of the remainder $Y=bX\setminus X$ is metrizable, then X is locally separable.*

Proof. Suppose X is not locally separable. Since X is homogeneous, the space X is nowhere locally separable if X is not locally separable. X is a metrizable space, there exists a σ-discrete base $\mathscr{B} = \bigcup\{\mathscr{B}_n\colon n \in \mathbb{N}\}$ of X. For each $n \in \mathbb{N}$, denote by F_n the set of all accumulation points for $\mathscr{B}_n$ in bX. The set F_n is a closed subset of bX and $F_n \subset Y$. The set F_n is a compact subset of $bX \setminus X$, so F_n is separable and metrizable. Thus $bX \setminus X$ is separable by Lemma 2.8. Thus the Souslin number of bX and X are all countable, and hence X is separable. A contradiction. Thus X is locally separable. □

3. A general result on topological groups and their remainders

By some known conclusions, we will give a more general result on the metrizable property of a non-locally compact topological group G and its remainder.

By the proof of Case 2 of Theorem 4.19 in [2], we can get the following lemma.

Lemma 3.1. *Let G be a non-locally compact topological group. If G is a paracompact p-space and has a compatification bG such that every compact subset of the remainder $Y = bG \setminus G$ is metrizable, then G is a metrizable space.*

Theorem 3.2. *Let G be a non-locally compact topological group. If G is a paracompact p-space and has a compatification bG such that every compact subset of the remainder $Y = bG \setminus G$ is metrizable, then G is a locally separable and metrizable space.*

Proof. G is a metrizable space by Lemma 3.1. By Proposition 1.1 in [26] every topological group G is homogeneous. Thus G is a locally separable by Theorem 2.9. □

Recall that a family $\mathscr{U}$ of non-empty open subsets of a space X is called a π-*base* of a point $x \in X$, if for any non-empty open subset V of X there is $U \in \mathscr{U}$ such that $U \subset V$. The π-character of x in X is defined by $\pi_\chi(x, X) = \min\{|\mathscr{U}|\colon \mathscr{U}$ is a π-base of the point $x\}$. If $\sup\{\pi_\chi(x, X)\colon x \in X\}$ is countable, then X is called to have *countable π-character*.

Lemma 3.3. *Let Y be a dense subspace of a regular space X. If the subspace Y is first countable (or has countable π-character), then every point of Y has a countable open neighborhood base (or has a countable π-base) in X, and if x is an accumulation point of a countable subset C of Y then the point x has a countable π-base in X.*

Proof. We only prove the case of the space Y being first countable. The proof of the case that Y has countable π-character is similar.

For any $y \in Y$ we let $\{V_n(y)\colon\ n \in \mathbb{N}\}$ be a countable open neighborhood base of the point y in Y. For each $n \in \mathbb{N}$ there is an open neighborhood $U_n(y)$ of y in X such that $U_n(y) \cap Y = V_n(y)$. If O is an open neighborhood of the point y in X, then there is an open subset O_1 of X such that $y \in O_1 \subset \overline{O_1} \subset O$ by the regularity property of X. So there is $n \in \mathbb{N}$ such that $y \in V_n(y) \subset O_1$, hence $\overline{V_n(y)} \subset \overline{O_1} \subset O$. Since $\overline{V_n(y)} = \overline{U_n(y)}$, the set $\overline{U_n(y)} \subset \overline{O_1} \subset O$. Thus $\{U_n(y)\colon\ n \in \mathbb{N}\}$ is a countable open neighborhood base of the point y in X.

Let x be an accumulation point of a countable subset C of Y. If W is an open neighborhood of the point x in X, then there are $y \in C$ and $n \in \mathbb{N}$ such that $y \in U_n(y) \subset W$. So $\{U_n(y)\colon\ n \in \mathbb{N},\ y \in C\}$ is a countable π-base of the point x in X. $\square$

Recall that a point x of a space X is said to have *countable pseudo-character in X* if the set $\{x\}$ is the intersection of countably many open subsets of X. A space X is said to have *countable pseudo-character*, if every point of X has countable pseudo-character in X.

Lemma 3.4 ([4, Theorem 5.1]). *Suppose that G is a topological group with a remainder of countable pseudo-character. Then at least one of the following conditions is satisfied:*
(1) *G is a paracompact p-space;*
(2) *the remainder $bG \setminus G$ is first countable.*

Lemma 3.5 ([6, Proposition 1.3]). *Let G be a topological group. If some point of G has a countable π-base, then G is metrizable.*

Lemma 3.6. *If a non-locally compact topological group G has a compatification bG such that the remainder $Y = bG \setminus G$ has countable pseudo-character, Y is not locally countably compact, and every compact subset of Y is metrizable, then G is a locally separable and metrizable space.*

P r o o f. By Lemma 3.4 G is a paracompact p-space or the remainder $bG \setminus G$ is first countable. If G is a paracompact p-space, then G is a locally separable and metrizable space by Theorem 3.2. Since Y is not locally countably compact, the space Y is not countably compact. There is a countable infinite subset $C \subset Y$ such that $\overline{C} \cap G \neq \emptyset$. If the remainder $bG \setminus G$ is first countable and $x \in \overline{C} \cap G$, then the point x has a countable π-base in bG by Lemma 3.3, hence the point x has a countable π-base in G. Thus G is metrizable by Lemma 3.5. So G is a locally separable and metrizable space by Theorem 3.2 $\square$

Theorem 3.7. *If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ belongs to $\mathscr{P}$, then G and $bG \setminus G$ are separable and metrizable spaces, where $\mathscr{P}$ is a class of spaces which satisfies the following conditions:*

(1) *if $X \in \mathscr{P}$, then every compact subset of the space X is a G_δ-set of X;*
(2) *if $X \in \mathscr{P}$ and X is not locally compact, then X is not locally countably compact;*
(3) *if $X \in \mathscr{P}$ and X is a Lindelöf p-space, then X is metrizable.*

Proof. Since $bG \setminus G$ has property $\mathscr{P}$, every compact subset of $bG \setminus G$ is a G_δ-set of $bG \setminus G$ by the condition (1) and is metrizable by the condition (3). The remainder $Y = bG \setminus G$ is not locally compact, thus it is not locally countably compact by the condition (2). By the condition (1) the remainder $Y = bG \setminus G$ has countable pseudo-character. So the conditions of Lemma 3.6 are satisfied, hence G is locally separable and metrizable. Thus G and $bG \setminus G$ are separable and metrizable spaces by Theorem 2.5. □

A space X is said to have a *locally G_δ-diagonal* if every point x of X has a neighborhood V_x which has a G_δ-diagonal.

Lemma 3.8. *If X has a locally G_δ-diagonal, then every compact subset of X is a G_δ-set of X.*

Proof. Let C be any compact subset of X. For each $x \in C$ there is an open neighborhood V_x of x such that V_x has a G_δ-diagonal. The set C is compact, there are some $n \in \mathbb{N}$ and a point x_i for each $i \leqslant n$ such that $C \subset \bigcup\{V_{x_i}\colon i \leqslant n\} = Y$. Since $\mathscr{P} = \{V_{x_i}\colon i \leqslant n\}$ is a finite open cover of the subspace Y and each element of $\mathscr{P}$ has a G_δ-diagonal, the subspace Y has a G_δ-diagonal by Lemma 11 in [18].

Thus the set C is a G_δ-set of Y by Lemma 2.6, hence it is a G_δ-set of X. □

In what follows, we denote $\mathscr{P}$ by a class of spaces which satisfies the conditions appearing in Theorem 3.7. By Lemma 2.2 and 2.6, we know that if a space X has a G_δ-diagonal then $X \in \mathscr{P}$. By Lemma 11 in [18] we know that if X is a regular Lindelöf space with a locally G_δ-diagonal then X has a G_δ-diagonal. Thus by Lemma 2.2 in this note we know that every Lindelöf p-space with a locally G_δ-diagonal is metrizable. If a space X has a locally G_δ-diagonal and X is not locally compact then X is not locally countably compact. By these conclusions and Lemma 3.8 we have that $X \in \mathscr{P}$ if X has a locally G_δ-diagonal.

Corollary 3.9 ([3, Theorem 5])**.** *If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ has a G_δ-diagonal, then G and $bG \setminus G$ are separable and metrizable.*

Corollary 3.10 ([6, Theorem 2.17; 18, Theorem 12]). *If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ has a locally G_δ-diagonal, then G and $bG \setminus G$ are separable and metrizable.*

In 1973, H. Martin introduced the class of CSS spaces [20]. Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{C}$ be the family of all non-empty compact subsets of X. If there exists a function $U\colon \mathbb{N} \times \mathscr{C} \longrightarrow \mathscr{T}$ such that:

(1) for every $C \in \mathscr{C}$, $C = \bigcap\{U(n, C)\colon n \in \mathbb{N}\}$ and $U(n+1, C) \subset U(n, C)$ for $n \in \mathbb{N}$;

(2) if $D \in \mathscr{C}$, $C \in \mathscr{C}$, and $C \subset D$, then $U(n, C) \subset U(n, D)$ for each $n \in \mathbb{N}$.

Then X is called a *c-semi-stratifiable* (CSS) *space.*

It is obvious that every subspace of a CSS space is CSS.

Lemma 3.11 ([8, Proposition 3.8]). *If X is a* CSS *countably compact space, then X is a compact metrizable space.*

Lemma 3.12 ([8, Proposition 3.8]). *If X is a* CSS *paracompact p-space, then X is metrizable.*

Lemma 3.13 ([23, Theorem 4]). *If $X = \bigcup\{X_n\colon n \in \mathbb{N}\}$ and X_n is a closed* CSS *subspace of X for each $n \in \mathbb{N}$, then X is a* CSS *space.*

Theorem 3.14. *If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ is a locally* CSS *space, then G and $bG \setminus G$ are separable and metrizable spaces.*

P r o o f. By Lemma 3.11 we know that if a locally CSS space X is not a locally compact space then X is not locally countably compact. Every regular Lindelöf locally CSS space is a CSS space by Lemma 3.13. Thus a Lindelöf locally CSS p-space is metrizable by Lemma 3.12.

Let X be a locally CSS regular space and let F be a non-empty compact subset of X. For each $x \in F$ there is an open neighborhood V_x of x such that $\overline{V_x}$ is CSS. There are $n \in \mathbb{N}$ and a point $x_i \in F$ for each $i \leqslant n$ such that $F \subset \bigcup\{V_{x_i}\colon i \leqslant n\} \subset \bigcup\{\overline{V_{x_i}}\colon i \leqslant n\}$. By Lemma 3.13 the subspace $Y = \bigcup\{\overline{V_{x_i}}\colon i \leqslant n\}$ is CSS. Thus the set F is a G_δ-set of Y, hence it is a G_δ-set of X.

So a locally CSS space belongs to $\mathscr{P}$. Thus G and $bG \setminus G$ are separable and metrizable spaces. □

Recall that a space X has a *quasi-$G_\delta(2)$-diagonal* provided there is a sequence $\{\mathscr{U}_n\colon n \in \mathbb{N}\}$ of collections of open subsets of X with the property that, given distinct points $x, y \in X$, there is $n \in \mathbb{N}$ with $x \in \operatorname{st}^2(x, \mathscr{U}_n) \subset X \setminus \{y\}$.

Proposition 3.15 ([23, Theorem 9]). *If X has a quasi-$G_\delta(2)$-diagonal, then X is a* CSS *space.*

By Theorem 3.14 and Proposition 3.15, we can obtain:

Corollary 3.16. *If a non-locally compact topological group G has a compactification bG such that the remainder $bG \setminus G$ has a locally quasi-$G_\delta(2)$-diagonal, then G and $bG \setminus G$ are separable and metrizable spaces.*

In [3, Theorem 10], it was proved that if G is a non-locally compact topological group and has a compactification bG such that the remainder $bG \setminus G$ has a point-countable base, then G and $bG \setminus G$ are separable and metrizable. Every Lindelöf p-space with a point-countable base is metrizable [14]. Every countably compact space with a point-countable base is compact and metrizable [11]. We can get the following proposition.

Proposition 3.17. *If X is a space such that every point of X has an open neighborhood which has a point-countable base, then the following conclusions hold:*
(1) *X has a point-countable base if X is meta-Lindelöf;*
(2) *X is metrizable if X is a Lindelöf p-space;*
(3) *a subset C of X is a G_δ-set of X if the set C is a compact subset of X;*
(4) *X is not locally countably compact if X is not locally compact.*

Proof. We just need to prove the item (3). Let C be a compact subset of X. For each $x \in C$ there is an open neighborhood V_x of x such that the subspace V_x has a point-countable base. There are $n \in \mathbb{N}$ and a point $x_i \in C$ for each $i \leqslant n$ such that $C \subset \bigcup\{V_{x_i} \colon i \leqslant n\}$. If $Y = \bigcup\{V_{x_i} \colon i \leqslant n\}$, then the subspace Y has a point-countable base $\mathscr{B}$. Thus C is metrizable. The subspace C is separable, since C is compact and metrizable. Let D be a countable dense subset of C. Thus $\mathscr{B}' = \{B \colon B \in \mathscr{B} \text{ and } B \cap C \neq \emptyset\}$ is countable. So $C = \bigcap\{\bigcup \mathscr{F} \colon \mathscr{F} \subset \mathscr{B}',\ C \subset \bigcup \mathscr{F}, \text{ and } |\mathscr{F}| < \omega\}$, and hence C is a G_δ-set of X. □

By Proposition 3.17 and Theorem 3.7, we have:

Theorem 3.18. *If G is a non-locally compact topological group and has a compactification bG such that every point of the remainder $bG \setminus G$ has a neighborhood in $bG \setminus G$, which has a point-countable base, then G and $bG \setminus G$ are separable and metrizable.*

Corollary 3.19 ([3, Theorem 10]). *If G is a non-locally compact topological group and has a compactification bG such that the remainder $bG \setminus G$ has a point-countable base, then G and $bG \setminus G$ are separable and metrizable.*

4. Results on some remainders of topological groups with point-countable weak bases

In this part, we will mainly discuss the properties of a non-locally compact topological group G which has a compactification bG such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset D such that every point of the set D has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace D has countable π-character).

Let us recall the definition of a weak base of a space X. A collection $\mathscr{B} = \bigcup\{\mathscr{B}_x \colon x \in X\}$ is called a *weak base* [25] of X, if for any $x \in X$ the following conditions hold:

(1) for each $x \in X$, $\mathscr{B}_x$ is closed under finite intersections and $x \in \bigcap \mathscr{B}_x$;
(2) a subset U of X is open if and only if for any $x \in U$ there is $B \in \mathscr{B}_x$ such that $x \in B \subset U$.

Recall that a space X is *Fréchet* if for any point x is the closure of a subset A of X, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of A which converges to the point x. A space X is *sequential* if a subset A of X is closed if and only if the set A contains all the limit points of the convergent sequences of A. Let X be a topological space, for a subset $A \subset X$, denote $[A]_\omega = \bigcup\{\overline{C} \colon C \subset A \text{ and } |C| \leqslant \omega\}$. Recall that a space X has *countable tightness* if for any point x in the closure of a subset A of X, there is a countable subset $C \subset A$ such that $x \in \overline{C}$. We denote this by $t(X) \leqslant \omega$. It is well known that a Fréchet space is sequential and a sequential space has countable tightness.

Lemma 4.1 ([25, Theorem 1.10]). *If $\mathscr{B} = \bigcup\{\mathscr{B}_x \colon x \in X\}$ is a weak base of a Hausdorff Fréchet space X, then $\mathscr{B}^* = \{B^o \colon B \in \mathscr{B}\}$ is a base of X.*

Lemma 4.2 ([21, Corollary 8]). *If X is a countably compact Hausdorff space with a point-countable weak base, then X is a compact metrizable space.*

Lemma 4.3 ([17, Lemma 2.1]). *If $\mathscr{P} = \bigcup\{\mathscr{P}_x \colon x \in X\}$ is a weak base of a space X and F is a closed subset of X, then $\mathscr{P}' = \bigcup\{\mathscr{P}'_x \colon x \in F\}$ is a weak base of the subspace F, where $\mathscr{P}'_x = \{F \cap P \colon P \in \mathscr{P}_x\}$ for each $x \in F$.*

Proposition 4.4. *Let X be a T_1-space and let A be a subset of X. If $t(X) \leqslant \omega$, then the set $[A]_\omega$ is a closed subset of X; if X is countably compact, then the set $[A]_\omega$ is countably compact.*

Proof. Suppose $t(X) \leqslant \omega$. If $x \in \overline{[A]_\omega}$, then there is a countable subset $B \subset [A]_\omega$ such that $x \in \overline{B}$. For each $b \in B$ there is a countable set $C_b \subset A$ such that $b \in \overline{C_b}$. So $x \in \overline{\bigcup\{C_b \colon b \in B\}} \subset [A]_\omega$. Thus $[A]_\omega$ is a closed subset of X if $t(X) \leqslant \omega$.

Suppose X is countably compact. For any infinite countable subset B of $[A]_\omega$, there is a countable subset $C \subset A$ such that $B \subset \overline{C} \subset [A]_\omega$. Thus the set B has an accumulation point in $\overline{C}$, hence $[A]_\omega$ is countably compact if X is countably compact. □

Lemma 4.5. *Let G be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of G in a compactification bG of G such that $Y = bG \setminus G$ has countable tightness. If there is an open subset U of Y such that every closed countably compact subset which is contained in U is compact and there is a subspace $M \subset Y$ such that $U \subset \overline{M}^Y$ and M has a dense subspace D which has countable π-character, then G is metrizable.*

Proof. Let U_0 be an open subset of bG such that $U_0 \cap Y = U$ and let U_1 be an open subset of bG such that $\overline{U_1} \subset U_0$, hence $\overline{U_1} \cap Y \subset U$. If $D_1 = U_1 \cap D$, then the set D_1 is dense in the subspace U_1. Denote $[D_1]_\omega = \bigcup\{\overline{C} \colon C \subset D_1$ and $|C| \leqslant \omega\}$. By Proposition 4.4 the set $[D_1]_\omega$ is a countably compact subspace of bG. Suppose $[D_1]_\omega \cap G = \emptyset$, then $[D_1]_\omega \subset \overline{U_1} \cap Y \subset U$. By Proposition 4.4 the set $[D_1]_\omega$ is closed in the subspace Y. Since $[D_1]_\omega \subset U$, the set $[D_1]_\omega$ is compact. Thus $\overline{U_1} = \overline{D_1} \subset [D_1]_\omega \subset Y$. This contradicts $U_1 \cap G \neq \emptyset$, so $[D_1]_\omega \cap G \neq \emptyset$. If $x \in [D_1]_\omega \cap G$, then there is a countable subset $C \subset D_1$ such that $x \in \overline{C}$.

The set $U_0 \cap D$ is dense in U_0, since $U \subset \overline{M}^Y$ and D is a dense subset of M. The subspace $U_0 \cap D$ is an open subspace of D, the subspace $U_0 \cap D$ has countable π-character. Thus every point of $U_0 \cap D$ has a countable π-base in U_0 by Lemma 3.3. The point $x \in \overline{C} \subset \overline{U_1} \subset U_0$. For each $z \in C$ let $\mathscr{V}_z$ be a countable π-base of the point z in U_0. If $\mathscr{B} = \bigcup\{\mathscr{V}_z \colon z \in C\}$, then $\mathscr{B}$ is a countable family of open subsets of U_0. Thus $\{B \cap G \colon B \in \mathscr{B}\}$ is a countable π-base of the point x in G. Thus G is metrizable by Lemma 3.5. □

Corollary 4.6. *Let G be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of G in a compactification bG of G such that $Y = bG \setminus G$ has countable tightness. If there is a point $y \in Y$ and an open neighborhood $U(y)$ of y in Y such that every closed countably compact subset which is contained in $U(y)$ is compact and $U(y)$ has a dense subspace D which has countable π-character, then G is metrizable.*

By the proof of Theorem 5.1 in [4], we have:

Lemma 4.7. *If a non-locally compact topological group G has a compatification bG such that the remainder $Y = bG \setminus G$ has a point y which has countable pseudo-character in Y, then G is a paracompact p-space or the point y has a countable open neighborhood base in bG.*

Theorem 4.8. *Let G be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of G in a compactification bG of G such that every compact subset of Y is metrizable and Y has countable tightness. If there is a point $y \in Y$ and an open neighborhood $U(y)$ of y in Y such that every closed countably compact subset which is contained in $U(y)$ is compact and there is a dense subspace D of $U(y)$ such that every point of D has countable pseudo-character in Y (or the subspace D has countable π-character), then G is locally separable and metrizable.*

Proof. If there is a dense subspace D of $U(y)$ such that every point d of D has countable pseudo-character in Y, then G is a paracompact p-space or every point d of D has a countable open neighborhood base in bG by Lemma 4.7. If G is a paracompact p-space, then G is a locally separable and metrizable space by Lemma 3.2. If every point d of D has a countable open neighborhood base in bG, then the subspace D has countable π-character. If the subspace D has countable π-character, then G is metrizable by Corollary 4.6, hence G is a locally separable and metrizable space by Lemma 3.2. □

Recall that a *neighborhood assignment* for a space X is a function φ from X to the topology of the space X such that $x \in \varphi(x)$ for any $x \in X$. A space X is called a *D-space* if for any neighborhood assignment φ for X there exists a closed discrete subset D of X such that $X = \bigcup\{\varphi(d)\colon d \in D\}$ [10]. Every metrizable space is a D-space.

Lemma 4.9 ([12], [22]). *If X is a countably compact space that is the union of a countable family of D-spaces, then X is compact.*

Theorem 4.10. *Let G be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of G in a compactification bG of G such that $Y = bG \setminus G$ has countable tightness. If the remainder Y is the union of a countable family $\{X_i\colon i \in \mathbb{N}\}$ of D-spaces such that for each $i \in \mathbb{N}$ there is a dense subspace D_i of X_i such that the subspace D_i has countable π-character (or every point of D_i has countable pseudo-character in Y), then G is a paracompact p-space.*

P r o o f. If there is a dense subspace D_i of X_i such that every point of D_i has countable pseudo-character in Y for each $i \in \mathbb{N}$, then G is a paracompact p-space or every point y of D_i has a countable open neighborhood base in bG by Lemma 4.7.

If every point y of D_i has a countable open neighborhood base in bG for each $i \in \mathbb{N}$, then the subspace D_i has countable π-character. In what follows, we show that G is a paracompact p-space if there is a dense subspace D_i in X_i such that the subspace D_i has countable π-character for each $i \in \mathbb{N}$.

Since every closed subspace of a D-space is a D-space, every closed countably compact subspace of Y is compact by Lemma 4.9.

If there is some $i \in \mathbb{N}$ and an open subset U of Y such that $U \subset \overline{X_i}^Y$, then G is metrizable by Lemma 4.5, otherwise, X_i is a nowhere dense subset of Y for each $i \in \mathbb{N}$. For $i \in \mathbb{N}$, assuming that there is an open subset U_j of bG for each $j \leqslant i$ such that $\overline{U_j} \subset U_{j-1} (U_0 = bG)$, $U_j \subset bG \setminus \bigcup\{\overline{X_m}\colon\ m \leqslant j\}$, and $U_j \cap Y \neq \emptyset$.

The set $(U_i \setminus \overline{X_{i+1}}) \cap Y \neq \emptyset$, there is an open subset U_{i+1} of bG such that $\overline{U_{i+1}} \subset U_i$ and $U_{i+1} \cap Y \neq \emptyset$. Thus $U_{i+1} \subset bG \setminus \bigcup\{\overline{X_m}\colon\ m \leqslant i+1\}$. So we have a sequence $\{U_i\colon\ i \in \mathbb{N}\}$ of open subsets of bG such that $\overline{U_{i+1}} \subset U_i$ and $U_i \subset bG \setminus \bigcup\{\overline{X_m}\colon\ m \leqslant i\}$. Thus $E = \bigcap\{\overline{U_i}\colon\ i \in \mathbb{N}\} = \bigcap\{U_i\colon\ i \in \mathbb{N}\} \neq \emptyset$, and $E \subset G$. Thus the family $\{U_i \cap G\colon\ i \in \mathbb{N}\}$ is a countable base of open neighborhoods of the set E in G. Every topological group that contains a non-empty compact subset with a countable base of open neighborhoods is a paracompact p-space [24]. Thus G is a paracompact p-space. □

In [21] Peng proved that every space with a point-countable weak base is a D-space.

Corollary 4.11. *Let G be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of G in a compactification bG of G such that $Y = bG \setminus G$ has countable tightness. If the remainder Y is the union of a countable family $\{X_i\colon\ i \in \mathbb{N}\}$ of spaces such that for each $i \in \mathbb{N}$ the space X_i has a point-countable weak base and there is a dense subspace D_i of X_i such that the subspace D_i has countable π-character (or every point of D_i has countable pseudo-character in Y), then G is a paracompact p-space.*

Theorem 4.12. *Let G be a non-locally compact topological group and let $Y = bG \setminus G$ be the remainder of G in a compactification bG of G such that $Y = bG \setminus G$ has countable tightness. If the remainder Y is the union of a finite family $\{X_i\colon\ i \leqslant n\}$ of D-spaces such that for each $i \leqslant n$ there is a dense subspace D_i of X_i such that the subspace D_i has countable π-character, then G is metrizable.*

P r o o f. Since every closed subspace of a D-space is a D-space, every closed countably compact subspace of Y is compact by Lemma 4.9. Since the family $\{X_i\colon$

$i \leqslant n\}$ is finite and is a cover of $bG \setminus G$, there are an open subset $U \subset Y$ and some $i \leqslant n$ such that $U \subset \overline{X_i}$, hence G is metrizable by Lemma 4.5. □

Corollary 4.13. *Let G be a non-locally compact topological group and let $Y = bG\setminus G$ be the remainder of G in a compactification bG of G such that $Y = bG\setminus G$ has countable tightness. If the remainder Y is the union of a finite family $\{X_i\colon i \leqslant n\}$ of spaces such that for each $i \leqslant n$ the space X_i has a point-countable weak base and there is a dense subspace D_i of X_i such that the subspace D_i has countable π-character, then G is metrizable.*

Lemma 4.14. *Let G be a non-locally compact topological group, and bG be a compactification of G such that the remainder $bG \setminus G$ has a point-countable weak base and has a dense subset D such that every point of the set D has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace D has countable π-character), then G is locally separable and metrizable.*

P r o o f. A space with a point-countable weak base is sequential, hence it has countable tightness. Thus the remainder $bG \setminus G$ has countable tightness. If every point of D has countable pseudo-character in Y, then G is a paracompact p-space or every point y of D has a countable open neighborhood base in bG by Lemma 4.7.

By Lemma 4.2 and Lemma 4.3 every compact subset of $bG\setminus G$ is metrizable. Thus G is locally separable and metrizable if G is a paracompact p-space by Lemma 3.2. If every point y of D has a countable open neighborhood base in bG, then the subspace D has countable π-character. If the subspace D has countable π-character in Y, then G is metrizable by Corollary 4.13. Since G is metrizable and every compact subset of $bG \setminus G$ is metrizable, G is locally separable and metrizable by Lemma 3.2. □

We recall that a space is a *M-space* if and only if it is the inverse image of a metric space by a quasi-perfect map.

Lemma 4.15 ([16, Corollary 13])**.** *Let $f\colon X \to Y$ be a closed map such that X has a point-countable weak base. If Y is a M-space, then Y is metrizable.*

By Lemma 4.15, we have:

Corollary 4.16. *If X is a Lindelöf p-space with a point-countable weak base, then X is metrizable.*

Theorem 4.17. *Let G be a non-locally compact topological group, and bG be a compactification of G such that the remainder $Y = bG \setminus G$ has a point-countable weak base and has a dense subset D such that every point of the set D has countable pseudo-character in the remainder $bG \setminus G$ (or the subspace D has countable π-character), then G and $bG \setminus G$ are separable and metrizable.*

P r o o f. G is locally separable and metrizable by Lemma 4.14.

If Y is a Fréchet space, then Y has a point-countable base by Lemma 4.1. Thus G and $bG \setminus G$ are separable and metrizable by Corollary 3.19.

Suppose Y is not a Fréchet space, there exists a subset A of Y such that the set $B = \bigcup\{C \cup \{x_C\}\colon C$ is a convergence sequence of A which converges to the point $x_C\}$ is not a closed subset of Y. Since Y has a point-countable weak base, the space Y is a sequential space. Since the set B is not a closed subset of Y, there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ of B such that the sequence $\{y_n\}_{n\in\mathbb{N}}$ converges to a point $y \notin B$. For each $n \in \mathbb{N}$ the point $y_n \in B$, so there exists a sequence $\{y_{nk}\}_{k\in\mathbb{N}}$ of A such that $\{y_{nk}\}_{k\in\mathbb{N}}$ converges to the point y_n. The point $y \notin B$, then there is no subsequence of $\{y_{nk}\colon n, k \in \mathbb{N}\}$ converging to the point y, otherwise $y \in B$.

G is locally separable and metrizable, hence $G = \bigoplus\limits_{\alpha\in\Lambda} G_\alpha$ by Proposition 2.3, where $\{G_\alpha\colon \alpha \in \Lambda\}$ is a discrete family of separable and metrizable subspaces of G.

Denote by F the set of all accumulation points for $\{G_\alpha\colon \alpha \in \Lambda\}$ in bG. Thus $F \subset Y$ and F is a compact subset of Y. Since Y has a point-countable weak base, the subspace F has a point-countable weak base by Lemma 4.3 and F is metrizable by Lemma 4.2.

If $\{y_{n_p}\}_{p\in\mathbb{N}}$ is a subsequence of the sequence $\{y_n\}_{n\in\mathbb{N}}$, then $\{y_{n_p}\}_{p\in\mathbb{N}}$ converges to y. Thus the point y is in the closure of $\{y_{n_p k}\colon p \in \mathbb{N},\ k \in \mathbb{N}\}$. So the point y is in the closure of $\{y_{mk}\colon m \in N_1,\ k \in \mathbb{N}\}$ if the subset N_1 of $\mathbb{N}$ is infinite.

Denote $L = \{m\colon m \in \mathbb{N}$ and $|\{k\colon y_{mk} \in F, k \in \mathbb{N}\}| = \omega\}$. Suppose $|L| = \omega$, then the point y is in the closure of the set $\{y_{mk}\colon m \in L$ and $y_{mk} \in F\}$. Thus $y \in F$. The set F is metrizable, so there is a sequence of the set $\{y_{mk}\colon m \in L$ and $y_{mk} \in F\}$ converging to the point y. A contradiction. Thus $|L| < \omega$.

Without loss of generality, we assume $\{y_{nk}\colon k \in \mathbb{N},\ n \in \mathbb{N}\} \subset Y \setminus F$. Then there exists an open subset U_{nk} of bG such that $y_{nk} \in U_{nk}$ and $\overline{U_{nk}} \cap F = \emptyset$ for each $k \in \mathbb{N}$ and for each $n \in \mathbb{N}$. Thus $|\{\alpha\colon U_{nk} \cap G_\alpha \neq \emptyset, \alpha \in \Lambda\}| < \omega$. If $U = \bigcup\{U_{nk}\colon n, k \in \mathbb{N}\}$, then U is an open subset of bG. The set U intersects with at most countably many G_α. We denote by $U \cap G = \bigcup\{U \cap G_{\alpha_i}\colon i \in \mathbb{N}\}$. If we let $M = \bigcup\{G_{\alpha_i}\colon i \in \mathbb{N}\}$, then M is separable and $U \cap G \subset M$. Since $\overline{G} = bG$, $\overline{U \cap G} = \overline{U}$. Thus $\overline{U} \subset \overline{M}$. The set M is a closed subset of G, so $\overline{M} \setminus M \subset bG \setminus G$, hence $\overline{M} \setminus M = \overline{M} \cap (bG \setminus G)$. The set $\overline{M} \setminus M$ has a point-countable weak base by Lemma 4.3. Since M is separable and metrizable, $\overline{M} \setminus M$ is a Lindelöf p-space. $\overline{M} \setminus M$ has a point-countable weak base, thus it is metrizable by Corollary 4.16. Since $y \in \overline{M} \setminus M$, there exists a subsequence of $\{y_{nk} \colon n, k \in \mathbb{N}\}$ which converges to y. Thus $y \in B$. This contradicts $y \notin B$.

Thus Y is a Fréchet space, hence G and $bG \setminus G$ are separable and metrizable. □

By the proof of Theroem 4.17, we have:

Theorem 4.18. *Let X be a locally separable and metrizable space. If bX is a compactification of X such that every Lindelöf p-subspace of the remainder $bX \setminus X$ is metrizable, then the remainder $bX \setminus X$ is a Fréchet space.*

References

[1] *A. V. Arhangel'skii*: On a class of spaces containing all metric and locally compact spaces. Mat. Sb. *67* (1965), 55–58 (In Russian.); English transl. in: Amer. Math. Soc. Transl. *(92)* (1970), 1–39.

[2] *A. V. Arhangel'skii*: Remainders in compactifications and generalized metrizability properties. Topology Appl. *150* (2005), 79–90.

[3] *A. V. Arhangel'skii*: More on remainders close to metrizable spaces. Topology Appl. *154* (2007), 1084–1088.

[4] *A. V. Arhangel'skii*: Fist countability, tightness, and other cardinal invariants in remainders of topological groups. Topology Appl. *154* (2007), 2950–2961.

[5] *A. V. Arhangel'skii*: Two types of remainders of topological groups. Commentat. Math. Univ. Carol. *49* (2008), 119–126.

[6] *A. V. Arhangel'skii*: A study of remainders of topological groups. Fundam. Math. *203* (2009), 165–178.

[7] *A. V. Arhangel'skii*: Some aspects of topological algebra and remainders of topological groups. Topol. Proc. *33* (2009), 13–28.

[8] *H. Bennett, R. Byerly, D. Lutzer*: Compact G_δ sets. Topology Appl. *153* (2006), 2169–2181.

[9] *J. Chaber*: Conditions which imply compactness in countably compact spaces. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. *24* (1976), 993–998.

[10] *E. K. van Douwen, W. F. Pfeffer*: Some properties of the Sorgenfrey line and related spaces. Pac. J. Math. *81* (1979), 371–377.

[11] *R. Engelking*: General Topology. Rev. and compl. ed. Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin 1989.

[12] *J. Gerlits, I. Juhász, Z. Szentmiklóssy*: Two improvements on Tkačenko's addition theorem. Commentat. Math. Univ. Carol. *46* (2005), 705–710.

[13] *G. Gruenhage*: Generalized Metric Spaces (K. Kunen and J. E. Vaughan, eds.). Handbook of set-theoretic topology. North-Holland, Amsterdam, 1984, pp. 423–501.

[14] *G. Gruenhage, E. Michael, Y. Tanaka*: Spaces determined by point-countable covers. Pac. J. Math. *113* (1984), 303–332.

[15] *M. Henriksen, J. R. Isbell*: Some properties of compactifications. Duke Math. J. *25* (1958), 83–105.

[16] *S. Lin, Y. Tanaka*: Point-countable k-networks, closed maps, and related results. Topology Appl. *59* (1994), 79–86.

[17] *Ch. Liu*: On weak bases. Topology Appl. *150* (2005), 91–99.

[18] *Ch. Liu*: Remainders in compactifications of topological groups. Topology Appl. *156* (2009), 849–854.

[19] *Ch. Liu, S. Lin*: Generalized metric spaces with algebaric structures. Topology Appl. *157* (2010), 1966–1974.
[20] *H. W. Martin*: Metrizability of M-spaces. Can. J. Math. *25* (1973), 840–841.
[21] *L.-X. Peng*: The D-property of some Lindelöf spaces and related conclusions. Topology Appl. *154* (2007), 469–475.
[22] *L.-X. Peng*: A note on D-spaces and infinite unions. Topology Appl. *154* (2007), 2223–2227.
[23] *L.-X. Peng, L. X. Wang*: On CSS spaces and related conclusions. Acta Math. Sci., Ser. A, Chin. Ed. *30* (2010), 358–363. (In Chinese.)
[24] *W. Roelcke, S. Dierolf*: Uniform Structures on Topological Groups and their Questions. McGraw-Hill International Book Company, New York, 1981.
[25] *F. Siwiec*: On defining a space by a weak base. Pac. J. Math. *52* (1974), 233–245.
[26] *M. Tkačenko*: Introduction to topological groups. Topology Appl. *86* (1998), 179–231.

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