Marko D. Petković; Paul Barry; Predrag Rajković

Closed-form expression for Hankel determinants of the Narayana polynomials

Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 1, 39–57

Czechoslovak Mathematical Journal, 62 (137) (2012), 39–57

CLOSED-FORM EXPRESSION FOR HANKEL DETERMINANTS OF THE NARAYANA POLYNOMIALS

Marko D. Petković, Niš, Paul Barry, Waterford,
Predrag Rajković, Niš

(Received July 15, 2010)

Abstract. We considered a Hankel transform evaluation of Narayana and shifted Narayana polynomials. Those polynomials arises from Narayana numbers and have many combinatorial properties. A mainly used tool for the evaluation is the method based on orthogonal polynomials. Furthermore, we provided a Hankel transform evaluation of the linear combination of two consecutive shifted Narayana polynomials, using the same method (based on orthogonal polynomials) and previously obtained moment representation of Narayana and shifted Narayana polynomials.

Keywords: Narayana numbers, Hankel transform, orthogonal polynomials

MSC 2010: 11Y55, 34A25

1. Introduction

The *Narayana numbers* were studied for the first time by P. A. MacMahon [19]. The name is given in honour to T. V. Narayana who later rediscovered them in [21].

Definition 1.1. Narayana numbers $(N(n,k))_{n,k\in\mathbb{N}_0}$ are defined by the relations [18], [23], [24]

$$N(0,k)=N(k,0)=\delta_{k0}\ \ (k\in\mathbb{N}_0),\quad N(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}\ \ (n,k\in\mathbb{N}). \tag{1.1}$$

Here δ_{ij} denotes the Kronecker delta function.

This research was supported by the Science Foundation of Republic Serbia, Project No. 174011 and Project No. 174013.

This sequence is denoted by A001263 in the On-Line Encyclopedia of Integer Sequences [23]. The corresponding sequence of *Narayana polynomials* $(a(n;r))_{n\in\mathbb{N}_0}$ is defined by

$$a(n;r) = \sum_{k=0}^{n} N(n,k) r^k. \tag{1.2}$$

Narayana numbers and polynomials actuate a lot of attention due to their various combinatorial interpretations (see [1], [3], [24]).

The connection between Narayana numbers and lattice paths was studied by R. A. Sulanke in [24]. Sulanke considered lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,-1)$ to (n,n), and proved that the number of such paths of length k with step size from $\mathbb{N}\times\mathbb{N}$, such that the whole path remains strictly above the line $y = x-1$, is equal to $N(n,k)$.

Furthermore, Brändén [3] considered *Dyck paths*, i.e. paths with the step size from the set $\{(0,1),(1,0)\}$ (i.e. in each step you can move up or forward by unit distance). It is proved that the number of colored Dyck paths, such that each corner of this path is colored in one of the r colors, is $a(n;r)$.

Various other interesting properties of Narayana numbers and polynomials can be found in [3], [18], [24], [25]. Narayana polynomials are refining the famous Catalan sequence $\mathcal{C}_n$, i.e. $a(n;1) = \mathcal{C}_n$ (see for example [18], [24]). A lot of identities involving Narayana polynomials and Catalan numbers have been recently published by Mansour and Sun in [20]. Moreover, $(a(n;2))_{n\in\mathbb{N}_0}$ is the sequence of *large Schröder numbers* [1], [4].

In order to provide an easier evaluation of the Hankel transform of Narayana polynomials, we introduce the sequences of *modified Narayana numbers and polynomials.*

Definition 1.2. The modified Narayana numbers $(N_1(n,k))_{n,k\in\mathbb{N}_0}$ are defined by the relations

$$N_1(0,k) = \delta_{k0}, \quad N_1(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1} \quad (n\in\mathbb{N};\ k\in\mathbb{N}_0).$$

The infinite matrix whose elements are Narayana numbers and the matrix whose members are modified Narayana numbers are given by

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & 0 & \\ 0 & 1 & 3 & 1 & 0 & \\ 0 & 1 & 6 & 6 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix}, \qquad \mathbf{N_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 0 & 0 & \\ 1 & 3 & 1 & 0 & 0 & \\ 1 & 6 & 6 & 1 & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

The corresponding *modified Narayana polynomials* are

$$a_1(n;r) = \sum_{k=0}^{n-1} N_1(n,k) r^k. \tag{1.3}$$

The connections between Narayana numbers (polynomials) and modified Narayana numbers (polynomials) are given by

$$N_1(n,k) = N(n,k+1); \quad a(0;r) = a_1(0;r) = 1, \quad a(n;r) = r \cdot a_1(n;r) \quad (n \in \mathbb{N}). \tag{1.4}$$

Remark 1.1. The sequence $((a_1(n;r))_{n\in\mathbb{N}_0}$ reduces to the sequence of *Catalan numbers* for $r = 1$, and to that of *little Schröder numbers* for $r = 2$.

The rest of the paper is organized as follows. In Section 2, we recall the definition and basic properties of the Hankel transform. Also, we introduce the method based on orthogonal polynomials for Hankel transform evaluation. That method is applied to the sequence of Narayana polynomials, in two steps. First, in Section 3, we evaluate the Hankel transform of the shifted Narayana polynomials. Then, in Section 4, we prove a few additional properties of the Hankel transform which are used for deriving the final result, together with the evaluation in the previous section. Section 5 provides further results concerning the Hankel transform evaluation of sums of two consecutive shifted Narayana polynomials.

2. Hankel transform and series reversion

An important transform on integer sequences that has been much studied recently is the Hankel transform [7], [17], [22]:

Definition 2.1. The *Hankel transform* of a given sequence $a = (a_n)_{n\in\mathbb{N}_0}$ is the sequence of Hankel determinants $(h_n)_{n\in\mathbb{N}_0}$ where $h_n = \det[a_{i+j-2}]_{i,j=1}^{n}$, i.e.

$$a = (a_n)_{n\in\mathbb{N}_0} \xrightarrow{\mathcal{H}} h = (h_n)_{n\in\mathbb{N}_0} : \ h_n = \det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{bmatrix}. \tag{2.1}$$

We denote the Hankel transform by $\mathcal{H}$ and hence we write $h = \mathcal{H}(a)$.

The Hankel transform has been widely investigated in numerous papers. For example, the papers [7], [22] deal with the Hankel transform of the sequences of the Catalan and adjusted Catalan numbers. Both the evaluations are done by using the method based on the orthogonal polynomials. Chamberland and French [5] generalized results from [7] providing the Hankel transform of sums of two consecutive generalized Catalan numbers. They used a technique different from [7]. Another Hankel transform evaluation concerning Catalan numbers is given by Barry and Hennessey in [2].

On the other hand, evaluation of Hankel determinants with orthogonal polynomial entries is investigated by M. E. H. Ismail in [13]. Also in paper [10], the authors provided the Hankel transform of central binomial coefficients. Many other Hankel transform evaluations are given in the papers [8], [9], [14], [26]. In this paper we also use a method based on orthogonal polynomials, as used in [7], [22].

Let $(a_n)_{n\in\mathbb{N}_0}$ be the moment sequence with respect to some measure $\mathrm{d}\lambda$. In other words, let

$$a_n = \int_{\mathbb{R}} x^n \,\mathrm{d}\lambda \quad (n = 0, 1, 2, \ldots). \tag{2.2}$$

Then the Hankel transform $h = \mathcal{H}(a)$ of the sequence $a = (a_n)_{n\in\mathbb{N}_0}$ can be expressed by the following relation known as the Heilermann formula (for example, see Krattenthaler [15]):

$$h_n = a_0^{n+1}\beta_1^n\beta_2^{n-1}\ldots\beta_{n-1}^2\beta_n. \tag{2.3}$$

The sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x-\alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \tag{2.4}$$

where $(Q_n(x))_{n\in\mathbb{N}_0}$ is the monic polynomial sequence, orthogonal with respect to the measure $\mathrm{d}\lambda$. The following theorem and corollary provide the way how to explicitly find the measure $\mathrm{d}\lambda$ with a prescribed moment sequence.

Theorem 2.1 (Stieltjes-Perron inversion formula [6], [16]). *Let $(\mu_n)_{n\in\mathbb{N}_0}$ be a sequence such that all elements of its Hankel transform are non-negative. Denote by $G(z) = \sum_{n=0}^{+\infty}\mu_n z^n$ the generating function of the sequence $(\mu_n)_{n\in\mathbb{N}_0}$ and $F(z) = z^{-1}G(z^{-1})$. Also let the function $\lambda(t)$ be defined by*

$$\lambda(t) - \lambda(0) = -\frac{1}{2\pi \mathrm{i}}\lim_{y\to 0^+}\int_0^t [F(x+\mathrm{i}y) - F(x-\mathrm{i}y)]\,\mathrm{d}x. \tag{2.5}$$

Then holds $\mu_n = \int_{\mathbb{R}} x^n\,\mathrm{d}\lambda$, i.e. sequence $(\mu_n)_{n\in\mathbb{N}_0}$ is the moment sequence of the measure $\lambda(t)$.

Corollary 2.2. *Under the assumptions of the previous theorem, let in addition* $F(\overline{z}) = \overline{F(z)}$ *hold. Then*

$$\lambda(t) - \lambda(0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^t \Im F(x + \mathrm{i}y) \, \mathrm{d}x. \tag{2.6}$$

We use Theorem 2.1 and Corollary 2.2 to determine the weight function (measure) such that the given sequence is the moment sequence to that measure. After that, by applying weight function transformations and using the Heilermann formula (2.3), we provide a closed-form expression for the Hankel transform. In order to efficiently use Theorem 2.1 and Corollary 2.2, we need the expression for the weight function of our sequence.

At last, for our further discussion we need the notion of series reversion (see [1]).

Definition 2.2. For a given function $v = f(u)$ with the property $f(0) = 0$, the series reversion is the sequence $\{s_k\}_{k \in \mathbb{N}_0}$ such that

$$u = f^{-1}(v) = s_0 + s_1 v + \ldots + s_n v^n + \ldots,$$

where $u = f^{-1}(v)$ is the inverse function of $v = f(u)$.

3. The Hankel transform of the shifted Narayana polynomials

Define the auxiliary sequence of *shifted Narayana polynomials* $a_{\mathrm{sh}}(n; r) = a_1(n+1; r) = r^{-1} a(n; r)$. That sequence is introduced since it is the moment sequence with respect to the absolutely continuous measure.

3.1. Evaluation of the weight function

The generating function $A_{\mathrm{sh}}(x; r)$ of the shifted Narayana polynomials can be expressed in terms of series reversion. For more details, see [1].

Lemma 3.1. *Denote by* $f(u)$ *the function*

$$v = f(u) = \frac{u}{1 + (r+1)u + ru^2}. \tag{3.1}$$

Series reversion of $f(u)$ *satisfies*

$$u = f^{-1}(v) = \sum_{n=0}^{+\infty} a_{\mathrm{sh}}(n; r) v^{n+1}. \tag{3.2}$$

Denote by $A_{\rm sh}(x;r)$ the generating function of the sequence $a_{\rm sh}(n;r)$, i.e.

$$A_{\rm sh}(x;r) = \sum_{n=0}^{+\infty} a_{\rm sh}(n;r)x^n.$$

From Lemma 3.1, we have that $A_{\rm sh}(x;r)$ satisfies the equation

$$\frac{xA_{\rm sh}(x;r)}{1+(r+1)xA_{\rm sh}(x;r)+rx^2A_{\rm sh}^2(x;r)} = x.$$

This equation has two solutions for $A_{\rm sh}(x;r)$, but only one satisfies the condition $A_{\rm sh}(0;r) = a_{\rm sh}(0;r) = 1$. Hence, we obtain

$$A_{\rm sh}(x;r) = \frac{1-(r+1)x-\sqrt{(1-(r+1)x)^2-4rx^2}}{2rx^2}.$$

Our next goal is to find the weight function $w_{\rm sh}(x)$ whose moments are $a_{\rm sh}(n;r)$, i.e.

$$a_{\rm sh}(n;r) = \int_{\mathbb{R}} x^n w_{\rm sh}(x)\,{\rm d}x. \tag{3.3}$$

Lemma 3.2. *Sequence $(a_{\rm sh}(n;r))_{n\in\mathbb{N}_0}$ is the moment sequence of the weight function*

$$w_{\rm sh}(x) = \begin{cases} \dfrac{\sqrt{4r-(x-r-1)^2}}{2\pi r}, & x \in \left(\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

P r o o f. The proof is based on the Stieltjes-Perron inversion formula (Theorem 2.1). First, let us denote by $F(z,r)$ the function

$$F(z;r) = z^{-1}A_{\rm sh}(z^{-1};r) = -\frac{(r+1)-z+\sqrt{(z-r-1)^2-4r}}{2r},$$

and by $\psi(x)$ the distribution function given by

$$\psi(x)-\psi(0) = -\frac{1}{\pi}\lim_{y\to 0^+}\int_0^t \Im F(x+{\rm i}y;r)\,{\rm d}x. \tag{3.5}$$

Consider the function $F(z;r)$ in the half-plane $\{z\in\mathbb{C}\colon\ \Im z>0\}$.

The function

$$\varrho(z;r) = \sqrt{(z-r-1)^2-4r}$$

has the branch points $(\sqrt{r}-1)^2$ and $(\sqrt{r}+1)^2$. We need to choose the regular branch of the square root such that it is positive when the expression under the square root is positive.

By explicit evaluation, we get that the integral of $F(z;r)$ is

$$\mathcal{F}(z;r) = \int F(z;r)\,\mathrm{d}z = \frac{1}{4r}(z^2 + (1+r-z)\varrho(z;r) - 2z(r+1)) + l_1(z;r), \tag{3.6}$$

where

$$l_1(z;r) = \ln(r+1-z+\varrho(z;r)).$$

Now equation (3.5) becomes

$$\psi(x) - \psi(0) = -\frac{1}{\pi}\lim_{y\to 0+}(\mathcal{F}(x+\mathrm{i}y;r) - \mathcal{F}(\mathrm{i}y;r)). \tag{3.7}$$

Let us now find $\lim_{y\to 0^+} \Im\mathcal{F}(x+\mathrm{i}y;r)$. It is obvious that:

$$\lim_{y\to 0^+} \Im\varrho(x+\mathrm{i}y;r) = \begin{cases} \sqrt{4r-(x-r-1)^2}, & x \in \left(\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases}$$

The function $l_1(z;r)$ has one more branch point $z = r+1$. We take a branch of $l_1(z;r)$ such that the imaginary part is 0 if the value under the logarithm is real and positive. We have

$$\lim_{y\to 0^+} \Im\, l_1(x+\mathrm{i}y;r) = \begin{cases} \pi + \arctan\dfrac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)}, & x \in \left(\left(\sqrt{r}-1\right)^2, r+1\right), \\ \arctan\dfrac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)}, & x \in \left(r+1, \left(\sqrt{r}+1\right)^2\right), \\ 0, & \text{otherwise.} \end{cases}$$

Including the two previous results on $\mathcal{F}(x+\mathrm{i}y;r)$ and using (3.7), we get the required expression for the weight function

$$w_{\mathrm{sh}}(x) = \psi'(x) = \frac{\sqrt{4r-(x-r-1)^2}}{2\pi r}, \quad x \in \left(\left(\sqrt{r}-1\right)^2, \left(\sqrt{r}+1\right)^2\right).$$

Outside the segment $[(\sqrt{r}-1)^2, (\sqrt{r}+1)^2]$, we have $w_{\mathrm{sh}}(x) = 0$. □

3.2. Evaluation of the coefficients of three-term recurrence relation

Denote by $(Q_n^{\mathrm{sh}}(x))_{n\in\mathbb{N}_0}$ the sequence of monic polynomials, orthogonal with respect to the weight function $w_{\mathrm{sh}}(x)$.

In order to evaluate explicitly the coefficients $(\alpha_n^{\mathrm{sh}})_{n\in\mathbb{N}_0}$ and $(\beta_n^{\mathrm{sh}})_{n\in\mathbb{N}_0}$ of the three-term recurrence relation

$$Q_{n+1}^{\mathrm{sh}}(x) = (x-\alpha_n^{\mathrm{sh}})Q_n^{\mathrm{sh}}(x) - \beta_n^{\mathrm{sh}}Q_{n-1}^{\mathrm{sh}}(x), \tag{3.8}$$

we apply the transformation formulas. These formulas connect the coefficients α_n^{sh} and β_n^{sh} of the original and transformed weight function.

Lemma 3.3. *Let $w(x)$ and $\overline{w}(x)$ be weight functions and denote by $(\pi_n(x))_{n\in\mathbb{N}_0}$ and $(\overline{\pi}_n(x))_{n\in\mathbb{N}_0}$ the corresponding orthogonal polynomials. Also denote by $(\alpha_n)_{n\in\mathbb{N}_0}$ $(\beta_n)_{n\in\mathbb{N}_0}$ and $(\overline{\alpha}_n)_{n\in\mathbb{N}_0}$, $(\overline{\beta}_n)_{n\in\mathbb{N}_0}$ the three-term relation coefficients corresponding to $w(x)$ and $\overline{w}(x)$ respectively. The following transformation formulas are valid:*

(1) *If $\overline{w}(x) = Cw(x)$ where $C>0$ then $\overline{\alpha}_n=\alpha_n$ for $n\in\mathbb{N}_0$ and $\overline{\beta}_0 = C\beta_0$, $\overline{\beta}_n=\beta_n$ for $n\in\mathbb{N}$. Moreover, we have $\overline{\pi}_n(x)=\pi_n(x)$ for all $n\in\mathbb{N}_0$.*

(2) *If $\overline{w}(x) = w(ax+b)$ where $a,b\in\mathbb{R}$ and $a\neq 0$ then $\overline{\alpha}_n = (\alpha_n-b)/a$ for $n\in\mathbb{N}_0$ and $\overline{\beta}_0=\beta_0/|a|$ and $\overline{\beta}_n=\beta_n/a^2$ for $n\in\mathbb{N}$. Moreover, we have $\overline{\pi}_n(x)=a^{-n}\pi_n(ax+b)$.*

P r o o f. In both cases, we directly check the orthogonality of $\overline{\pi}_n(x)$ and obtain the coefficients $\overline{\alpha}_n$ and $\overline{\beta}_n$ by putting $\overline{\pi}_n(x)$ in the three-term recurrence relation for $\pi_n(x)$. □

Lemma 3.4. *The coefficients α_n^{sh} and β_n^{sh} $(n\in\mathbb{N}_0)$ in the three-term recurrence relation* (3.8) *are given by*

$$\beta_0^{\mathrm{sh}}=1,\;\; \beta_n^{\mathrm{sh}}=r\;\;(n\in\mathbb{N});\quad \alpha_n^{\mathrm{sh}}=r+1\;\;(n\in\mathbb{N}_0).$$

P r o o f. The monic Chebyshev polynomials of the second kind

$$Q_n^{(1)}(x)=S_n(x)=\frac{\sin((n+1)\arccos x)}{2^n\cdot\sqrt{1-x^2}}$$

are orthogonal with respect to the weight $w^{(1)}(x)=\sqrt{1-x^2}$. The corresponding coefficients in the three-term relation are

$$\beta_0^{(1)}=\frac{\pi}{2},\;\; \beta_n^{(1)}=\frac{1}{4}\;\;(n\geqslant 1),\quad \alpha_n^{(1)}=0\;\;(n\geqslant 0).$$

Let us introduce a new weight function $w^{(2)}(x) = \sqrt{4r - (x - r - 1)^2}$. It satisfies $w^{(2)}(x) = w^{(1)}(ax + b)$, where $a = 1/(2\sqrt{r})$ and $b = -(r+1)/(2\sqrt{r})$. Hence we get (see Lemma 3.3):

$$\beta_0^{(2)} = \pi\sqrt{r}, \ \ \beta_n^{(2)} = r \ \ (n \in \mathbb{N}), \quad \alpha_n^{(2)} = r + 1 \ \ (n \in \mathbb{N}_0).$$

Finally, since $w_{\mathrm{sh}}(x) = w^{(2)}(x)/(\pi\sqrt{r})$, we conclude that $\beta_n^{\mathrm{sh}} = \beta_n^{(2)} = r$ for $n \geqslant 1$ and $\alpha_n^{\mathrm{sh}} = \alpha_n^{(2)} = r + 1$. The coefficient β_0^{sh} can be obtained by direct evaluation of the integral $\beta_0^{\mathrm{sh}} = \int_{\mathbb{R}} w_{\mathrm{sh}}(x)\,\mathrm{d}x = 1$. □

Corollary 3.5. *The squared norms of monic orthogonal polynomials $Q_n(x)$ have values*

$$\|Q_n^{\mathrm{sh}}\|^2 = r^n \quad (n \in \mathbb{N}_0).$$

P r o o f. By using the statement of Lemma 3.4 and formula (2.3), the conclusion follows. □

3.3. Proof of the main result

Denote by $(h_{\mathrm{sh}}(n;r))_{n\in\mathbb{N}_0}$ the Hankel transform of the sequence $(a_{\mathrm{sh}}(n;r))_{n\in\mathbb{N}_0}$. Now using the Heilermann formula (2.3), we obtain

$$h_{\mathrm{sh}}(n;r) = a_{\mathrm{sh}}(0;r)^{n+1} \prod_{i=1}^{n} (\beta_i^{\mathrm{sh}})^{n+1-i} = 1^{n+1} \prod_{i=1}^{n} r^{n+1-i} = r^{\binom{n+1}{2}}.$$

This completes the proof of the main result of this section. □

Theorem 3.6. *The Hankel transform of the shifted Narayana polynomials $(a_{\mathrm{sh}}(n;r))_{n\in\mathbb{N}_0}$ is given by*

$$h_{\mathrm{sh}}(n;r) = r^{\binom{n+1}{2}}.$$

4. The Hankel transform of the Narayana and modified Narayana polynomials

Denote by $(h(n;r))_{n\in\mathbb{N}_0}$ and $(h_1(n;r))_{n\in\mathbb{N}_0}$, respectively, the Hankel transforms of the Narayana and modified Narayana polynomials, i.e. $(a(n;r))_{n\in\mathbb{N}_0}$ and $(a_1(n;r))_{n\in\mathbb{N}_0}$.

From the relation

$$a_1(n;r) = a_{\mathrm{sh}}(n-1;r) \quad (n \geqslant 1), \tag{4.1}$$

we conclude that $(a_1(n;r))_{n\geqslant 1}$ can be expressed as the moment sequence in the form

$$a_1(n;r) = \int_{\mathbb{R}} x^n \tilde{w}(x)\,\mathrm{d}x \ \ (n \geqslant 1), \quad \text{where } \tilde{w}(x) = \frac{w_{\mathrm{sh}}(x)}{x}. \tag{4.2}$$

On the other hand, we have

$$\int_{\mathbb{R}} \frac{w_{\mathrm{sh}}(x)}{x}\,\mathrm{d}x = r^{-1} \neq a_1(0;r).$$

Hence, we introduce the sequence $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ defined by

$$\tilde{a}(n;r) := \begin{cases} r^{-1}, & n = 0,\\ a_1(n;r), & n \geqslant 1, \end{cases} \tag{4.3}$$

which is the moment sequence for $\tilde{w}(x)$. The next lemma shows the connection between the Hankel transforms of the sequences $(a_1(n;r))_{n\in\mathbb{N}_0}$ and $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$.

Lemma 4.1. *For the sequences $(a_k)_{n\in\mathbb{N}_0}$ and $(\tilde{a}_k)_{n\in\mathbb{N}_0}$ such that $a_k = \tilde{a}_k$ for all $k \geqslant 1$, their Hankel transforms $(h_n)_{n\in\mathbb{N}_0}$ and $(\tilde{h}_n)_{n\in\mathbb{N}_0}$ are related by*

$$h_n = \tilde{h}_n + (a_0 - \tilde{a}_0)\hat{h}_{n-1} \quad (n \in \mathbb{N}_0),$$

where $\{\hat{h}_n\}_{n\in\mathbb{N}_0}$ is the Hankel transform of the sequence $\{\hat{a}_k\}_{k\in\mathbb{N}_0}$ given by $\hat{a}_k = a_{k+2}$ $(k \geqslant 0)$ and $\hat{h}_{-1} = 1$.

P r o o f. By expanding the determinants over the first row, we get

$$h_n = \sum_{k=0}^{n-1} a_k M_{1,k+1}, \quad \tilde{h}_n = \sum_{k=0}^{n-1} \tilde{a}_k \tilde{M}_{1,k+1}, \tag{4.4}$$

where $M_{1,k}$ and $\tilde{M}_{1,k}$ are the minors. Note that the minors $M_{1,k+1}$ and $\tilde{M}_{1,k+1}$ are equal for every $k \in \mathbb{N}_0$. Hence, we have

$$h_n - \tilde{h}_n = (a_0 - \tilde{a}_0) M_{1,1}.$$

But on the other hand, $M_{1,1}$ is the $(n-1)$st member of the Hankel transform of the sequence $\{a_2, a_3, \ldots\}$, i.e. we have $M_{1,1} = \det[a_{i+j}]_{1 \leqslant i,j \leqslant n-1}$ and we denote it by $\hat{h}_{n-1}$. □

Applying Lemma 4.1 to the sequences $(a_1(n;r))_{n \in \mathbb{N}_0}$ and $(\tilde{a}(n;r))_{n \in \mathbb{N}_0}$, we obtain the following corollary.

Corollary 4.2. *If $h_1(n;r)$, $\tilde{h}(n;r)$ and $\hat{h}(n;r)$ are the Hankel transforms of the sequences $a_1(n;r)$, $\tilde{a}(n;r)$ and $\hat{a}(n;r) = a_1(n+2;r)$ respectively, then*

$$h_1(n;r) = \tilde{h}(n;r) + (1 - r^{-1})\hat{h}(n-1;r).$$

Now we have to evaluate the Hankel transform of the sequences $(\hat{a}(n;r))_{n \in \mathbb{N}_0}$ and $(\tilde{a}(n;r))_{n \in \mathbb{N}_0}$.

4.1. The Hankel transform of $(\hat{a}(n;r))_{n \in \mathbb{N}_0}$

Note that $\hat{a}(n;r) = a_{\mathrm{sh}}(n+1;r)$, i.e. $(\hat{a}(n;r))_{n \in \mathbb{N}_0}$ is the shifted sequence of $(a_{\mathrm{sh}}(n;r))_{n \in \mathbb{N}_0}$. Hence $(\hat{a}(n;r))_{n \in \mathbb{N}_0}$ is the moment sequence of the weight function $\hat{w}(x) = x w_{\mathrm{sh}}(x)$. The following lemma connects the Hankel transforms of the original and the shifted sequences.

Lemma 4.3. *Let $\omega(x)$ be the weight function and $a_n = \int_{\mathbb{R}} x^n \omega(x)\,\mathrm{d}x$ its nth moment. Denote $\overline{\omega}(x) = x\omega(x)$ and let $(Q_n(x))_{n \in \mathbb{N}_0}$ be the sequence of monic orthogonal polynomials corresponding to the weight $\omega(x)$. Then the Hankel transforms $(h_n)_{n \in \mathbb{N}_0}$ and $(\bar{h}_n)_{n \in \mathbb{N}_0}$ of the sequences $(a_n)_{n \in \mathbb{N}_0}$ and $(\bar{a}_n)_{n \in \mathbb{N}_0}$ are related by*

$$\bar{h}_n = h_n (-1)^{n+1} \lambda_{n+1} \quad (n \in \mathbb{N}_0), \tag{4.5}$$

where $\lambda_n = Q_n(0)$ $(n \in \mathbb{N}_0)$.

P r o o f. The coefficients β_n and $\overline{\beta}_n$ satisfy (see [11] or [12])

$$\overline{\beta}_n = \beta_n \frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2}.$$

Inserting it in the Heilermann formula (2.3), we obtain

$$\bar{h}_{n+1} = \bar{h}_n \bar{a}_0 \prod_{i=1}^{n+1} \bar{\beta}_i = \bar{h}_n \bar{a}_0 \prod_{i=1}^{n+1} \beta_i \frac{\lambda_{i-1}\lambda_{i+1}}{\lambda_i^2} = h_{n+1} \frac{\bar{h}_n}{h_n} \Big(\frac{\lambda_0 \bar{a}_0}{\lambda_1 a_0}\Big) \frac{\lambda_{n+2}}{\lambda_{n+1}}. \tag{4.6}$$

By applying the previous equation n times, we have

$$\frac{\bar{h}_{n+1}}{h_{n+1}} = \frac{\bar{h}_0}{h_0} \Big(\frac{\bar{a}_0}{a_0}\Big)^{n+1} \Big(\frac{\lambda_0}{\lambda_1}\Big)^{n+1} \frac{\lambda_{n+2}}{\lambda_1}. \tag{4.7}$$

Now, since $\bar{h}_0 = \bar{a}_0 = a_1$, $h_0 = a_0$ and $\lambda_0 = 1$, we have

$$\bar{h}_{n+1} = h_{n+1} \Big(\frac{a_1}{\lambda_1 a_0}\Big)^{n+2} \lambda_{n+2}.$$

Also note that $\lambda_1 = Q_1(0) = -a_1/a_0$, so $\bar{h}_{n+1} = (-1)^{n+2} h_{n+1} \lambda_{n+1}$. After decreasing the index by 1 we finally get by desired relation (4.5). Note that for $n = 0$ the relation (4.5) trivially holds. □

By applying Lemma 4.3 to the sequences $(a_{\mathrm{sh}}(n;r))_{n\in\mathbb{N}_0}$ and $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$, we evaluate the Hankel transform of $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$.

Corollary 4.4. *The Hankel transform of the sequence* $(\hat{a}(n;r))_{n\in\mathbb{N}_0}$ *is given by*

$$\hat{h}(n;r) = r^{\binom{n+1}{2}} \frac{r^{n+2}-1}{r-1} \quad (n \in \mathbb{N}_0).$$

P r o o f. Recall that from Lemma 3.4 we have

$$\beta_0^{\mathrm{sh}} = 1, \ \beta_n^{\mathrm{sh}} = r \ (n \in \mathbb{N}); \quad \alpha_n^{\mathrm{sh}} = r+1 \ (n \in \mathbb{N}_0).$$

Denote $\lambda_n = Q_n^{\mathrm{sh}}(0)$. Then

$$\lambda_{n+1} + (r+1)\lambda_n + r\lambda_{n-1} = 0 \ (n \in \mathbb{N}_0), \quad \lambda_0 = 1, \quad \lambda_1 = -(r+1).$$

The previous equation has a unique solution $\lambda_n = (-1)^n(r^{n+1}-1)/(r-1)$. Now we obtain the required formula for $\hat{h}(n;r)$ using $h_{\mathrm{sh}}(n;r) = r^{\binom{n+1}{2}}$ (Theorem 3.6) and Lemma 4.3. □

4.2. The Hankel transform of $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$

Recall that $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ is defined as the moment sequence of $\tilde{w}(x) = w_{\mathrm{sh}}(x)/x$. The following lemma is proved in [12] and establishes the required transformation formulas in this case.

Lemma 4.5 ([12]). *Consider the same notation as in Lemma 3.3. Let the sequence $(r_n)_{n\in\mathbb{N}_0}$ be defined by*

$$r_{-1} = -\int_{\mathbb{R}} \overline{w}(x)\,\mathrm{d}x, \quad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0). \tag{4.8}$$

If $\overline{w}(x) = w(x)/(x-c)$ where $c < \inf\operatorname{supp}(w)$ then

$$\begin{aligned} \overline{\alpha}_0 &= \alpha_0 + r_0, & \overline{\alpha}_n &= \alpha_n + r_n - r_{n-1}, \\ \overline{\beta}_0 &= -r_{-1}, & \overline{\beta}_n &= \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \quad (n \in \mathbb{N}). \end{aligned} \tag{4.9}$$

Now, we are ready to evaluate the Hankel transform of the sequence $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$.

Lemma 4.6. *The coefficients $\tilde{\beta}_n$ are given by*

$$\tilde{\beta}_n = \begin{cases} \beta_{n-1}, & n > 1, \\ r^{-1}, & n = 0. \end{cases}$$

The Hankel transform $(\tilde{h}(n;r))_{n\in\mathbb{N}_0}$ of $(\tilde{a}(n;r))_{n\in\mathbb{N}_0}$ is given by $\tilde{h}(n;r) = r^{\binom{n}{2}-1}$.

Proof. We apply Lemma 4.5. The coefficients $\tilde{\beta}_n$ are given by $\tilde{\beta}_n = \beta^{\mathrm{sh}}_{n-1} r_{n-1}/r_{n-2}$, where the sequence $(r_n)_{n\in\mathbb{N}_0}$ is determined by the recurrence relation:

$$r_n = -\alpha^{\mathrm{sh}}_n - \frac{\beta^{\mathrm{sh}}_n}{r_{n-1}}, \quad r_{-1} = -a_0. \tag{4.10}$$

After interchanging α^{sh}_n and β^{sh}_n in Lemma 3.4, we have

$$r_n + r + 1 + \frac{r}{r_{n-1}} = 0 \ \ (n \geqslant 1), \quad r_0 = -1, \quad r_{-1} = -r^{-1}. \tag{4.11}$$

It can be proved by mathematical induction that $r_n = -1$ for all $n \geqslant 1$. Hence

$$\tilde{\beta}_n = \beta^{\mathrm{sh}}_{n-1}\frac{r_{n-1}}{r_{n-2}} = r \ \ (n > 1), \quad \tilde{\beta}_1 = \beta^{\mathrm{sh}}_0 \frac{r_0}{r_{-1}} = r, \quad \tilde{\beta}_0 = \int_{\mathbb{R}} \frac{w_{\mathrm{sh}}(x)}{x}\,\mathrm{d}x = r^{-1}.$$

Now, the Hankel transform is

$$\tilde{h}(n;r) = \tilde{a}_0^{n+1}\tilde{\beta}_1^n\tilde{\beta}_2^{n-1}\ldots\tilde{\beta}_{n-1}^2\tilde{\beta}_n = \frac{1}{r^{n+1}} r^{\binom{n+1}{2}} = r^{\binom{n}{2}-1}.$$

□

4.3. Proof of the main result

From the previous facts, we can formulate and prove the main statement od this section.

Theorem 4.7. *The Hankel transform* $(h_1(n;r))_{n\in\mathbb{N}_0}$ *of the sequence of modified Narayana polynomials* $(a_1(n;r))_{n\in\mathbb{N}_0}$ *is given by*

$$h_1(n;r) = r^{\binom{n+1}{2}}. \tag{4.12}$$

Proof. From Corollary 4.2 we have

$$\begin{aligned} h_1(n;r) &= \tilde{h}(n;r) + (1-r^{-1})\hat{h}(n-1;r) \\ &= r^{\binom{n+1}{2}-n} + (1-r^{-1})r^{\binom{n}{2}}\frac{r^{n+1}-1}{r-1} \\ &= r^{\binom{n}{2}-1} + \frac{r-1}{r}r^{\binom{n}{2}}\frac{r^{n+1}-1}{r-1} = r^{\binom{n}{2}+n} = r^{\binom{n+1}{2}}. \end{aligned}$$

□

The evaluation of the Hankel transform of the sequence of Narayana polynomials $(a(n;r))_{n\in\mathbb{N}_0}$, now goes straightforward.

Theorem 4.8. *The Hankel transform* $(h(n;r))_{n\in\mathbb{N}_0}$ *of the sequence of Narayana polynomials* $(a(n;r))_{n\in\mathbb{N}_0}$ *is given by*

$$h(n;r) = r^{\binom{n+1}{2}}. \tag{4.13}$$

Proof. Notice that $a(n;r) = r\tilde{a}(n;r)$ for any $n \geqslant 0$. Therefore it can be easily concluded that the Hankel transform of Narayana polynomials $(a(n;r))_{n\in\mathbb{N}_0}$ is given by

$$h(n;r) = r^{n+1}\tilde{h}(n;r) = r^{\binom{n}{2}+n} = r^{\binom{n+1}{2}}. \tag{4.14}$$

□

4.4. Special cases

Now let us recall a few special cases mentioned in the first section.

Example 4.1. In the case $r = 1$ we confirm the known result that the Hankel transform of the Catalan sequence is $\mathcal{C}_n = a_1(n;1) = a(n;1)$. After inserting $r = 1$ in (4.13) we can easily establish the well-known result ([23] for example): $h_1(n;1) = h(n;1) = 1$ $(n \in \mathbb{N}_0)$.

Example 4.2. Recall also that in the case $r = 2$ we have that $(a_1(n;2))_{n\in\mathbb{N}_0}$ is the sequence of little Schröder numbers $(1, 1, 3, 11, 45, 197, 903, 4279, \ldots)$. Again, by inserting $r = 2$ in (4.13) we have the following result: $h_1(n;2) = 2^{\binom{n+1}{2}}$ $(n \in \mathbb{N}_0)$.

Example 4.3. The numbers $(a(n;2))_{n\in\mathbb{N}_0}$ are the large Schröder numbers: $(1,2,6,22,90,394,\ldots)$. By inserting $r=2$ in (4.14) we also obtain a well-known result [4]: $h(n;2)=2^{\binom{n+1}{2}}$ $(n\in\mathbb{N}_0)$.

5. The Hankel transform of linear combinations of two consecutive shifted Narayana numbers

In this section we find the closed-form expression for the Hankel transform of the sequence $s(n;r,\gamma,\delta)=\gamma a_{\rm sh}(n;r)+\delta a_{\rm sh}(n+1;r)$. This result is similar to the one derived in [22].

Let us start with a simplified case $s(n;r,C)=s(n;r,1,C)=a_{\rm sh}(n;r)+Ca_{\rm sh}(n+1;r)$. It is clear that the elements of the sequence $s(n;r,C)$ are the moments corresponding to the weight function

$$\begin{aligned} w_s(x) &= (x+C)w_{\rm sh}(x)\\ &= \begin{cases} \dfrac{(x+C)\sqrt{4r-(x-r-1)^2}}{2\pi r}, & x\in\left(\left(\sqrt{r}-1\right)^2,(\sqrt{r}+1)^2\right),\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

To find the Hankel transform of the sequence $(s(n;r))_{n\in\mathbb{N}_0}$, we need to construct orthogonal polynomials $(Q_n^s(x))_{n\in\mathbb{N}_0}$ corresponding to the weight $w_s(x)$ and to obtain the closed form of the coefficients of the three-terms recurrence relation satisfied by these polynomials.

Lemma 5.1. *The coefficients α_n^s and β_n^s in the three-term relation satisfied by $(Q_n^s(x))_{n\in\mathbb{N}_0}$ are*

$$\begin{aligned} \beta_n^s &= r\,\frac{(t_2^n-t_1^n)(t_2^{n+2}-t_1^{n+2})}{(t_2^{n+1}-t_1^{n+1})^2},\\ \alpha_n^s &= -C-\frac{t_2^{n+2}-t_1^{n+2}}{t_2^{n+1}-t_1^{n+1}}-r\,\frac{t_2^{n+1}-t_1^{n+1}}{t_2^{n+2}-t_1^{n+2}}, \end{aligned} \tag{5.1}$$

where

$$t_{1,2}=\frac{-r-1-C\pm\sqrt{(r+1+C)^2-4r}}{2},\quad \beta_0^s=s(0;r;C)=1+C(r+1).$$

P r o o f. By introducing a new weight function $w_s(x)=(x+C)w_{\rm sh}(x)$ we can derive the coefficients α_n^s and β_n^s by using the relations

$$\mu_n=Q_n^s(-C),\quad \alpha_n^s=-C-\frac{\mu_{n+1}}{\mu_n}-\beta_{n+1}^{\rm sh}\frac{\mu_n}{\mu_{n+1}},\quad \beta_n^s=\beta_n^{\rm sh}\frac{\mu_{n-1}\mu_{n+1}}{\mu_n^2}. \tag{5.2}$$

To obtain the values $\mu_n = Q_n^s(-C)$, let us rewrite the three-term recurrence relation for the polynomials $Q_n(x)$ and take $x = -C$:

$$\mu_{n+1} + (r+1+C)\mu_n + r\mu_{n-1} = 0. \tag{5.3}$$

Trivially we have the initial conditions $\mu_{-1} = 0$ and $\mu_0 = 1$. Relation (5.3), together with the stated initial conditions is a second order difference equation with constant coefficients. Hence, the solution of (5.3) is given by

$$\mu_n = \frac{t_2^{n+1} - t_1^{n+1}}{t_2 - t_1}. \tag{5.4}$$

By inserting (5.4) in (5.2), we finally obtain expressions (5.1). □

Now we are ready to apply the Heilermann formula and evaluate the Hankel transform $h_s(n;r,C)$ of the sequence $s(n;r,C)$:

$$\begin{aligned} &h_s(n;r,C) \\ &\quad = h_s(n-1;r,C)\cdot s(0;r,C)\cdot \beta_n^s \dots \beta_1^s \\ &\quad = h_s(n-1;r,C)\cdot s(0;r,C)\cdot \beta_n^{\mathrm{sh}} \dots \beta_1^{\mathrm{sh}} \cdot \frac{\mu_{n+1}\mu_{n-1}}{\mu_n^2}\cdot\frac{\mu_n\mu_{n-2}}{\mu_{n-1}^2}\cdot \dots \cdot \frac{\mu_2\mu_0}{\mu_1^2} \\ &\quad = h_s(n-1;r,C)\cdot \frac{h_{sh}(n;r)}{h_{\mathrm{sh}}(n-1;r)}\cdot\frac{\mu_{n+1}}{\mu_n}\cdot\frac{s(0;r,C)\mu_0}{a_{\mathrm{sh}}(0;r)\mu_1}. \end{aligned}$$

By successive application of the previous equation we obtain

$$h_s(n;r,C) = h_s(0;r,C)\cdot\frac{h_{\mathrm{sh}}(n;r,C)}{h_{\mathrm{sh}}(0;r,C)}\cdot\frac{\mu_{n+1}}{\mu_1}\cdot\Big(\frac{s(0;r,C)\mu_0}{a_{\mathrm{sh}}(0;r)\mu_1}\Big)^n,$$

wherefrom

$$h_s(n;r,C) = h_{\mathrm{sh}}(n;r,C)\cdot\mu_{n+1}\cdot\Big(\frac{s(0;r,C)\mu_0}{a_{\mathrm{sh}}(0;r)\mu_1}\Big)^{n+1}. \tag{5.5}$$

Now replacing

$$\begin{aligned} h_s(0;r,C) = s(0;r,C) = a_{\mathrm{sh}}(0;r) + Ca_{\mathrm{sh}}(1;r) = 1 + C(r+1), \\ \mu_1 = t_1 + t_2 = -(r+1+C), \quad \mu_0 = 1, \end{aligned}$$

and using formulas (4.13) and (5.4), we obtain

$$h_s(n;r,C) = (-1)^{n+1} r^{\binom{n+1}{2}} \frac{t_2^{n+2} - t_1^{n+2}}{t_2 - t_1}.$$

Inserting the values for $t_{1,2}$ from Lemma 5.1, we complete the proof of the theorem:

Theorem 5.2. *The Hankel transform $h_s = h_s(n;r,C)$ of the sequence $(s(n;r,C))_{n\in\mathbb{N}_0}$ is*

$$h_s = \frac{r^{\binom{n+1}{2}}}{2^{n+2}\zeta}\big((r+1+C+\zeta)^{n+2} - (r+1+C-\zeta)^{n+2}\big), \quad \zeta = \sqrt{(r+1+C)^2 - 4r}. \tag{5.6}$$

The next lemma generalizes the above approach and can be proved analogously to Theorem 5.2.

Lemma 5.3. *Let $\omega(x)$ be a weight function and $a_n = \int_{\mathbb{R}} x^n\omega(x)\,\mathrm{d}x$ its moments. Denote by $(Q_n(x))_{n\in\mathbb{N}_0}$ the sequence of orthogonal polynomials with respect to the weight $\omega(x)$. Also denote by $(h_n)_{n\in\mathbb{N}_0}$ and $(h_n^s)_{n\in\mathbb{N}_0}$, the Hankel transforms of the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(a_n + Ca_{n+1})_{n\in\mathbb{N}_0}$, respectively. The following relation is true:*

$$h_n^s = h_n\mu_{n+1}\Big(\frac{Ca_0+a_1}{a_0\mu_1}\Big)^{n+1}, \quad \textit{where } \mu_n = Q_n(-C). \tag{5.7}$$

Now return to the proof of the main result of the section, i.e. the evaluation of the Hankel transform $h_s(n;r,\gamma,\delta)$ of the sequence $s(n;r,\gamma,\delta)$. Assume that $\gamma \neq 0$. Since

$$s(n;r,\gamma,\delta) = \gamma\Big(a_{\mathrm{sh}}(n;r) + \frac{\delta}{\gamma}a_{\mathrm{sh}}(n+1;r)\Big) = \gamma s\Big(n;r,\frac{\delta}{\gamma}\Big),$$

we directly obtain that

$$h_s(n;r,\gamma,\delta) = \gamma^{n+1}h_s\Big(n;r,\frac{\delta}{\gamma}\Big).$$

Now inserting $C = \delta/\gamma$ in (5.6) and using the previous expression we finally derive the main result of this section, i.e. the closed form expression for $h_s = h_s(n;r,\gamma,\delta)$:

$$h_s = \frac{r^{\binom{n+1}{2}}}{2^{n+2}\theta}((\gamma(r+1)+\delta+\theta)^{n+2} - (\gamma(r+1)+\delta-\theta)^{n+2}), \quad \theta = \sqrt{(\gamma(r+1)+\delta)^2 - 4r\gamma^2}. \tag{5.8}$$

Note that the formula (5.8) also holds in the case $\gamma = 0$, since $h_s(n;r,\gamma,\delta)$ is the polynomial function of γ and δ. This follows either from (5.8) or from the fact that $s(n;r,\gamma,\delta)$ is a linear function of γ and δ.

Acknowledgement. The authors wish to thank the anonymous referee for valuable comments improving the quality of the paper.

References

[1] *P. Barry*: On integer-sequences-based constructions of generalized Pascal triangles. J. Integer Seq. *59* (2006). Article 06.2.4. Electronic only.

[2] *P. Barry, A. Hennessy*: Notes on a family of Riordan arrays and associated integer Hankel transforms. J. Integer Seq. *12* (2009). Article ID 09.5.3. Electronic only.

[3] *P. Brändén*: q-Narayana numbers and the flag h-vector of $J(2 \times \mathbf{n})$. Discrete Math. *281* (2004), 67–81.

[4] *R. A. Brualdi, S. Kirkland*: Aztec diamonds and digraphs, and Hankel determinants of Schröder numbers. J. Comb. Theory Ser. B *94* (2005), 334–351.

[5] *M. Chamberland, C. French*: Generalized Catalan numbers and generalized Hankel transformations. J. Integer Seq. *10* (2007).

[6] *T. S. Chihara*: An Introduction to Orthogonal Polynomials. Gordon and Breach, New York, 1978.

[7] *A. Cvetković, P. M. Rajković, M. Ivković*: Catalan Numbers, the Hankel transform, and Fibonacci numbers. J. Integer Seq. *5* (2002).

[8] *O. Eğecioğlu, T. Redmond, C. Ryavec*: Almost product evaluation of Hankel determinants. Electron. J. Comb. *15, #R6* (2008).

[9] *O. Eğecioğlu, T. Redmond, C. Ryavec*: A multilinear operator for almost product evaluation of Hankel determinants. J. Comb. Theory Ser. A *117* (2010), 77–103.

[10] *M. Garcia Armas, B. A. Sethuraman*: A note on the Hankel transform of the central binomial coefficients. J. Integer Seq. *11* (2008). Article ID 08.5.8.

[11] *W. Gautschi*: Orthogonal polynomials: Applications and computation. In: Acta Numerica Vol. 5 (A. Iserles, ed.). Cambridge University Press, Cambridge, 1996, pp. 45–119.

[12] *W. Gautschi*: Orthogonal Polynomials. Computation and Approximation. Oxford University Press, Oxford, 2004.

[13] *M. E. H. Ismail*: Determinants with orthogonal polynomial entries. J. Comput. Appl. Math. *178* (2005), 255–266.

[14] *A. Junod*: Hankel determinants and orthogonal polynomials. Expo. Math. *21* (2003), 63–74.

[15] *C. Krattenthaler*: Advanced determinant calculus: A complement. Linear Algebra Appl. *411* (2005), 68–166.

[16] *W. Lang*: On sums of powers of zeros of polynomials. J. Comput. Appl. Math. *89* (1998), 237–256.

[17] *J. W. Layman*: The Hankel transform and some of its properties. J. Integer Seq. *4* (2001). Article 01.1.5. Electronic only.

[18] *L. L. Liu, Y. Wang*: A unified approach to polynomial sequences with only real zeros. Adv. Appl. Math. *38* (2007), 542–560.

[19] *P. A. MacMahon*: Combinatorial Analysis, Vols. 1 and 2. Cambridge University Press, Cambridge, 1915, 1916, reprinted by Chelsea, 1960.

[20] *T. Mansour, Y. Sun*: Identities involving Narayana polynomials and Catalan numbers. Discrete Math. *309* (2009), 4079–4088.

[21] *T. V. Narayana*: Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités. C. R. Acad. Sci. *240* (1955), 1188–1189. (In French.)

[22] *P. M. Rajković, M. D. Petković, P. Barry*: The Hankel transform of the sum of consecutive generalized Catalan numbers. Integral Transforms Spec. Funct. *18* (2007), 285–296.

[23] *N. J. A. Sloane*: The On-Line Encyclopedia of Integer Sequences; available at `http://www.research.att.com/~njas/sequences`.

[24] *R. A. Sulanke*: Counting lattice paths by Narayana polynomials. Electron. J. Comb. *7* (2000). #R40.

[25] *R. A. Sulanke*: The Narayana distribution. J. Statist. Plann. Inference *101* (2002), 311–326.
[26] *G. Xin*: Proof of the Somos-4 Hankel determinants conjecture. Adv. Appl. Math. *42* (2009), 152–156.

Authors' addresses: Marko D. Petković, University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia, e-mail: dexterofnis@gmail.com; Paul Barry, Waterford Institute of Technology, School of Science, Waterford, Ireland, e-mail: pbarry@wit.ie; Predrag Rajković, University of Niš, Faculty of Mechanical Engineering, Aleksandra Medvedeva 14, 18000 Niš, Serbia, e-mail: pedja.rajk@yahoo.com.