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ON LINEAR OPERATORS STRONGLY PRESERVING INVARIANTS OF BOOLEAN MATRICES

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Abstract. Let $\mathbb{B}_k$ be the general Boolean algebra and T a linear operator on $M_{m,n}(\mathbb{B}_k)$. If for any A in $M_{m,n}(\mathbb{B}_k)$ ($M_n(\mathbb{B}_k)$, respectively), A is regular (invertible, respectively) if and only if $T(A)$ is regular (invertible, respectively), then T is said to strongly preserve regular (invertible, respectively) matrices. In this paper, we will give complete characterizations of the linear operators that strongly preserve regular (invertible, respectively) matrices over $\mathbb{B}_k$. Meanwhile, noting that a general Boolean algebra $\mathbb{B}_k$ is isomorphic to a finite direct product of binary Boolean algebras, we also give some characterizations of linear operators that strongly preserve regular (invertible, respectively) matrices over $\mathbb{B}_k$ from another point of view.

Keywords: linear operator, invariant, regular matrix, invertible matrix, general Boolean algebra

MSC 2010: 15A04, 15A09, 16Y60

1. Introduction and preliminaries

A semiring means a type $\langle 2,2,0,0\rangle$ algebra $(R,+,\cdot,0,1)$ satisfying the following identities:

- $x+(y+z)=(x+y)+z$;
- $x(yz)=(xy)z$;
- $x(y+z)=xy+xz$, $(x+y)z=xz+yz$;
- $x+0=x$;
- $x+y=y+x$;

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- $x \cdot 1 = 1 \cdot x = x$;
- $x0 = 0x = 0$.

Semirings constitute a fairly natural generalization of rings and distributive lattices.

Let $M_{m,n}(R)$ denote the set of all $m \times n$ matrices with entries in a semiring R. If $m = n$, we use the notation $M_n(R)$ instead of $M_{n,n}(R)$. Algebraic operations on $M_{m,n}(R)$ and such notions as linearity are also defined as if the underlying scalars were in a field.

The theory of matrices over semirings have broad applications in optimization theory, models of discrete event networks and graph theory (refer to [9], [10], [11], [17]). Regular matrices and invertible matrices are two special types of matrices, they play a central role in the theory of matrices (refer to [1], [9], [11], [17], [19]).

Recall that a $m \times n$ matrix A over a semiring R is called regular if there is a $n \times m$ matrix B over R such that $ABA = A$. A matrix A in $M_n(R)$ is said to be invertible if there is a matrix B in $M_n(R)$ such that $AB = BA = I_n$, where I_n is the $n \times n$ identity matrix.

Let T be a linear operator on $M_{m,n}(R)$ and $\mathscr{R}(R)$ denote the set of all the regular matrices in $M_{m,n}(R)$. We say that

(i) T preserves regularity (or T preserves $\mathscr{R}(R)$) if $T(A) \in \mathscr{R}(R)$ whenever $A \in \mathscr{R}(R)$;

(ii) T strongly preserves regularity (or T strongly preserves $\mathscr{R}(R)$) when $T(A) \in \mathscr{R}(R)$ if and only if $A \in \mathscr{R}(R)$ for all $A \in M_{m,n}(R)$.

Let T be a linear operator on $M_n(R)$ and $GL_n(R)$ denote the set of all the invertible matrices in $M_n(R)$. We say that

(i) T preserves invertibility (or T preserves $GL_n(R)$) if $T(A) \in GL_n(R)$ whenever $A \in GL_n(R)$;

(ii) T strongly preserves invertibility (or T strongly preserves $GL_n(R)$) when $T(A) \in GL_n(R)$ if and only if $A \in GL_n(R)$ for all $A \in M_n(R)$.

During the past 100 years, one of the most active and fertile subjects in matrix theory is the linear preserver problem (LPP for short), which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. The first paper can be traced down to Frobenius's work in 1897. Since then, a number of works in the area have been published. Among these works, although the linear operators concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings (refer to [2]–[8], [12]–[14], [16], [18], [21]–[26]).

In the last decades, a number of characterizations of idempotent, nilpotent, regular and invertible matrices over semirings are given (refer to [1], [10], [11], [14], [15],

[17]–[20]). Also, the linear operators preserving a number of other invariants of matrices over semirings (many of them are antinegative semirings without zero divisors) have been characterized (refer to [2]–[8], [12]–[14], [21]–[26]). Among these papers, Song, Kang and Beasley etc. studied the linear operators strongly preserving regular matrices over antinegative commutative semirings with no zero divisors, including the binary Boolean algebra, the nonnegative reals, the nonnegative integers and the fuzzy scalars (refer to [12], [23], [25]); Li, Tan, and Tang [14] characterized the linear operators strongly preserving invertible matrices over some antinegative commutative semirings with no zero divisors. Besides, Song, Kang, and Jun etc. (refer to [22], [24], [26]) studied and characterized the linear operators that preserve (or strongly preserve) column ranks, nilpotent matrices and idempotent matrices over general Boolean algebras.

For a fixed positive integer k, let $\mathbb{B}_k$ be the general Boolean algebra of subsets of a k-element set $\mathbb{S}_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote the singleton subsets of $\mathbb{S}_k$. Union is denoted by $+$ and intersection by $\cdot$; the null set is denoted by 0 and the set $\mathbb{S}_k$ by 1. Under these operations, $\mathbb{B}_k$ becomes a commutative semiring, i.e., a semiring satisfying the additional identity $xy = yx$. In particular, if $k = 1$, $\mathbb{B}_1$ is called the binary Boolean algebra. The matrices over general Boolean algebra $\mathbb{B}_k$ will be called Boolean matrices.

In this paper we are going to study the linear operators strongly preserving regular matrices over $\mathbb{B}_k$ and the linear operators strongly preserving invertible matrices over $\mathbb{B}_k$, respectively. We will give complete characterizations of the linear operators that strongly preserve regular matrices over $\mathbb{B}_k$ in Section 2. Noticing that a general Boolean algebra $\mathbb{B}_k$ is isomorphic to a finite direct product of binary Boolean algebras, we also give some other characterizations of linear operators that strongly preserve regular matrices over $\mathbb{B}_k$ from another point of view, i.e., we will give the characterizations of linear operators that strongly preserve regular matrices over the finite direct product of binary Boolean algebras. In Section 3, we will firstly characterize the linear operators that strongly preserve invertible matrices over $\mathbb{B}_k$. Next, we also give some other characterizations of linear operators that strongly preserve invertible matrices over the finite direct product of binary Boolean algebras.

For notations and terminologies occurring but not mentioned in this paper, the readers can refer to [10], [11], [23].

2. Linear operators strongly preserving regular matrices over $\mathbb{B}_k$

For any matrix $A = [a_{ij}] \in M_{m,n}(\mathbb{B}_k)$, the lth constituent, A_l, of A is the $m \times n$ binary Boolean matrix whose (i,j)th entry is 1 if and only if $a_{ij} \supseteq \sigma_l$. Via the constituents, A can be written uniquely as

$$A = \sum_{l=1}^{k} \sigma_l A_l,$$

which is called the canonical form of A (refer to [13]).

It follows from the uniqueness of the above decomposition and the fact that the singletons are mutually orthogonal idempotents that for all matrices $A \in M_{m,n}(\mathbb{B}_k)$, $B, C \in M_{n,q}(\mathbb{B}_k)$ and all $\alpha \in \mathbb{B}_k$,

$$(AB)_l = A_l B_l; \quad (B + C)_l = B_l + C_l; \quad (\alpha A)_l = \alpha_l A_l$$

for all $1 \leqslant l \leqslant k$.

Proposition 2.1. *Let A be a matrix in $M_{m,n}(\mathbb{B}_k)$ with $k \geqslant 1$. Then A is regular if and only if all constituents of A are regular in $M_{m,n}(\mathbb{B}_1)$.*

Proof. Assume that $A \in M_{m,n}(\mathbb{B}_k)$ and A is regular. Then there exists $X \in M_{m,n}(\mathbb{B}_k)$ such that $AXA = A$. Since A and X have unique canonical forms respectively, say $A = \sum_{l=1}^{k} \sigma_l A_l$ and $X = \sum_{l=1}^{k} \sigma_l X_l$, we have that

$$\left(\sum_{l=1}^{k} \sigma_l A_l\right)\left(\sum_{l=1}^{k} \sigma_l X_l\right)\left(\sum_{l=1}^{k} \sigma_l A_l\right) = AXA = A = \sum_{l=1}^{k} \sigma_l A_l.$$

That is, $\sum_{l=1}^{k} \sigma_l (A_l X_l A_l) = \sum_{l=1}^{k} \sigma_l A_l$. And then we have $A_l X_l A_l = A_l$ for all $1 \leqslant l \leqslant k$. Note that $X_l \in M_{m,n}(\mathbb{B}_1)$, so A_l is regular in $M_{m,n}(\mathbb{B}_1)$ for all $1 \leqslant l \leqslant k$.

Conversely, assume that $A = \sum_{l=1}^{k} \sigma_l A_l \in M_{m,n}(\mathbb{B}_k)$ and all lth constituents of A are regular in $M_{m,n}(\mathbb{B}_1)$. Then for each $1 \leqslant l \leqslant k$, there exists $X_l \in M_{m,n}(\mathbb{B}_1)$ such that $A_l X_l A_l = A_l$. Now, take $X = \sum_{l=1}^{k} \sigma_l X_l \in M_{m,n}(\mathbb{B}_k)$, we have

$$AXA = \left(\sum_{l=1}^{k} \sigma_l A_l\right)\left(\sum_{l=1}^{k} \sigma_l X_l\right)\left(\sum_{l=1}^{k} \sigma_l A_l\right) = \sum_{l=1}^{k} \sigma_l A_l = A.$$

Hence, A is regular in $M_{m,n}(\mathbb{B}_k)$ with $k \geqslant 1$. $\square$

Lemma 2.2 ([24]). *Let A be a matrix in $M_n(\mathbb{B}_k)$ with $k \geqslant 1$. Then A is invertible if and only if all its constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^t$.*

The above lemma also shows that $A \in M_n(\mathbb{B}_k)$ ($k \geqslant 1$) is invertible if and only if all its constituents are invertible, since the permutation matrices are the only invertible matrices in $M_n(\mathbb{B}_1)$.

Now, assume that T is a linear operator on $M_{m,n}(\mathbb{B}_k)$ with $k \geqslant 1$. For each $1 \leqslant l \leqslant k$, its lth constituent operator, T_l, is defined (refer to [13], [26]) by

$$(\forall\, B \in M_{m,n}(\mathbb{B}_k)) \quad T_l(B) = (T(B))_l.$$

By the linearity of T, we have

$$T(A) = \sum_{l=1}^{k} \sigma_l T_l(A_l)$$

for any matrix $A \in M_{m,n}(\mathbb{B}_k)$.

Proposition 2.3. *Let T be a linear operator on $M_{m,n}(\mathbb{B}_k)$. Then T strongly preserves regularity if and only if its each lth constituent operator, T_l, strongly preserves regularity on $M_{m,n}(\mathbb{B}_1)$.*

Proof. ($\Rightarrow$) For any $A_l \in M_{m,n}(\mathbb{B}_1)$, assume that $A_l \in \mathscr{R}(\mathbb{B}_1)$. Take $A = A_l = \sum\limits_{i=1}^{k} \sigma_i A_l$. Clearly, $A \in \mathscr{R}(\mathbb{B}_k)$. Since T is a linear operator that strongly preserves regularity on $M_{m,n}(\mathbb{B}_k)$, we have $T(A) \in \mathscr{R}(\mathbb{B}_k)$. And then we get $T_l(A_l) \in \mathscr{R}(\mathbb{B}_1)$ by Proposition 2.1.

On the other hand, for any $A \in M_{m,n}(\mathbb{B}_1)$, assume that $T_l(A) \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Note that $T(A) = T\Big(\sum\limits_{i=1}^{k} \sigma_i A_l\Big) = \sum\limits_{i=1}^{k} \sigma_i T_l(A_l)$, we have $T(A) \in \mathscr{R}(\mathbb{B}_k)$ by Proposition 2.1. And then we can get $A \in \mathscr{R}(\mathbb{B}_k)$ since T is a linear operator that strongly preserves regularity on $M_{m,n}(\mathbb{B}_k)$. By Proposition 2.1 again, we have $A (= A_l) \in \mathscr{R}(\mathbb{B}_1)$.

($\Leftarrow$) For any $A = \sum\limits_{l=1}^{k} \sigma_l A_l \in \mathscr{R}(\mathbb{B}_k)$, by Proposition 2.1, $A_l \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Since each lth constituent operator, T_l, strongly preserves regularity on $M_{m,n}(\mathbb{B}_1)$, we immediately have $T_l(A_l) \in \mathscr{R}(\mathbb{B}_1)$. By Proposition 2.1 again, $T(A) = T\Big(\sum\limits_{l=1}^{k} \sigma_l A_l\Big) = \sum\limits_{i=1}^{k} \sigma_l T_l(A_l) \in \mathscr{R}(\mathbb{B}_k)$.

On the other hand, for any $A = \sum\limits_{l=1}^{k} \sigma_l A_l \in M_{m,n}(\mathbb{B}_k)$, assume that $T(A) = T\Big(\sum\limits_{l=1}^{k} \sigma_l A_l\Big) = \sum\limits_{i=1}^{k} \sigma_l T_l(A_l) \in \mathscr{R}(\mathbb{B}_k)$. By Proposition 2.1, we have $T_l(A_l) \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Since T_l strongly preserves regularity on $M_{m,n}(\mathbb{B}_1)$, we immediately get $A_l \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. And then we have $A = \sum\limits_{l=1}^{k} \sigma_l A_l \in \mathscr{R}(\mathbb{B}_k)$ by Proposition 2.1. $\square$

For a given semiring R, the (factor) rank, $\mathrm{fr}(A)$, of a nonzero $A \in M_{m,n}(R)$ is defined as the least integer r for which there are $B \in M_{m,r}(R)$ and $C \in M_{r,n}(R)$ such that $A = BC$ (refer to [5], [6]). The rank of a zero matrix is zero. Also we can easily obtain

$$(*) \qquad 0 \leqslant \mathrm{fr}(A) \leqslant \min\{m, n\} \quad \text{and} \quad \mathrm{fr}(AB) \leqslant \min\{\mathrm{fr}(A), \mathrm{fr}(B)\}$$

for all $A \in M_{m,n}(R)$ and for all $B \in M_{n,q}(R)$.

Lemma 2.4 ([23]). *Let $\min\{m, n\} \leqslant 2$. If T is an operator (that need not be linear) on $M_{m,n}(\mathbb{B}_1)$, then T strongly preserves regularity.*

Theorem 2.5. *Let $\min\{m, n\} \leqslant 2$. If T is a linear operator on $M_{m,n}(\mathbb{B}_k)$ with $k \geqslant 1$, then T strongly preserves regularity.*

Proof. Assume that $A \in M_{m,n}(\mathbb{B}_k)$ is regular. By Proposition 2.1, all lth constituents of A are regular in $M_{m,n}(\mathbb{B}_1)$. Notice that $T(A) = \sum\limits_{l=1}^{k} \sigma_l T_l(A_l)$, and T_l strongly preserves regularity by Lemma 2.4, we immediately obtain that $T(A)$ is regular by Proposition 2.1.

Conversely, for any $A \in M_{m,n}(\mathbb{B}_k)$, assume that $T(A)$ is regular. Since $T(A) = \sum\limits_{l=1}^{k} \sigma_l T_l(A_l)$, by Proposition 2.1, all $T_l(A_l)$ are regular in $M_{m,n}(\mathbb{B}_1)$. And then, it follows from Lemma 2.4 that all A_l are regular. So, by Proposition 2.1, A is regular.

Thus, summing up the above discussions, we have shown that T strongly preserves regularity. $\square$

In the following we will continue to characterize the linear operators that strongly preserve regular matrices over $\mathbb{B}_k$ when $\min\{m, n\} \geqslant 3$.

The proposition and lemma below are needed.

Proposition 2.6. *Let A be a matrix in $M_{m,n}(\mathbb{B}_k)$ with $k \geqslant 1$. If $U \in M_m(\mathbb{B}_k)$ and $V \in M_n(\mathbb{B}_k)$ are invertible, then the following are equivalent:*

(i) *A is regular in $M_{m,n}(\mathbb{B}_k)$;*

(ii) *UAV is regular in $M_{m,n}(\mathbb{B}_k)$;*
(iii) *A^t is regular in $M_{n,m}(\mathbb{B}_k)$.*

Proof. Since (i) $\Leftrightarrow$ (iii) is clear, we only need to show that (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii) Assume that A is regular in $M_{m,n}(\mathbb{B}_k)$. Then there exists $X \in M_{n,m}(\mathbb{B}_k)$ such that $AXA = A$. And then since $U \in M_m(\mathbb{B}_k)$ and $V \in M_n(\mathbb{B}_k)$ are invertible, we have $(UAV)(V^{-1}XU^{-1})(UAV) = UAV$. This means that UAV is regular in $M_{m,n}(\mathbb{B}_k)$.

(ii) $\Rightarrow$ (i) Assume that UAV is regular in $M_{m,n}(\mathbb{B}_k)$. Then there exists $Y \in M_{n,m}(\mathbb{B}_k)$ such that $(UAV)Y(UAV) = UAV$. Since $U \in M_m(\mathbb{B}_k)$ and $V \in M_n(\mathbb{B}_k)$ are invertible, we immediately get $A(VYU)A = U^{-1}(UAV)Y(UAV)V^{-1} = U^{-1}UAVV^{-1} = A$. Hence, A is regular in $M_{m,n}(\mathbb{B}_k)$. $\square$

Lemma 2.7 ([23]). *Let T be a linear operator on $M_{m,n}(\mathbb{B}_1)$ with $\min\{m,n\} \geqslant 3$. Then T strongly preserves regularity if and only if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in M_{m,n}(\mathbb{B}_1)$, or $m = n$ and $T(X) = PX^tQ$ for all $X \in M_n(\mathbb{B}_1)$.*

Theorem 2.8. *Let $\min\{m,n\} \geqslant 3$ and T be a linear operator on $M_{m,n}(\mathbb{B}_k)$ with $k \geqslant 1$. Then T strongly preserves regularity if and only if there are invertible matrices P and Q such that $T(X) = PXQ$ for all $X \in M_{m,n}(\mathbb{B}_k)$, or $m = n$ and $T(X) = P\Big(\sum\limits_{l=1}^{k} \sigma_l Y_l\Big)Q$ for all $X = \sum\limits_{l=1}^{k} \sigma_l X_l \in M_n(\mathbb{B}_k)$, where $Y_l = X_l$ or $Y_l = X_l^t$.*

Proof. ($\Rightarrow$) Assume that T strongly preserves regularity. By Proposition 2.3, it is known that each lth constituent operator, T_l, strongly preserves the regularity on $M_{m,n}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Now, for any $X = \sum\limits_{l=1}^{k} \sigma_l X_l \in M_{m,n}(\mathbb{B}_k)$, notice that $T(X) = \sum\limits_{l=1}^{k} \sigma_l T_l(X_l)$ and by Lemma 2.7, each lth constituent operator, T_l, has the form

$$T_l(X_l) = P_l X_l Q_l \quad \text{or} \quad m = n \quad \text{and} \quad T_l(X_l) = P_l X_l^t Q_l,$$

where $X_l \in M_{m,n}(\mathbb{B}_1)$, and P_l and Q_l are permutation matrices for $l = 1, 2, \ldots, k$. Thus we immediately get

$$\begin{aligned} T(X) &= \sum_{l=1}^{k} \sigma_l T_l(X_l) = \sum_{l=1}^{k} \sigma_l P_l Y_l Q_l \\ &= \Big(\sum_{l=1}^{k} \sigma_l P_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Q_l\Big) \triangleq PXQ \end{aligned}$$

for all $X = \sum_{l=1}^{k} \sigma_l X_l \in M_{m,n}(\mathbb{B}_k)$, or $m = n$ and

$$\begin{aligned} T(X) &= \sum_{l=1}^{k} \sigma_l T_l(X_l) = \sum_{l=1}^{k} \sigma_l P_l Y_l Q_l \\ &= \Big(\sum_{l=1}^{k} \sigma_l P_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Q_l\Big) \\ &\triangleq P\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)Q, \end{aligned}$$

where $Y_l = X_l$ or $Y_l = X_l^t$ for $l = 1, 2, \ldots, k$. Also, by Lemma 2.2, $P = \sum_{l=1}^{k} \sigma_l P_l$, $Q = \sum_{l=1}^{k} \sigma_l Q_l$ are clearly invertible in $M_n(\mathbb{B}_k)$. Thus, we have shown the necessity.

($\Leftarrow$) We will show the sufficiency as follows:

For any $X = \sum_{l=1}^{k} \sigma_l X_l \in M_{m,n}(\mathbb{B}_k)$, if $X \in \mathscr{R}(\mathbb{B}_k)$, then by Proposition 2.6, $T(X) = PXQ$ is clearly regular.

For $m = n$, assume $X = \sum_{l=1}^{k} \sigma_l X_l \in \mathscr{R}(\mathbb{B}_k)$, then by Proposition 2.1, we have $X_l \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. And then, $\sum_{l=1}^{k} \sigma_l Y_l$ is also regular, where $Y_l = X_l$ or $Y_l = X_l^t$. Hence, by Proposition 2.6, $T(X) = P\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)Q \in \mathscr{R}(\mathbb{B}_k)$.

Assume that $X = \sum_{l=1}^{k} \sigma_l X_l \in M_{m,n}(\mathbb{B}_k)$ and $T(X) \in \mathscr{R}(\mathbb{B}_k)$. Then $T(X) = PXQ$ is regular. By Proposition 2.6, $X \in \mathscr{R}(\mathbb{B}_k)$.

For $m = n$, assume that $X = \sum_{l=1}^{k} \sigma_l X_l \in M_n(\mathbb{B}_k)$ and $T(X) \in \mathscr{R}(\mathbb{B}_k)$. Note that $T(X) = P\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)Q$, where P and Q are invertible matrices, $Y_l = X_l$ or $Y_l = X_l^t$, by Proposition 2.6, we have $Y = \sum_{l=1}^{k} \sigma_l Y_l \in \mathscr{R}(\mathbb{B}_k)$, and then $Y_l \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Since $Y_l = X_l$ or $Y_l = X_l^t$, we immediately get $X_l \in \mathscr{R}(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. It follows from Proposition 2.1 that $X \in \mathscr{R}(\mathbb{B}_k)$. $\square$

So we have obtained the characterizations of linear operators which strongly preserve regular matrices over general Boolean algebras $\mathbb{B}_k$.

Next, we will give two examples.

Example 2.9. Let

$$P = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_1 \\ \sigma_3 & \sigma_1 & \sigma_2 \end{pmatrix} \in M_3(\mathbb{B}_3), \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathbb{B}_3).$$

By Lemma 2.2, P, Q are clearly invertible matrices. Now, define a linear operator T on $M_{3,4}(\mathbb{B}_3)$ by

$$T(X) = PXQ$$

for all $X \in M_{3,4}(\mathbb{B}_3)$. Then it is not hard for us to check that T strongly preserves $\mathscr{R}(\mathbb{B}_3)$.

Example 2.10. Let

$$P = \begin{pmatrix} \sigma_3 & \sigma_1 & \sigma_2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_3 & \sigma_1, \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{B}_3).$$

By Lemma 2.2, $P, Q \in GL_3(\mathbb{B}_3)$. Define a linear operator T on $M_3(\mathbb{B}_3)$ by

$$T(X) = P(\sigma_1 X_1^t + \sigma_2 X_2 + \sigma_3 X_3^t)Q$$

for all $X = \sum\limits_{l=1}^{3} \sigma_l X_l$. Then it is also routine to check that T strongly preserves $\mathscr{R}(\mathbb{B}_3)$.

Notice that a general Boolean algebra $\mathbb{B}_k$ is isomorphic to a finite direct product of binary Boolean algebras $\mathbb{B}_1$. In the following, we will also give some other characterizations of linear operators that strongly preserve regular matrices over $\mathbb{B}_k$ from another point of view, i.e., we will study the characterizations of linear operators that strongly preserve regular matrices over the finite direct product of binary Boolean algebras $\mathbb{B}_1$.

The propositions below are needed.

Proposition 2.11. *Let $S = \prod\limits_{i=1}^{k} S_i$, where $S_i = \mathbb{B}_1$. If $T\colon M_{m,n}(S) \to M_{m,n}(S)$ is a linear operator, then for any $i \in \{1, 2, \ldots, k\}$, there exists a unique linear operator $T_i\colon M_{m,n}(\mathbb{B}_1) \to M_{m,n}(\mathbb{B}_1)$ such that $T(A)_i = T_i(A_i)$ for any $A \in M_{m,n}(S)$.*

Proof. For any $B \in M_{m,n}(\mathbb{B}_1)$, define $T_i(B) := T(C)_i$, where $C \in M_{m,n}(S)$ is such that $C_i = B$ and $C_j = O$ for any $i \neq j$. Let $s \in S$ with $s_i = 1$ and $s_j = 0$ for any $j \neq i$. Then for any $A \in M_{m,n}(S), T(A)_i = s_i T(A)_i = (sT(A))_i = T(sA)_i = T_i(A_i)$.

For any $A_1, A_2 \in M_{m,n}(S_i)$, assume that $T_i(A_1) = T(\bar{A}_1)_i, T_i(A_2) = T(\bar{A}_2)_i$, where $(\bar{A}_1)_i = A_1$, $(\bar{A}_2)_i = A_2$ and $(\bar{A}_1)_j = O$, $(\bar{A}_2)_j = O$ for any $j \neq i$. And then we have

$$\begin{aligned}T_i(A_1) + T_i(A_2) &= T(\bar{A}_1)_i + T(\bar{A}_2)_i = (T(\bar{A}_1) + T(\bar{A}_2))_i \\ &= T(\bar{A}_1 + \bar{A}_2)_i = T_i((\bar{A}_1 + \bar{A}_2)_i) \\ &= T_i((\bar{A}_1)_i + (\bar{A}_2)_i) = T_i(A_1 + A_2).\end{aligned}$$

Also, for any $a \in S_i$, there exists $\bar{a} \in S$ such that $(\overline{a})_i = a$. And then

$$aT_i(A_1) = (\bar{a})_i T(\bar{A}_1)_i = (\overline{a}T(\bar{A}_1))_i = T(\overline{a}\bar{A}_1)_i = T_i((\overline{a}\bar{A}_1)_i) = T_i(aA_1).$$

Thus, T_i is a linear operator.

Suppose that $\overline{T}_i \colon M_{m,n}(S_i) \to M_{m,n}(S_i)$ is also a linear operator such that $T(A)_i = \overline{T}_i(A_i)$ for any $A \in M_{m,n}(S)$. For any $B \in M_{m,n}(S_i)$, there exists $\overline{B} \in M_{m,n}(S)$ such that $(\overline{B})_i = B$. Then

$$\overline{T}_i(B) = \overline{T}_i((\overline{B})_i) = T(\overline{B})_i = T_i((\overline{B})_i) = T_i(B).$$

Thus, $\overline{T}_i = T_i$. □

In the following, the linear operator T_i will be called ith constituent operator of T for convenience.

Proposition 2.12. *Let $S = \prod_{i=1}^{k} S_i$, where $S_i = \mathbb{B}_1$. Then for any $A, B \in M_{m,n}(S)$, A is regular if and only if A_i is regular for any $i \in \{1, 2, \ldots, k\}$.*

Proof. It is routine to check this proposition. □

Proposition 2.13. *Let $S = \prod_{i=1}^{k} S_i$, where $S_i = \mathbb{B}_1$. Let T be a linear operator on $M_{m,n}(S)$. Then T strongly preserves regularity if and only if its ith constituent operator, T_i, strongly preserves regularity.*

Proof. Let T be a linear operator on $M_{m,n}(S)$ and T_i be its ith constituent operator on $M_{m,n}(\mathbb{B}_1)$.

($\Rightarrow$) For any $A_l \in M_{m,n}(\mathbb{B}_1)$, assume that $A_l \in \mathscr{R}(\mathbb{B}_1)$. Take $A = (A_l, A_l, \ldots, A_l) \in \prod_{i=1}^{k} M_{m,n}(S_i)$. We have $A \in \mathscr{R}(S)$ by Proposition 2.12. Since T is a linear operator that strongly preserves regularity on $M_{m,n}(S)$, we have $T(A) = (T_1(A_l), \ldots, T_l(A_l), \ldots, T_k(A_l)) \in \mathscr{R}(S)$. And then we get $T_i(A_l) \in \mathscr{R}(\mathbb{B}_1)$.

On the other hand, for any $A_l \in M_{m,n}(\mathbb{B}_1)$, assume that $T_i(A_l) \in \mathscr{R}(\mathbb{B}_1)$ for $i = 1, 2, \ldots, k$. Take $A = (A_l, A_l, \ldots, A_l) \in \prod_{i=1}^{k} M_{m,n}(S_i)$. We have

$$T(A) = (T_1(A_l), \ldots, T_l(A_l), \ldots, T_k(A_l)) \in \mathscr{R}(S)$$

by Proposition 2.12. And then $A \in \mathscr{R}(\mathbb{B}_k)$ since T is a linear operator that strongly preserves regularity on $M_{m,n}(\mathbb{B}_k)$. By Proposition 2.12 again, we have $A_l \in \mathscr{R}(\mathbb{B}_1)$.

($\Leftarrow$) For any $A = (A_1, A_2, \ldots, A_k) \in \prod_{i=1}^{k} M_{m,n}(S_i)$, by Proposition 2.12, $A_i \in \mathscr{R}(\mathbb{B}_1)$ for $i = 1, 2, \ldots, k$. Since each ith constituent operator, T_i, strongly preserves regularity on $M_{m,n}(\mathbb{B}_1)$, we immediately have $T_i(A_i) \in \mathscr{R}(\mathbb{B}_1)$. And then

$$T(A) = (T_1(A_1), \ldots, T_2(A_2), \ldots, T_k(A_k)) \in \mathscr{R}(S)$$

by Proposition 2.12 again.

On the other hand, for any $A = (A_1, A_2, \ldots, A_k) \in \prod_{i=1}^{k} M_{m,n}(S_i)$, assume that $T(A) = (T_1(A_1), \ldots, T_2(A_2), \ldots, T_k(A_k)) \in \mathscr{R}(S)$. By Proposition 2.12, we have $T_i(A_i) \in \mathscr{R}(\mathbb{B}_1)$ for $i = 1, 2, \ldots, k$. Since T_i strongly preserves regularity on $M_{m,n}(\mathbb{B}_1)$, we immediately get $A_i \in \mathscr{R}(\mathbb{B}_i)$ for $i = 1, 2, \ldots, k$. And then we have $A = (A_1, A_2, \ldots, A_k) \in \mathscr{R}(S)$ by Proposition 2.12. $\square$

By virtue of Lemma 2.7, we can easily have the following result:

Theorem 2.14. *Let* $S = \prod_{i=1}^{k} S_i$, *where* $S_i = \mathbb{B}_1$. *Let* $\min\{m, n\} \leqslant 2$. *If* T *is a linear operator on* $M_{m,n}(S)$, *then* T *strongly preserves regularity.*

In the following we will continue to characterize the linear operators that strongly preserve regular matrices over $S = \prod_{i=1}^{k} \mathbb{B}_1$ when $\min\{m, n\} \geqslant 3$.

Theorem 2.15. *Let* $S = \prod_{i=1}^{k} S_i$, *where* $S_i = \mathbb{B}_1$, *and* T *be a linear operator on* $M_{m,n}(S)$, *where* $\min\{m, n\} \geqslant 3$. *Then the following statements are equivalent:*

(i) *T strongly preserves regularity;*

(ii) *there exist invertible matrices $P = (P_1, P_2, \ldots, P_k) \in M_m(S)$, $Q = (Q_1, Q_2, \ldots, Q_k) \in M_n(S)$ where P_i, Q_i are permutation matrices such that either $T(X) = PXQ$ for any $X \in M_{m,n}(S)$ or $m = n$ and $T(X) = P(s_1 X + s_2 X^t)Q$ for any $X \in M_n(S)$, where $s_1, s_2 \in S$ satisfy $(s_1)_i, (s_2)_i \in \{0, 1\}$ and $(s_1)_i \neq (s_2)_i$ for any $i \in \{1, 2, \ldots, k\}$.*

Proof. (i) $\Rightarrow$ (ii) Let $S = \prod_{i=1}^{k} S_i$, where $S_i = \mathbb{B}_1$, and T be a linear operator on $M_{m,n}(S)$, where $\min\{m, n\} \geqslant 3$. Suppose that T strongly preserves regularity. By Proposition 2.11, T_i strongly preserves regularity for any $i \in \{1, 2, \ldots, k\}$. It follows from Lemma 2.7 that there exist permutation matrices P_i and Q_i such that either

$$T_i(X_i) = P_i X_i Q_i \quad \text{for any } X_i \in M_{m,n}(S_i) \tag{1}$$

or $m = n$ and

$$T_i(X_i) = P_i X_i^t Q_i \quad \text{for any } X_i \in M_n(S_i). \tag{2}$$

Take $P = (P_1, P_2, \ldots, P_k) \in M_m(S)$, $Q = (Q_1, Q_2, \ldots, Q_k) \in M_n(S)$. If $m \neq n$, then for any $i \in \{1, 2, \ldots, k\}$, $T_i(X_i) = P_i X_i Q_i$ for any $X_i \in M_n(S_i)$. Thus $T(A) = PAQ$ for any $A \in M_{m,n}(S)$. If $m = n$, let

$$I_1 := \{i \colon T_i \text{ is the form of (1)}\},$$
$$I_2 := \{i \colon T_i \text{ is the form of (2)}\}.$$

It is clear that $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, 2, \ldots, k\}$. Let $s_1, s_2 \in S$ satisfy $(s_1)_i = 1$ if $i \in I_1$, 0 otherwise; $(s_2)_i = 1$ if $i \in I_2$, 0 otherwise. It follows that

$$T(A) = P(s_1 A + s_2 A^t)Q$$

for any $A \in M_{m,n}(S)$.

(ii) $\Rightarrow$ (i) Assume that $T(X) = PXQ$ for any $X \in M_{m,n}(S)$. Since P and Q are invertible, it is easy to see that T strongly preserves regularity in this case. Another case is $m = n$ and $T(X) = P(s_1 X + s_2 X^t)Q$ for any $X \in M_n(S)$. For any $i \in \{1, 2, \ldots, k\}$ and $A_i \in M_n(S_i)$, there exists $\bar{A} \in M_n(S)$ such that $A_i = (\bar{A})_i$,

$$T_i(A_i) = T_i((\bar{A})_i) = T(\bar{A})_i = (P(s_1 \bar{A} + s_2 \bar{A}^t)Q)_i.$$

If $(s_1)_i = 1, (s_2)_i = 0$, then

$$T_i(A_i) = (P(s_1 \bar{A} + s_2 (\bar{A})^t)Q)_i = P_i A_i Q_i$$

for any $A_i \in M_n(S_i)$. Otherwise,

$$T_i(A_i) = P_i A_i^t Q_i$$

for any $A_i \in M_n(S_i)$. Noticing that P_i and Q_i are permutation matrices, by Lemma 2.7, T_i strongly preserves regularity, and then T strongly preserves regularity by Proposition 2.13. $\square$

Hence, we have obtained characterizations of linear operators that strongly preserve regular matrices over the finite direct product of binary Boolean algebras $\mathbb{B}_1$.

3. Linear operators strongly preserving invertible matrices over $\mathbb{B}_k$

In this section, analogous with the discussions in Section 2, we will continue to study the characterizations of the linear operators strongly preserving invertible matrices over general Boolean algebras $\mathbb{B}_k$.

First, we will introduce the following lemma and proposition.

Lemma 3.1 ([14]). *Let T be a linear operator on $M_n(\mathbb{B}_1)$. Then T strongly preserves $GL_n(\mathbb{B}_1)$ if and only if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in M_n(\mathbb{B}_1)$, or $T(X) = PX^tQ$ for all $X \in M_n(\mathbb{B}_1)$.*

Proposition 3.2. *Let T be a linear operator on $M_n(\mathbb{B}_k)$ with $k \geqslant 1$. Then T strongly preserves $GL_n(\mathbb{B}_k)$ if and only if its each lth constituent operator, T_l, strongly preserves $GL_n(\mathbb{B}_1)$.*

Proof. ($\Rightarrow$) For any $A_l \in GL_n(\mathbb{B}_1)$, take $A = A_l = \sum\limits_{i=1}^{k} \sigma_i A_l$. Clearly, $A \in GL_n(\mathbb{B}_k)$. Since T strongly preserves $GL_n(\mathbb{B}_k)$ with $k \geqslant 1$, we have $T(A) \in GL_n(\mathbb{B}_k)$. And then $T_l(A_l) \in GL_n(\mathbb{B}_1)$ by Lemma 2.2.

On the other hand, for any $A \in M_n(\mathbb{B}_1)$, assume that $T_l(A) \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Note that $T(A) = T\Bigl(\sum\limits_{i=1}^{k} \sigma_i A_l\Bigr) = \sum\limits_{i=1}^{k} \sigma_i T_l(A_l)$, so we have $T(A) \in GL_n(\mathbb{B}_k)$ by Lemma 2.2. Since T is a linear operator that strongly preserves invertibility on $M_n(\mathbb{B}_k)$, we have $A \in GL_n(\mathbb{B}_k)$. And then it follows from Lemma 2.2 that $A(= A_l) \in GL_n(\mathbb{B}_1)$.

($\Leftarrow$) For any $A = \sum\limits_{l=1}^{k} \sigma_l A_l \in GL_n(\mathbb{B}_k)$, by Lemma 2.2, $A_l \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Since each lth constituent operator, T_l, strongly preserves invertibility on $M_n(\mathbb{B}_1)$, we have $T_l(A_l) \in GL_n(\mathbb{B}_1)$. By Lemma 2.2 again, $T(A) = T\Bigl(\sum\limits_{l=1}^{k} \sigma_l A_l\Bigr) = \sum\limits_{i=1}^{k} \sigma_l T_l(A_l) \in GL_n(\mathbb{B}_k)$.

On the other hand, assume that for any $A = \sum\limits_{l=1}^{k} \sigma_l A_l \in M_{m,n}(\mathbb{B}_k)$, $T(A) = T\Bigl(\sum\limits_{l=1}^{k} \sigma_l A_l\Bigr) = \sum\limits_{i=1}^{k} \sigma_l T_l(A_l) \in GL_n(\mathbb{B}_k)$. Then by Lemma 2.2, we have $T_l(A_l) \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots\ldots, k$. Noting that T_l strongly preserves invertibility on $M_n(\mathbb{B}_1)$, we immediately get $A_l \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. And then by Lemma 2.2, we have $A = \sum\limits_{l=1}^{k} \sigma_l A_l \in GL_n(\mathbb{B}_k)$. $\square$

Now, we will give the main theorem of this section.

Theorem 3.3. *Let T be a linear operator on $M_n(\mathbb{B}_k)$ with $k \geqslant 1$. Then T strongly preserves $GL_n(\mathbb{B}_k)$ if and only if there are invertible matrices P and Q such that $T(X) = P\Big(\sum\limits_{l=1}^{k} \sigma_l Y_l\Big)Q$ for all $X = \sum\limits_{l=1}^{k} \sigma_l X_l \in M_n(\mathbb{B}_k)$, where $Y_l = X_l$ or $Y_l = X_l^t$.*

Proof. ($\Rightarrow$) Assume that T strongly preserves $GL_n(\mathbb{B}_k)$. It follows from Proposition 3.2 that its each lth constituent operator, T_l, also strongly preserves $GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Now, for any $X = \sum\limits_{l=1}^{k} \sigma_l X_l \in M_n(\mathbb{B}_k)$, $T(X) = \sum\limits_{l=1}^{k} \sigma_l T_l(X_l)$. By Lemma 3.1, each lth constituent operator, T_l, has the form

$$T_l(X_l) = P_l X_l Q_l \quad \text{or} \quad T_l(X_l) = P_l X_l^t Q_l,$$

where P_l and Q_l are permutation matrices for all $l = 1, 2, \ldots, k$, so we can immediately obtain that

$$T(X) = \sum_{l=1}^{k} \sigma_l T_l(X_l) = \sum_{l=1}^{k} \sigma_l P_l Y_l Q_l = \Big(\sum_{l=1}^{k} \sigma_l P_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Y_l\Big)\Big(\sum_{l=1}^{k} \sigma_l Q_l\Big)$$

where $Y_l = X_l$ or $Y_l = X_l^t$ for $l = 1, 2, \ldots, k$. Take $\sum\limits_{l=1}^{k} \sigma_l P_l = P$, $\sum\limits_{l=1}^{k} \sigma_l Q_l = Q$, then by Lemma 2.2, P, Q are clearly invertible in $M_n(\mathbb{B}_k)$. Hence, we have shown the necessity.

($\Leftarrow$) Assume that for any $X = \sum\limits_{l=1}^{k} \sigma_l X_l \in M_n(\mathbb{B}_k)$, $X \in GL_n(\mathbb{B}_k)$. By Lemma 2.2, $X_l \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$, and so $Y_l \in GL_n(\mathbb{B}_1)$, where $Y_l = X_l$ or $Y_l = X_l^t$. By Lemma 2.2 again, $\sum\limits_{l=1}^{k} \sigma_l Y_l \in GL_n(\mathbb{B}_k)$. Note that P and Q are invertible matrices, so we immediately get $T(X) = P\Big(\sum\limits_{l=1}^{k} \sigma_l Y_l\Big)Q \in GL_n(\mathbb{B}_k)$.

On the other hand, assume that $X = \sum\limits_{l=1}^{k} \sigma_l X_l \in M_n(\mathbb{B}_k)$ and $T(X) \in GL_n(\mathbb{B}_k)$. Note that $T(X) = P\Big(\sum\limits_{l=1}^{k} \sigma_l Y_l\Big)Q$, where P and Q are invertible matrices, $Y_l = X_l$ or $Y_l = X_l^t$, so we have $Y = \sum\limits_{l=1}^{k} \sigma_l Y_l \in GL_n(\mathbb{B}_k)$. And then $Y_l \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. It follows from $Y_l = X_l$ or $Y_l = X_l^t$ that $X_l \in GL_n(\mathbb{B}_1)$ for $l = 1, 2, \ldots, k$. Consequently, we have $X \in GL_n(\mathbb{B}_k)$ by Lemma 2.2. $\square$

Thus, we have also obtained complete characterizations of the linear operators that strongly preserve invertible matrices over $\mathbb{B}_k$. An example will be given in the following.

Example 3.4. Let

$$P=\begin{pmatrix}\sigma_1 & \sigma_2 & \sigma_3\\ \sigma_2 & \sigma_3 & \sigma_1\\ \sigma_3 & \sigma_1 & \sigma_2\end{pmatrix},\quad Q=\begin{pmatrix}\sigma_3 & \sigma_1 & \sigma_2\\ \sigma_1 & \sigma_2 & \sigma_3\\ \sigma_2 & \sigma_3 & \sigma_1\end{pmatrix}\in M_3(\mathbb{B}_3).$$

By Lemma 2.2, $P, Q \in GL_3(\mathbb{B}_3)$.

Now, define a linear operator T on $M_3(\mathbb{B}_3)$ by

$$T(X)=P(\sigma_1X_1+\sigma_2X_2^t+\sigma_3X_3)Q$$

for all $X=\sum\limits_{l=1}^{3}\sigma_lX_l$. Then it is easy to check that T is a linear operator that strongly preserves $GL_3(\mathbb{B}_3)$.

In the following, we will also from another perspective give some other characterizations of linear operators that strongly preserve invertible matrices over $\mathbb{B}_k$. That is, we will study the characterizations of linear operators that strongly preserve invertible matrices over the finite direct product of binary Boolean algebras $\mathbb{B}_1$.

Proposition 3.5. *Let $S=\prod\limits_{i=1}^{k}S_i$, where $S_i=\mathbb{B}_1$. Then for any $A, B\in M_n(S)$, A is invertible if and only if A_i is invertible for any $i\in\{1,2,\dots,k\}$.*

Proof. It is routine to check this proposition. □

Proposition 3.6. *Let $S=\prod\limits_{i=1}^{k}S_i$, where $S_i=\mathbb{B}_1$. Let T be a linear operator on $M_n(S)$. Then T strongly preserves $GL_n(S)$ if and only if its ith constituent operator, T_i, strongly preserves $GL_n(\mathbb{B}_1)$ for any $i\in\{1,2,\dots,k\}$.*

Proof. Let T be a linear operator on $M_n(S)$ and T_i be its ith constituent operator on $M_n(\mathbb{B}_1)$.

($\Rightarrow$) For any $A_l\in M_n(\mathbb{B}_1)$, assume that $A_l\in GL_n(\mathbb{B}_1)$. Take $A=(A_l, A_l,\dots, A_l)\in\prod\limits_{i=1}^{k}M_n(S_i)$. We have $A\in GL_n(S)$ by Proposition 3.5. Since T is a linear operator that strongly preserves invertibility on $M_n(S)$, we have $T(A)=(T_1(A_l),\dots,T_l(A_l),\dots,T_k(A_l))\in GL_n(S)$. And then we get $T_i(A_l)\in GL_n(\mathbb{B}_1)$.

On the other hand, for any $A_l\in M_n(\mathbb{B}_1)$, assume that $T_i(A_l)\in GL_n(\mathbb{B}_1)$ for $i=1,2,\dots,k$. Take $A=(A_l, A_l,\dots,A_l)\in\prod\limits_{i=1}^{k}M_n(S_i)$. We have

$$T(A)=(T_1(A_l),\dots,T_i(A_l),\dots,T_k(A_l))\in GL_n(S)$$

by Proposition 3.5. And then $A \in GL_n(S)$ since T is a linear operator that strongly preserves invertibility on $M_n(S)$. By Proposition 3.5 again, we have $A_l \in GL_n(\mathbb{B}_1)$.

($\Leftarrow$) For any $A = (A_1, A_2, \ldots, A_k) \in \prod_{i=1}^{k} M_n(S_i)$, by Proposition 3.5, $A_i \in GL_n(\mathbb{B}_1)$ for $i = 1, 2, \ldots, k$. Since each ith constituent operator, T_i, strongly preserves the invertibility on $M_n(\mathbb{B}_1)$, we immediately have $T_i(A_i) \in GL_n(\mathbb{B}_1)$. By Proposition 3.5 again, $T(A) = (T_1(A_1), \ldots, T_2(A_2), \ldots, T_k(A_k)) \in GL_n(S)$.

On the other hand, for any $A = (A_1, A_2, \ldots, A_k) \in \prod_{i=1}^{k} M_n(S_i)$, assume that $T(A) = (T_1(A_1), \ldots, T_2(A_2), \ldots, T_k(A_k)) \in GL_n(S)$. By Proposition 3.5, we have $T_i(A_i) \in GL_n(\mathbb{B}_1)$ for $i = 1, 2, \ldots, k$. Since T_i strongly preserves invertibility on $M_n(\mathbb{B}_1)$, we immediately get $A_i \in GL_n(\mathbb{B}_1)$ for $i = 1, 2, \ldots, k$. And then we have $A = (A_1, A_2, \ldots, A_k) \in GL_n(S)$ by Proposition 3.5. $\square$

Theorem 3.7. *Let $S = \prod_{i=1}^{k} S_i$, where $S_i = \mathbb{B}_1$, and T be a linear operator on $M_n(S)$. Then the following statements are equivalent:*

(i) *T strongly preserves $GL_n(S)$;*

(ii) *there exist invertible matrices $P = (P_1, P_2, \ldots, P_k), Q = (Q_1, Q_2, \ldots, Q_k) \in M_n(S)$, where P_i, Q_i are permutation matrices, such that $T(X) = P(s_1 X + s_2 X^t)Q$ for any $X \in M_n(S)$, where $s_1, s_2 \in S$ satisfy $(s_1)_i, (s_2)_i \in \{0, 1\}$ and $(s_1)_i \neq (s_2)_i$ for any $i \in \{1, 2, \ldots, k\}$.*

Proof. (i) $\Rightarrow$ (ii) Let $S = \prod_{i=1}^{k} S_i$, where $S_i = \mathbb{B}_1$, and T be a linear operator on $M_n(S)$. Suppose that T strongly preserves $GL_n(S)$. By Proposition 3.6, T_i strongly preserves $GL_n(\mathbb{B}_1)$ for any $i \in \{1, 2, \ldots, k\}$. It follows from Lemma 3.1 that there exist permutation matrices P_i and Q_i such that either

$$T_i(X_i) = P_i X_i Q_i \tag{3}$$

or

$$T_i(X_i) = P_i X_i^t Q_i \tag{4}$$

for any $X_i \in M_n(S_i)$. Take $P = (P_1, P_2, \ldots, P_k), Q = (Q_1, Q_2, \ldots, Q_k) \in M_n(S)$. Let

$$I_1 := \{i\colon\ T_i \text{ is the form of (3)}\},$$
$$I_2 := \{i\colon\ T_i \text{ is the form of (4)}\}.$$

It is clear that $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, 2, \ldots, k\}$. Let $s_1, s_2 \in S$ satisfy $(s_1)_i = 1$ if $i \in I_1$, 0 otherwise; $(s_2)_i = 1$ if $i \in I_2$, 0 otherwise. It follows that

$$T(A) = P(s_1 A + s_2 A^t)Q$$

for any $A \in M_n(S)$.

(ii) $\Rightarrow$ (i) Assume that there exist invertible matrices $P = (P_1, P_2, \ldots, P_k), Q = (Q_1, Q_2, \ldots, Q_k) \in M_n(S)$ with P_i, Q_i are permutation matrices such that

$$T(X) = P(s_1 X + s_2 X^t)Q$$

for any $X \in M_n(S)$. For any $i \in \{1, 2, \ldots, k\}$ and $A_i \in M_n(S_i)$, there exists $\bar{A} \in M_n(S)$ such that $A_i = (\bar{A})_i$. And then

$$T_i(A_i) = T_i((\bar{A})_i) = T(\bar{A})_i = (P(s_1\bar{A} + s_2\bar{A}^t)Q)_i.$$

If $(s_1)_i = 1, (s_2)_i = 0$, then

$$T_i(A_i) = (P(s_1\bar{A} + s_2(\bar{A})^t)Q)_i = P_i A_i Q_i$$

for any $A_i \in M_n(S_i)$. Otherwise,

$$T_i(A_i) = P_i A_i^t Q_i$$

for any $A_i \in M_n(S_i)$. Noticing that P_i and Q_i are permutation matrices, by Lemma 3.1, T_i strongly preserves invertibility, and then T strongly preserves invertibility by Proposition 3.6. $\square$

Hence, we have obtained characterizations of linear operators that strongly preserve invertible matrices over the finite direct product of binary Boolean algebras $\mathbb{B}_1$. This also means that we get some other characterizations of linear operators that strongly preserve invertible matrices over $\mathbb{B}_k$ from another point of view.

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