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CONTINUITY OF SOLUTIONS OF A QUASILINEAR HYPERBOLIC EQUATION WITH HYSTERESIS*

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Abstract. This paper is devoted to the investigation of quasilinear hyperbolic equations of first order with convex and nonconvex hysteresis operator. It is shown that in the nonconvex case the equation, whose nonlinearity is caused by the hysteresis term, has properties analogous to the quasilinear hyperbolic equation of first order. Hysteresis is represented by a functional describing adsorption and desorption on the particles of the substance. An existence result is achieved by using an approximation of implicit time discretization scheme, a priori estimates and passage to the limit; in the convex case it implies the existence of a continuous solution.

Keywords: hysteresis, quasilinear hyperbolic equations, generalized play operator, discontinuous solution

MSC 2010: 35L60, 35S30

1. Introduction

We deal with the quasilinear hyperbolic equation with hysteresis

$$\begin{gathered}\frac{\partial(u+v)}{\partial t}+\frac{\partial u}{\partial x}=f, \quad v=\mathcal{F}(u) \text{ in } \Omega\times[0,T],\\ u(x,0)=u_0(x),\\ u(\alpha,t)=0,\end{gathered}\tag{1.1}$$

where α is a fixed real number. In what follows, we denote for simplicity $\Omega=(\alpha,\infty)$. We study equation (1.1) as a model of adsorption-desorption. We assume a thin tube x filled up with uniformly distributed substance. The symbol u denotes the

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concentration of solution, v is concentration of the adsorbed species on the surface of the particles. Here $\mathcal{F}(\cdot)$ is a functional describing adsorption and desorption on the particles of the substance.

Adsorption is the adhesion of molecules of a substance, such as a gas or liquid, to the surface of another substance, such as a solid. This process creates a film of molecules attracted to the surface (of the adsorbent). Desorption is the reverse of adsorption. It is a surface phenomenon. It means that at some point x at time t the concentration of the chemical in the solution is $u(x,t)$ and the resulting concentration $v(x,t)$ is obtained on the output. The adsorption-desorption functional exhibits hysteresis, i.e., the relations between u and v for the cases when u is increasing or decreasing follow different curves. The motivation for our study comes from applications in chemical and geological engineering (see [5], [14], [15]).

We investigate the smoothness of solutions of the initial value problem coupled with a nonconvex generalized play operator and with a suitably restricted class of hysteresis models, whose hysteresis loops are convex. This branch of hysteresis is represented by a generalized play operator and a generalized Prandtl-Ishlinskii operator in its convexity domain. We find out that they prevent formation of shocks. On the other hand, the nonconvex hysteresis operators cause a discontinuity of the solution.

If we consider the quasilinear hyperbolic equation of the first order without hysteresis, its typical feature is characteristics crossing. It means that the characteristic curves with different values of solution meet. At these points we get discontinuity of the solution. To overcome this lack of existence of a classical solution, the concept of a weak solution is introduced. A weak solution may contain discontinuities, need not be differentiable, and will require less smoothness for being considered a solution than a classical solution. A moving discontinuity of the first kind is called a shock. Its speed is given by the Rankine-Hugonoit condition. More details about quasilinear hyperbolic equations of the first order one can find for example in [16].

If we expand our class of solutions to include weak solutions, we no longer have uniqueness of the solution of the initial value problem. We need an additional criterion for selecting the physically correct weak solution. This criterion is called an entropy condition [16].

Geometrically, a shock satisfies the entropy condition if the characteristics enter into the discontinuity curve. The main result of the paper is a theorem showing that for hysteresis models satisfying convexity of hysteresis loops we obtain existence of a smooth solution, and shocks do not occur. The result is proved by the method of implicit time discretization. Equation (1.1) can be set in the form of the Cauchy problem [17] and has one and only one integral solution in the sense of the nonlinear semigroup theory [8], [17]. Such solution satisfies an entropy condition [6]. The smoothness of solution which follows from the existence theorem implies that this is actually a strong solution (in a sense of nonlinear semigroup theory [17]), thus an integral solution which satisfies the entropy condition [6].

We conclude with some examples. We compute the exact solution for the classical play and a slightly modified generalized play operator using the method of characteristics. In order to preserve continuity for classical play operator we have to use the method of rarefaction waves. If we do not employ it we obtain a discontinuous solution which does not satisfy the entropy condition. The same approach is used for the investigation of a modified nonconvex generalized play operator. In this case we get a discontinuous solution, i.e., the characteristics with different values of solution cross.

The paper is organized as follows. In Section 2 we briefly introduce the concept of hysteresis, define a generalized play operator (see [2], [9], [17]), Prandtl-Ishlinskii operator of play type [10] and give some special examples considered later. In Section 3 the proof of the existence result is led through three steps: approximation, a priori estimates and limit procedure. We compute in Section 4 an explicit solution with a classical play operator. It is continuous and no shocks occur, because all loops are convex. We describe each solution's region and characteristics in detail and sketch them. Section 5 is devoted to a special example of a generalized play operator which is nonconvex. It is shown that here a continuous solution does not exist.

2. Hysteresis operators

Hysteresis is a phenomenon in which the response of a physical system to an external influence depends not only on the present magnitude of the influence but also on the previous history of the system. Hysteresis operators are characterized by two main properties—memory effect and rate independence.

In this section we define the generalized play operator, the classical play operator being a special example, and the Prandtl-Ishlinskii operator of play type.

2.1. Generalized play operator. Let $\gamma_l, \gamma_r \colon \mathbb{R} \to [-\infty, \infty]$ be continuous and nondecreasing functions with

$$\gamma_r(u) \leqslant \gamma_l(u) \quad \forall\, u \in \mathbb{R}.$$

Let u be a continuous piecewise linear function $[0, T] \to \mathbb{R}$ (namely, a function whose graph is a polygonal), and let $v_0 \in [\gamma_r(u(0)), \gamma_l(u(0))]$ be given. For any $t \in [0, T]$, let $t_0 = 0 < t_1 < \ldots < t_N = T$ be such that u is linear (or, more precisely, affine) in

$[t_{i-1}, t_i]$ for $i = 1, 2, \ldots$ Then we set recursively

$$
(2.1) \qquad \mathcal{P}(u(t)) = v(t) = \begin{cases} \min\{\gamma_l(u(0)), \max\{\gamma_r(u(0)), v_0\}\} \\ \quad \text{if } t = 0, \\ \min\{\gamma_l(u(t)), \max\{\gamma_r(u(t)), v(t_{i-1})\}\} \\ \quad \text{if } t \in (t_{i-1}, t_i],\ i = 1, 2, \ldots \end{cases}
$$

The mapping $\mathcal{P}$ that associates v with u according to the above rule is called generalized play (Fig. 1) [9, § 2].

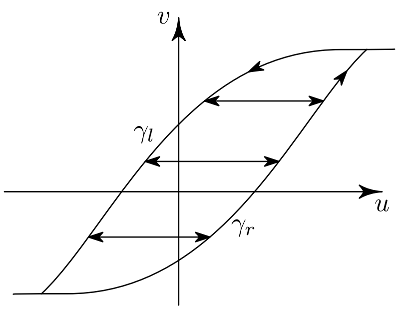

Figure 1. Generalized play.

The inequality (proved in [9, § 2])

$$
|\mathcal{P}(u_1)(t) - \mathcal{P}(u_2)(t)| \leqslant L \max_{0 \leqslant s \leqslant t} |u_1(s) - u_2(s)| \quad \forall\, t \in [0, T]
$$

implies that $\mathcal{P}$ can be extended to a Lipschitz continuous mapping $\mathcal{C}([0,T]) \to \mathcal{C}([0,T])$, provided both γ_r and γ_l are Lipschitz continuous with a constant L.

Now we mention some special examples which we will consider later.

(1) We assume a special example of a generalized play operator, where $\gamma_r(u) = u - r$, $\gamma_l(u) = u + r$ for a given parameter $r > 0$. Then $\mathcal{P}$ is called the classical play operator (Fig. 2) and denoted as $\mathcal{P}_r$.

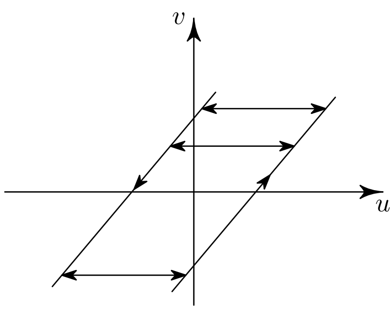

Figure 2. Classical play operator.

(2) We consider a special example of a generalized play operator (Fig. 3). The left hysteresis boundary curve is given by the function

$$\gamma_l(u) = \begin{cases} u+1 & \text{if } \; -2 \leqslant u \leqslant 0, \\ 1 & \text{if } \; u \geqslant 0, \\ -1 & \text{if } \; u \leqslant -2 \end{cases}$$

and the right boundary curve is given by

$$\gamma_r(u) = \begin{cases} u-1 & \text{if } \; 0 \leqslant u \leqslant 2, \\ 1 & \text{if } \; u \geqslant 2, \\ -1 & \text{if } \; u \leqslant 0. \end{cases}$$

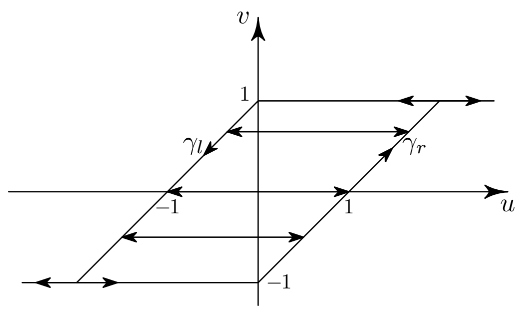

Figure 3. As example of generalized play.

An alternative definition of the classical play operator is given in the following way: For a given input function $u \in C^0([0,T])$ and initial condition $x_r^0 \in [-r,r]$, we define the output $v := \mathcal{P}_r(x_r^0, u) \in C^0([0,T]) \cap BV(0,T)$ of the play operator

$$\mathcal{P}_r\colon \; [-r,r] \times C^0([0,T]) \to C^0([0,T]) \cap BV(0,T)$$

as the solution of the Stieltjes integral variational inequality

$$\begin{gathered} \int_0^T [u(t) - v(t) - y(t)]\, \mathrm{d}v(t) \geqslant 0 \quad \forall\, y \in C^0([0,T]), \; \max_{0 \leqslant t \leqslant T} |y(t)| \leqslant r, \\ |u(t) - v(t)| \leqslant r \quad \forall\, t \in [0,T], \\ v(0) = u(0) - x^0. \end{gathered} \tag{2.2}$$

Let us consider now the whole family of play operators $\mathcal{P}_r$ parameterized by r, $r > 0$, which can be interpreted as a memory variable. Accordingly, we introduce the configuration space

$$\Lambda := \Big\{ \lambda \in W^{1,\infty}(0,\infty); \; \Big| \frac{\mathrm{d}\lambda(r)}{\mathrm{d}r} \Big| \leqslant 1 \text{ a.e. in } (0,\infty) \Big\},$$

as well as its subspaces

$$\Lambda_R := \{\lambda \in \Lambda;\ \lambda(r) = 0 \text{ for } r \geqslant R\}, \quad \Lambda_0 := \bigcup_{R>0} \Lambda_R.$$

Elements $\lambda \in \Lambda$ are called memory configurations, see [10]. For a given $\lambda \in \Lambda$, it is convenient to define the initial condition x_r^0 by the formula

$$x_r^0 := Q_r(u(0) - \lambda(r)),$$

where $Q_r\colon \mathbb{R} \to [-r, r]$ is the projection

$$Q_r(x) := \operatorname{sign}(x) \min\{r, |x|\} = \min\{r, \max\{-r, x\}\}.$$

For any $\lambda \in \Lambda$, $u \in C^0([0,T])$ and $r > 0$ we set

$$p_r(\lambda, u) := \mathcal{P}_r(x_r^0, u), \quad p_0(\lambda, u) = u. \tag{2.3}$$

Moreover, the operator $p_r\colon \Lambda \times C^0([0,T]) \to C^0([0,T])$ is Lipschitz continuous in the following sense:

Lemma 2.1. *For every $u, w \in C^0([0,T])$, $\lambda, \mu \in \Lambda$, and $r > 0$ we have*

$$|p_r(\lambda, u) - p_r(\mu, w)|_\infty \leqslant \max\{|\lambda(r) - \mu(r)|, |u - w|_\infty\}.$$

2.2. Prandtl-Ishlinskii operator. The play operator can be used to construct more complex hysteresis models such as the Prandtl-Ishlinskii operator of play type, see [17].

Definition 2.1. Let a constant $a \geqslant 0$ and a function $h \in BV_{\mathrm{loc}}(0, \infty)$ be given such that $\lim\limits_{s \to 0^+} h(s) = a$. We set

$$\varphi(r) := \int_0^r h(s)\,\mathrm{d}s \quad \text{for } r > 0.$$

Then the operator $\mathcal{F}_\varphi\colon \Lambda_0 \times C^0([0,T]) \to C^0([0,T])$ defined by the formula

$$\mathcal{F}_\varphi(\lambda, u) = au + \int_0^\infty p_r(\lambda, u)\,\mathrm{d}h(r), \quad \lambda \in \Lambda_0,\ u \in C^0([0,T]), \tag{2.4}$$

where p_r is the play operator (2.3), is called the Prandtl-Ishlinskii operator generated by the function φ.

It can be shown ([10]) that for every $\lambda \in \Lambda_0$, $u \in C^0([0,T])$, and $t \in [0,T]$ there exists $R(t) < \infty$ such that $p_r(\lambda, u)(t) = 0$ for all $r \geqslant R(t)$. The Stieltjes integral in (2.4) is therefore always finite.

The following result can be found in [10, Section II.3].

Theorem 2.1. *The operator $\mathcal{F}_\varphi$ is*

i) *causal, i.e.,*

$$\begin{cases} \forall\, u_1, u_2 \in C^0([0,T]),\ \forall\, t \in [0,T],\ \forall\, \lambda \in \Lambda_0, \\ \textit{if } u_1 = u_2 \textit{ in } [0,t],\ \textit{then } [\mathcal{F}_\varphi(\lambda, u_1)](t) = [\mathcal{F}_\varphi(\lambda, u_2)](t); \end{cases} \tag{2.5}$$

ii) *rate independent, i.e.,*

$$\begin{cases} \forall\, u \in C^0([0,T]),\ \forall\, t \in [0,T],\ \forall\, \lambda \in \Lambda_0,\ \textit{if } s\colon [0,T] \to [0,T] \textit{ is an} \\ \textit{increasing homeomorphism, then } [\mathcal{F}_\varphi(\lambda, u \circ s)](t) = [\mathcal{F}_\varphi(\lambda, u)](s(t)); \end{cases} \tag{2.6}$$

iii) *if the function h is nonnegative and monotone, then $\mathcal{F}_\varphi$ is piecewise monotone in the following sense:*

$$\begin{cases} \forall\, u \in C^0([0,T]),\ \forall\, [t_1, t_2] \subset [0,T],\ \forall\, \lambda \in \Lambda_0,\ \textit{if } u \textit{ is either nondecreasing} \\ \textit{or nonicreasing in } [t_1, t_2],\ \textit{then so is } \mathcal{F}_\varphi(\lambda, u); \end{cases} \tag{2.7}$$

iv) *locally Lipschitz continuous in the following sense: for all $t \in [0,T]$, for all $v_1, v_2 \in C^0([0,T])$, for all $\lambda_1, \lambda_2 \in \Lambda_R$, where $R > 0$ is given we have*

$$\begin{cases} |\mathcal{F}_\varphi(\lambda_1, v_1) - \mathcal{F}_\varphi(\lambda_2, v_2)|(t) \leqslant |h(0)||v_1 - v_2|(t) \\ \qquad + (\operatorname{Var}_{[0,R(t)]} h) \max\{\|\lambda_1(r) - \lambda_2(r)\|_{[0,R]}, \|v_1 - v_2\|_{C^0([0,t])}\} \end{cases} \tag{2.8}$$

where $R(t) := \max\{R, \|v_1\|_{C^0([0,T])}, \|v_2\|_{C^0([0,T])}\}$.

We assume Prandtl-Ishlinskii operators of the play type, i.e., such that the function h is positive and nondecreasing in $(0,\infty)$. These operators fulfil the additional hypothesis of convexity of the hysteresis loops, which can be described in the following way: if the input u increases (decreases) in a suitable neighbourhood of 0, then the input-output couple $(u, \mathcal{F}_\varphi(u))$ moves along the graph of some convex (concave) nondecreasing function. The main result of this is the so called second order energy inequality. If we set

$$v(t) := \mathcal{F}_\varphi(\lambda, u)(t), \quad V(t) := \frac{1}{2}\,\dot{v}(t)\dot{u}(t) \qquad \text{a.e. in } [0,T],$$

then this energy inequality can be summarized as

$$\ddot{v}(t)\dot{u}(t) - \dot{V}(t) \geqslant 0 \quad \text{in the sense of distributions.} \tag{2.9}$$

In order to get the existence result we will need a discrete version of (2.9) which can be proved directly from the geometrical properties of hysteresis loops. For the precise statement of (2.9) and for more details on this topic, see [10].

For the purpose of dealing with partial differential equations we consider both the input and the initial memory configuration λ that additionally depend on the space variable x. If for instance $\lambda(x,\cdot)$ belongs to Λ_0 and $u(x,\cdot)$ belongs to $C^0([0,T])$ for (almost) every x, then we define

$$\begin{aligned}\mathcal{F}(u)(x,t) &:= \mathcal{F}_\varphi(\lambda(x), u(x,\cdot))(t)\\ &:= au(x,t) + \int_0^\infty p_r(\lambda(x), u(x,\cdot))(t)\,\mathrm{d}h(r).\end{aligned} \tag{2.10}$$

3. An existence result

Let us set $Q = \Omega \times (0,T)$. We assume that $u_0 \in L^2(\Omega)$ is a given initial condition, $f \in L^2(Q)$ is a given function. We keep an initial configuration $\lambda \in L^\infty(\Omega, \Lambda_R)$ fixed and write for simplicity $\mathcal{F}(u(x,\cdot))$ instead of $\mathcal{F}(\lambda(x), u(x,\cdot))$ in the sequel. We set $v_0(x) = [\mathcal{F}(u(x,\cdot))](0)$ a.e. in Ω. Let

$$\mathcal{F}\colon\ \mathcal{M}(\Omega; C^0([0,T])) \to \mathcal{M}(\Omega; C^0([0,T]))$$

be a hysteresis operator, where we denote by $\mathcal{M}(\Omega; C^0([0,T]))$ the Fréchet space of measurable functions $\Omega \to C^0([0,T])$.

Problem 3.1. We search for a function $u \in \mathcal{M}(\Omega; C^0([0,T])) \cap L^2(Q)$ such that $\mathcal{F}(u) \in L^2(Q)$ and

$$\begin{aligned}\int_0^T\int_\Omega (u + \mathcal{F}(u))\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega u\frac{\partial\psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}t\\ = -\int_0^T\int_\Omega f\psi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \psi(x,0)[u_0(x)+v_0(x)]\,\mathrm{d}x\end{aligned} \tag{3.1}$$

for any $\psi \in H^1(Q)$ with $\psi(\cdot,T) = 0$ a.e. in Ω.

I n t e r p r e t a t i o n. The variational equation (3.1) yields

$$\frac{\partial}{\partial t}[u+\mathcal{F}(u)]+\frac{\partial u}{\partial x}=f \quad \text{in } \mathcal{D}'(Q) \quad \text{(in the sense of distributions)},$$

whence

$$\frac{\partial}{\partial t}[u+\mathcal{F}(u)]=f-\frac{\partial u}{\partial x} \quad \text{in } L^2(Q).$$

Thus $u+\mathcal{F}(u)\in H^1(0,T;L^2(\Omega))$. Hence, integrating by parts in time in (3.1), we get

$$[u+\mathcal{F}(u)]|_{t=0}=u_0+v_0 \quad \text{in } L^2(\Omega), \qquad u(\alpha,t)=0 \quad \text{for } t>0.$$

Now we are ready to state and prove existence of a solution of Problem 3.1.

Theorem 3.1 (Existence). *Assume that the operator $\mathcal{F}$ satisfies the hypotheses* (2.5)–(2.8). *Moreover, let $f\in W^{1,1}(0,T;L^2(\Omega))$, $u_0\in H^1(\Omega)$, and $u_0(\alpha)=0$. Then Problem* 3.1 *has at least one solution such that*

$$u\in W^{1,\infty}(0,T;L^2(\Omega))\cap L^\infty(\Omega;H^1(0,T))\cap L^\infty(0,T;H^1(\Omega)),$$
$$\mathcal{F}(u)\in H^1(0,T;L^2(\Omega)).$$

P r o o f. (i) Approximation. Let us fix $m\in N$, set $k:=T/m$. For any $n=1,\dots,m$ let us consider $u_m^0(x):=u_0(x)$, $v_m^0(x):=v_0(x)$ and $f_m^n(x):=k^{-1}\int_{(n-1)k}^{nk}f(x,t)\,\mathrm{d}t$ a.e. in Ω. We approximate our problem by an implicit time discretization scheme.

P r o b l e m 3.2. To find $u_m^n\in L^2(\Omega)$ for $n=1,\dots,m$ such that, if $u_m(x,\cdot)$ is the linear time interpolate of $u_m(x,nk):=u_m^n(x)$ for $n=1,\dots,m$ a.e. in Ω, $v_m^n(x):=[\mathcal{F}(u_m)](x,nk)$ for $n=1,\dots,m$ a.e. in Ω, then for any $\psi\in L^2(\Omega)$

$$(3.2)\quad \frac{1}{k}\int_\Omega(u_m^n-u_m^{n-1})\psi\,\mathrm{d}x+\frac{1}{k}\int_\Omega(v_m^n-v_m^{n-1})\psi\,\mathrm{d}x+\int_\Omega\frac{\mathrm{d}u_m^n}{\mathrm{d}x}\,\psi\,\mathrm{d}x=\int_\Omega f_m^n\psi\,\mathrm{d}x.$$

For any $n\in\{1,\dots,m\}$, assume that $u_m^1,\dots,u_m^{n-1}\in L^2(\Omega)$ are known, and consider the problem of determining u_m^n. For almost any $x\in\Omega$, $u_m(x,\cdot)$ is affine in $[(n-1)k,nk]$; therefore $[\mathcal{F}(u_m)](x,nk)$ depends only on $u_m(x,\cdot)|_{[0,(n-1)k]}$ which is known, and on $u_m^n(x)$ which must be determined. Hence, there exists a function $F_m^n\colon\ \mathbb{R}\times\Omega\to\mathbb{R}$ such that

$$v_m^n(x)=[\mathcal{F}(u_m)](x,nk):=F_m^n(u_m^n(x),x) \quad \text{a.e. in } \Omega.$$

This allows us to introduce the operator $\tilde{F}_m^n\colon\ \mathcal{M}(\Omega)\to\mathcal{M}(\Omega)\colon\ w\mapsto F_m^n(w(\cdot),\cdot)$. Working as in [17, Section IX.1], it is possible to show that $\tilde{F}_m^n\colon\ L^2(\Omega)\to L^2(\Omega)$ is strongly continuous and affinely bounded.

Now, the main idea how to get the solution of Problem 3.2 is to rewrite (3.2) in the following way:

$$\frac{u_m^n - u_m^{n-1}}{k} + \frac{v_m^n - v_m^{n-1}}{k} + \frac{\mathrm{d}u_m^n}{\mathrm{d}x} = f_m^n \quad \text{in } \mathcal{D}'(\Omega) \tag{3.3}$$

with an initial condition

$$u_m^n(\alpha) = 0.$$

If we separate the terms which are known from the ones we have to determine, the equation (3.3) can be rewritten in the form

$$u_m^n + v_m^n + k\frac{\mathrm{d}u_m^n}{\mathrm{d}x} = kf_m^n + u_m^{n-1} + v_m^{n-1} \quad \text{a.e. in } \Omega,$$

where the right-hand side of the previous equation is a known function. We can use a standard procedure from ordinary differential equations to conclude that this equation has one and only one solution in Ω.

(ii) A priori estimates. We fix any $j \in \{1, \dots, m\}$ and set

$$q_m^n := \frac{u_m^n - u_m^{n-1}}{k}, \quad z_m^n := \frac{v_m^n - v_m^{n-1}}{k} \quad \text{for } n = 1, \dots, j;$$

and we define

$$q_m^0 + z_m^0 := f_m^0 - \frac{\mathrm{d}u_m^0}{\mathrm{d}x} \in L^2(\Omega). \tag{3.4}$$

By taking the incremental ratio in time in (3.3), we have

$$\frac{q_m^n - q_m^{n-1}}{k} + \frac{z_m^n - z_m^{n-1}}{k} + \frac{\mathrm{d}q_m^n}{\mathrm{d}x} = \frac{f_m^n - f_m^{n-1}}{k}. \tag{3.5}$$

First of all, the fact that the operator $\mathcal{F}$ is locally Lipschitz continuous (with Lipschitz constant, say $\mathcal{L}$) yields

$$\exists\, \tau \in L^2(\Omega)\colon\ |v_m^n(x)| \leqslant \mathcal{L} \max_{j=1,\dots,n} |u_m^j(x)| + \tau(x) \tag{3.6}$$

a.e. in Ω.

At this point, we claim that the following discrete version of the second order inequality holds:

$$\begin{aligned}\Big(\frac{q_m^n - q_m^{n-1}}{k} + \frac{z_m^n - z_m^{n-1}}{k}\Big)(u_m^n - u_m^{n-1}) \\ \geqslant \frac{1}{2}(q_m^n + z_m^n)q_m^n - \frac{1}{2}(q_m^{n-1} + z_m^{n-1})q_m^{n-1} \\ \text{a.e. in } \Omega, \text{ for any } n = 2, \dots, j.\end{aligned} \tag{3.7}$$

For more details and the detailed proof, see [4]. Let us multiply (3.5) by $kq_m^n = u_m^n - u_m^{n-1}$, integrate in (α,β) for any $\beta \in (\alpha,\infty)$ and then sum for $n = 1,\dots,j$. First we have

$$\Big[\Big(\frac{q_m^n - q_m^{n-1}}{k} + \frac{z_m^n - z_m^{n-1}}{k}\Big) + \frac{\mathrm{d}q_m^n}{\mathrm{d}x}\Big](u_m^n - u_m^{n-1}) = \frac{(f_m^n - f_m^{n-1})}{k}(u_m^n - u_m^{n-1}),$$
$$\Big(\frac{q_m^n - q_m^{n-1}}{k} + \frac{z_m^n - z_m^{n-1}}{k}\Big)(u_m^n - u_m^{n-1}) + \frac{k}{2}\,\frac{\mathrm{d}(q_m^n)^2}{\mathrm{d}x} = (f_m^n - f_m^{n-1})q_m^n.$$

Then

$$\begin{aligned}
&\sum_{n=1}^{j}\int_\alpha^\beta \Big(\frac{q_m^n - q_m^{n-1}}{k} + \frac{z_m^n - z_m^{n-1}}{k}\Big)(u_m^n - u_m^{n-1})\,\mathrm{d}x + \frac{k}{2}\sum_{n=1}^{j}\int_\alpha^\beta \frac{\mathrm{d}(q_m^n)^2}{\mathrm{d}x}\,\mathrm{d}x\\
&= \int_\alpha^\beta (q_m^1 + z_m^1)q_m^1\,\mathrm{d}x - \int_\alpha^\beta (q_m^0 + z_m^0)q_m^1\,\mathrm{d}x\\
&\quad + \sum_{n=2}^{j}\int_\alpha^\beta \Big(\frac{q_m^n - q_m^{n-1}}{k} + \frac{z_m^n - z_m^{n-1}}{k}\Big)(u_m^n - u_m^{n-1})\,\mathrm{d}x + \frac{k}{2}\sum_{n=1}^{j}\int_\alpha^\beta \frac{\mathrm{d}(q_m^n)^2}{\mathrm{d}x}\,\mathrm{d}x\\
&\overset{(3.7)}{\geqslant} \int_\alpha^\beta (q_m^1 + z_m^1)q_m^1\,\mathrm{d}x + \frac{1}{2}\sum_{n=2}^{j}\int_\alpha^\beta [(q_m^n + z_m^n)q_m^n - (q_m^{n-1} + z_m^{n-1})q_m^{n-1}]\,\mathrm{d}x\\
&\quad - \frac{1}{2}\int_\alpha^\beta |q_m^1|^2\,\mathrm{d}x - \frac{1}{2}\|q_m^0 + z_m^0\|^2_{L^2(\Omega)} + \frac{k}{2}\sum_{n=1}^{j}[q_m^n(\beta)^2 - q_m^n(\alpha)^2]\\
&\overset{(2.7)}{\geqslant} \frac{1}{2}\int_\alpha^\beta |q_m^j|^2\,\mathrm{d}x - \frac{1}{2}\|q_m^0 + z_m^0\|^2_{L^2(\Omega)} + \frac{k}{2}\sum_{n=1}^{j}[q_m^n(\beta)^2 - q_m^n(\alpha)^2].
\end{aligned}$$

On the other hand,

$$\sum_{n=1}^{j}\int_\alpha^\beta (f_m^n - f_m^{n-1})q_m^n\,\mathrm{d}x \leqslant \Big\|\frac{\partial f}{\partial t}\Big\|_{L^1(0,T;L^2(\Omega))}\max_{n=1,\dots,j}\Big(\int_\alpha^\beta q_m^n(x)^2\,\mathrm{d}x\Big)^{1/2}.$$

From the previous two chains of inequalities, together with (3.4), using compatibility of the initial condition, we deduce

$$\begin{aligned}
\frac{1}{2}\int_\alpha^\beta |q_m^j(x)|^2\,\mathrm{d}x + \frac{k}{2}\sum_{n=1}^{j} q_m^n(\beta)^2 &\leqslant \Big\|\frac{\partial f}{\partial t}\Big\|_{L^1(0,T;L^2(\Omega))}\max_{n=1,\dots,j}\Big(\int_\alpha^\beta q_m^n(x)^2\,\mathrm{d}x\Big)^{1/2}\\
&\quad + \frac{1}{2}\|q_m^0 + z_m^0\|^2_{L^2(\Omega)} + \frac{k}{2}\sum_{n=1}^{j} q_m^n(\alpha)^2,
\end{aligned}$$

which yields the a priori estimate

$$\int_\Omega |q_m^j|^2\,\mathrm{d}x + k\sum_{n=1}^{j}|q_m^n|^2 \leqslant \text{constant (independent of } m). \tag{3.8}$$

(iii) Limit procedure. We denote by $v_m(x,\cdot)$ the linear time interpolate of $v_m(x,nk) := v_m^n(x)$ for $n=0,\dots,m$ a.e. in Ω; moreover, we set $\bar{u}_m(x,t) := u_m^n(x)$ if $(n-1)k < t \leqslant nk$, for $n=1,\dots,m$ a.e. in Ω and define $\overline{f}_m$ in a similar way. Thus (3.3) becomes

$$\frac{\partial u_m}{\partial t} + \frac{\partial v_m}{\partial t} + \frac{\partial \bar{u}_m}{\partial x} = \overline{f}_m, \tag{3.9}$$

while (3.8) yields

$$\begin{aligned} \|u_m\|_{W^{1,\infty}(0,T;L^2(\Omega))\cap L^\infty(\Omega;H^1(0,T))} &\leqslant \text{constant (independent of } m),\\ \|\bar{u}_m\|_{L^\infty(0,T;H^1(\Omega))} &\leqslant \text{constant (independent of } m). \end{aligned} \tag{3.10}$$

The a priori estimates we have found allow us to conclude that there exists u such that, possibly taking $m \to \infty$ along a subsequence,

$$\begin{aligned} u_m \to u \quad &\text{weakly star in } W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(\Omega;H^1(0,T))\\ &\qquad\qquad\qquad\qquad \cap L^\infty(0,T;H^1(\Omega)),\\ \bar{u}_m \to u \quad &\text{weakly star in } L^\infty(0,T;H^1(\Omega)). \end{aligned} \tag{3.11}$$

Moreover, as $H^1(0,T;L^2(\Omega)) = L^2(\Omega;H^1(0,T)) \subset L^2(\Omega;C^0([0,T]))$ with continuous injection, by (3.6) and by (3.10) we easily obtain

$$\|v_m\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \text{constant (independent of } m);$$

this allows us to conclude that there exists v such that, possibly taking $m \to \infty$ along a subsequence,

$$v_m \to v \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)). \tag{3.12}$$

The three above formulas yield

$$\begin{aligned} &\|u_m + v_m\|_{H^1(0,T;L^2(\Omega))}\\ &\qquad \leqslant c\|u_m+v_m\|_{L^2(Q)} + \left\|\frac{\partial}{\partial t}(u_m+v_m)\right\|_{L^2(0,T;L^2(\Omega))}\\ &\qquad \leqslant c\|u_m\|_{L^2(Q)} + c\sqrt{T}\|v_m\|_{L^2(\Omega;C^0([0,T]))}\\ &\qquad\quad + \left\|\overline{f}_m - \frac{\partial \bar{u}_m}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} \leqslant \text{constant.} \end{aligned} \tag{3.13}$$

So, by taking $m \to \infty$ along a subsequence, we get

$$u_m + v_m \to u + v \quad \text{weakly in } H^1(0,T;L^2(\Omega)).$$

Let us show that $v = \mathcal{F}(u)$. As a consequence of the estimate (3.10), we see that $\partial u/\partial x \in L^\infty(0,T;L^2(\Omega))$.

The space

$$\Big\{z \in L^1(\Omega \times (0,T));\ \frac{\partial u}{\partial x} \in L^\infty(0,T;L^2(\Omega)),\ \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))\Big\}$$

is compactly embedded in $C^0(\bar\Omega \times [0,T])$. Hence, we have, passing to a subsequence,

$$u_m \to u \quad \text{uniformly in } C^0(\bar\Omega \times [0,T]).$$

Using the continuity of the operator $\mathcal{F}(u)$, we deduce

$$\mathcal{F}(u_m) \to \mathcal{F}(u) \quad \text{uniformly in } [0,T],\ \text{a.e. in } \Omega.$$

As $v_m(x,\cdot)$ is the linear time interpolate of $v_m(x,nk) = [\mathcal{F}(u_m)](x,nk)$ for $n = 0,\dots,m$ a.e. in Ω, we have

$$v_m - \mathcal{F}(u_m) \to 0 \quad \text{uniformly in } [0,T],\ \text{a.e. in } \Omega.$$

Therefore by (3.12) we conclude $v = \mathcal{F}(u)$ a.e. in Q.

Hence, taking $m \to \infty$ in (3.9), we get

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = f.$$

This completes the proof. □

Remark 3.1. The assumptions of Theorem 3.1 are satisfied e.g. by the play operator (see [10, Sections II.1, II.2]) and the Prandtl-Ishlinskii operator (see [10, Section II.3]).

Remark 3.2. The Problem 3.1 corresponding to a generalized play operator or a generalized Prandtl-Ishlinskii operator can be set in the form of an abstract Cauchy problem [17]. For such a system we dispose of the notion of an integral solution in the sense of nonlinear semigroup theory [17]. Our weak solution (u,v) has the regularity

$$u \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(\Omega;H^1(0,T)) \cap L^\infty(0,T;H^1(\Omega)), \quad v \in H^1(0,T;L^2(\Omega)).$$

Hence, (u,v) coincides with the strong solution and therefore with the integral solution of the Cauchy problem which satisfies the entropy condition. This condition is proved in [7] and can be stated as follows.

Let us denote by $\mathcal{L}_\gamma$ the hysteresis region, i.e., the subset of $\mathbb{R}^2$ of admissible pairs (u,v) such that $v=\mathcal{F}(u,v_0)$ for arbitrary v_0.

Theorem 3.2 (Entropy condition)**.** *Assume that $u_0\in L^2(\Omega)$. Then the solution of Problem* 3.1 *satisfies*

$$\int\!\!\int_Q \Big[(|u-\theta|+|v-\hat\theta|)\frac{\partial\varphi}{\partial t}+|u-\theta|\frac{\partial\varphi}{\partial x}+\operatorname{sign}(u-\theta)f\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t\geqslant 0 \tag{3.14}$$

for all $\varphi\in C_0^\infty(Q)$ such that $\varphi\geqslant 0$ and this holds for all $(\theta,\hat\theta)\in\mathcal{L}_\gamma$.

4. Example 1

We study the partial differential equation

$$\frac{\partial(u+v)}{\partial t}+\frac{\partial u}{\partial x}=0, \tag{4.1}$$

with the initial conditions

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad x\in(\alpha,\infty),$$

and the boundary condition

$$u(\alpha,t)=0.$$

Here $v=\mathcal{P}_r(\cdot)$ is a classical play operator defined in Section 2.

The equation can be rewritten as

$$u_t+u_x+\left\{\begin{array}{l} 0 \text{ if } u-1<v<u+1\\ u_t \text{ if } v=u+1 \text{ decreasing}\\ u_t \text{ if } v=u-1 \text{ increasing}\end{array}\right\}=0.$$

We obtain the following equation from the definition of the play operator:

a) $v=u-1$, v is increasing. The values of v_t are then allowed to take any positive value and $v_t=u_t$.
b) $v=u+1$, v is decreasing. The values of v_t are then allowed to take any negative value and $v_t=u_t$.
c) $u-1<v<u+1$, then $v_t=0$ because v is constant.

For an explicit example, let us take the initial conditions

$$u(x,0) = u_0(x) \equiv \begin{cases} x & \text{for } -3 \leqslant x \leqslant 0, \\ -6-x & \text{for } -6 < x < -3, \\ 0 & \text{for } x \leqslant -6, \end{cases}$$

$$v(x,0) = v_0(x) \equiv \begin{cases} 0 & \text{for } -1 \leqslant x \leqslant 0, \\ x+1 & \text{for } -3 \leqslant x < -1, \\ \frac{1}{3}x - 1 & \text{for } -\frac{9}{2} \leqslant x < -3, \\ -x-7 & \text{for } -6 \leqslant x < -\frac{9}{2}, \\ -1 & \text{for } x < -6. \end{cases}$$

In order to compute the exact solution, we use the method of characteristics. If our original equation is $u_t + \kappa u_x = 0$, then the solution subject to the above initial condition would preserve its shape and travel with speed κ and $u(x,t) = u_0(x - \kappa t)$. In our case, with different values of κ, the characteristics must cross. The solution itself remains continuous.

The computations:

$$\frac{\partial u}{\partial t} + \kappa \frac{\partial u}{\partial x} = 0.$$

We consider the characteristics:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = \kappa,$$
$$\frac{\mathrm{d}}{\mathrm{d}s}(\kappa t - x) = 0,$$
$$u(x,t) = u_0(x - \kappa t).$$

Characteristics are in the form: $x - \kappa t = k$.

We describe the solution in detail. Results are summed up in a table (Tab. 1) and in a graph (Fig. 4). The solution is described in different regions where the characteristics are changing.

The initial condition is increasing for $x \in [-3, 0]$.

A: We are here inside the hysteresis loop. This means $v = 0$, $v_t = 0$ and we have the equation $u_t + u_x = 0$, i.e., $\kappa = 1$ in the above computation. The solution is determined by the initial condition $u_0(x,t)$. So $u = u_0(x-t) = x - t$. This will be our solution until $u = -1$, because then we hit the left hysteresis boundary curve and therefore the equation will be changed. Thus $t < x+1$. The boundary for the region A is $0 < t < x+1$.

B: The same situation as above, but now we hit the left boundary curve of the play operator and stay there ($u = -1$, $v = 0$). We are above the line $t = x+1$ and the solution is determined by the continuity of the solution. The boundaries are determined by $x + 1 < t$.

In regions A and B, κ is equal to one, u values are decreasing and v remains constant. In these cases the play operator does not play any role yet.

C: Now we start considering the play operator. This means $v = u + 1$, $v_t = u_t$ and we have the equation $u_t + \frac{1}{2}u_x = 0$, i.e., $\kappa = \frac{1}{2}$ in the above computations. The solution is determined by the initial condition: $u(x,t) = u_0(x - \kappa t) = x - \frac{1}{2}t$ and it must satisfy $u < -1$. So $x - \frac{1}{2}t < -1 \Rightarrow 2x + 2 < t$. Because u is decreasing in t, we move on the left boundary curve of the play operator and so $v(x,t) = u(x,t) + 1 = x - \frac{1}{2}t + 1$. The boundaries are determined by $2x + 2 < t$.

The initial condition is decreasing for $x \in (-6, -3)$. So we have to move inside the hysteresis loop. When we stay inside the loop, then $\kappa = 1$.

D: We are inside the hysteresis loop again, so $\kappa = 1$. The solution is determined by the initial condition: $u(x,t) = u_0(x - t) = -6 - (x - t) = -6 - x + t$. We search for a continuous solution, i.e. the regions C and D have to be divided by a line on which both solutions coincide. This means $x - \frac{1}{2}t = t - 6 - x$ and $t = 4 + \frac{4}{3}x$, $v(x,t) = x - \frac{2}{3}x - 2 + 1 = -1 + \frac{1}{3}x$. The boundaries are given by $\frac{4}{3}x + 4 < t$.

F: Now we move on the right boundary curve of the play operator, so $\kappa = \frac{1}{2}$. The solution is $u(x,t) = u_0(x - \frac{1}{2}t) = -6 - x + \frac{1}{2}t$, $v(x,t) = u(x,t) - 1 = \frac{1}{2}t - 7 - x$. We search for a continuous solution. The boundaries are calculated from $u(x,t) = -6 - x + \frac{1}{2}t < 0 \Rightarrow t < 12 + 2x$ and from the continuity of $v(x,t):\ -1 + \frac{1}{3}x = \frac{1}{2}t - 7 - x \Rightarrow t = 12 + \frac{8}{3}x$. The boundaries are determined by $\frac{8}{3}x + 12 < t < 2x + 12$.

When we are in the region D, we move inside the hysteresis loop and compute only the lower bound of this region. Then we move to the right boundary curve of the region F, and compute the lower and upper bounds. But there is a problem, because we assume that the lower bound of the region F is the upper one of the region D, hence the continuity is broken. If we try to find a continuous solution, we have to try to fit a new region E between D and F.

E: In order to preserve continuity of the solution, we calculate the missing boundary lines from the equation $bt - bx + c = u$. We replace t by one known boundary line, i.e. $t = \frac{8}{3}x + 12$. Hence, we get $\frac{5}{3}bx + 12b + c = \frac{1}{3}x$. We deduce from the equation that $12b + c = 0$ and $\frac{5}{3}b = \frac{1}{3}$. Then we express the coefficient b, i.e., $b = \frac{1}{5}$. After inserting it into the equation we have $c = -\frac{12}{5}$, so that $u(x,t) = -bx + bt + c = \frac{1}{5}(t - x - 12)$.

We search for a continuous solution. Therefore, the regions E and C have to be divided by a line on which both solutions coincide. This means $\frac{1}{5}(t - x - 12) = x - \frac{1}{2}t \Rightarrow t = \frac{12}{7}x + \frac{24}{7}$.

Consequently, the regions E and D have to be divided by a line on which both solutions coincide. This means $\frac{1}{5}(t-x-12) = -6-x+t \Rightarrow t = x+\frac{9}{2}$.

Therefore, the regions E and B have to be divided by a line on which both solutions coincide. This means $\frac{1}{5}(t-x-12) = -1 \Rightarrow t = x+7$.

For $x \leqslant \frac{3}{2}$ the value of $v(x,t) = \frac{1}{3}x - 1$ is the same as in the region D. For $\frac{3}{2} < x \leqslant 5$ we get the value of $v(x,t)$ from the continuity between the regions E and C, i.e. from the continuous line $t = \frac{12}{7}x + \frac{24}{7}$. So $v(x,t) = x - \frac{1}{2}(\frac{12}{7}x + \frac{24}{7}) + 1 = \frac{x}{7} - \frac{5}{7}$. For $x > 5$ the value of $v(x,t) = 0$ is the same as in the region B for $x \leqslant 5$.

Now we use continuity of the solution to obtain an upper boundary line of the region E. Hence, $bt - bx + c = 0$, i.e., $\frac{1}{5}t - \frac{1}{5}x - \frac{12}{5} = 0 \Rightarrow t = x+12$.

The boundaries are determined by $x + \frac{9}{2} < t < \frac{8}{3}x + 12,\ x \leqslant 0,\ x + \frac{9}{2} < t < x+12,\ 0 < x \leqslant \frac{3}{2},\ \frac{12}{7}x + \frac{24}{7} < t < x+12,\ \frac{3}{2} < x \leqslant 5,\ x+7 < t < x+12,\ x > 5$.

G: We stop when $u(x,t) = 0$. Then $v(x,t) = \begin{cases} -1 & \text{if } x \leqslant 0, \\ -1 + \frac{1}{3}x & \text{if } 0 < x \leqslant \frac{3}{2}, \\ \frac{x}{7} - \frac{5}{7}x & \text{if } \frac{3}{2} < x \leqslant 5, \\ 0 & \text{if } x > 5. \end{cases}$

The boundaries are determined by $2x + 12 < t,\ x \leqslant 0,\ x + 12 < t,\ x > 0$.

We present obtained results summed up in Tab. 1 and Fig. 4.

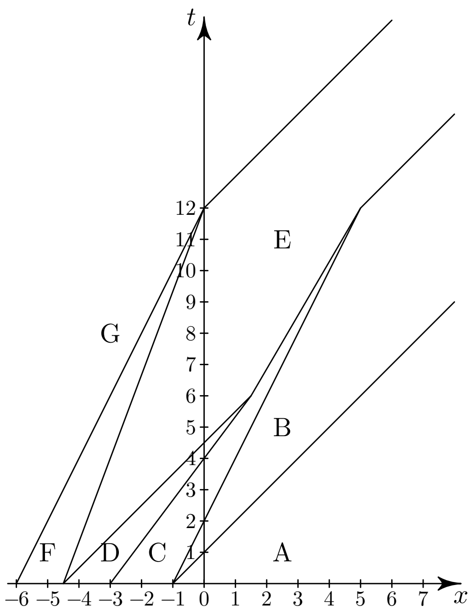

Figure 4. Regions of solution in $u(x,t)$ plane.

region	description	$u(x,t)$	$v(x,t)$
A	$0 < t < x+1$	$x-t$	0
B	$x+1 < t < 2x+2,\ x \leqslant 5$ $x+1 < t < x+7,\ x > 5$	-1	0
C	$2x+2 < t < \frac{4}{3}x+4,\ x \leqslant \frac{3}{2}$ $2x+2 < t < \frac{12}{7}x + \frac{24}{7},\ x > \frac{3}{2}$	$x - \frac{1}{2}t$	$x - \frac{1}{2}t + 1$
D	$\frac{4}{3}x + 4 < t < \frac{9}{2} + x$	$-6 - x + t$	$-1 + \frac{1}{3}x$
E	$x + \frac{9}{2} < t < \frac{8}{3}x + 12,\ x \leqslant 0$ $x + \frac{9}{2} < t < x + 12,\ 0 < x \leqslant \frac{3}{2}$ $\frac{12}{7}x + \frac{24}{7} < t < x + 12,\ \frac{3}{2} < x \leqslant 5$ $x + 7 < t < x + 12,\ x > 5$	$\frac{1}{5}(t - x - 12)$	$-1 + \frac{1}{3}x$ $-1 + \frac{1}{3}x$ $\frac{x}{7} - \frac{5}{7}$ 0
F	$\frac{8}{3}x + 12 < t < 2x + 12$	$-6 - x + \frac{1}{2}t$	$-7 - x + \frac{1}{2}t$
G	$2x + 12 < t,\ x \leqslant 0$ $x + 12 < t,\ x > 0$	0	$-1,\ x \leqslant 0$ $-1 + \frac{1}{3}x,\ 0 < x \leqslant \frac{3}{2}$ $\frac{x}{7} - \frac{5}{7},\ \frac{3}{2} < x \leqslant 5$ $0,\ x > 5$

Table 1. Computations of solution.

5. Example 2—discontinuity

Now we consider a special example of a generalized play operator. The initial condition is for simplicity

$$u(x,0) = u_0(x) = x \quad \text{and} \quad v(x,0) = x - 1, \qquad x \in (\alpha, \infty).$$

The equation (4.1) can be rewritten for this operator as

$$u_t + u_x + \left\{ \begin{array}{ll} 0 & \text{if } u - 1 < v < u + 1 \\ u_t & \text{if } v = u + 1 \text{ decreasing} \\ u_t & \text{if } v = u - 1 \text{ increasing} \\ 0 & \text{if } v = 1 \\ 0 & \text{if } v = -1 \end{array} \right\} = 0.$$

We obtain this equation from the definition of the operator:

a) $v = u - 1$, $-1 < v < 1$, v is increasing. The values of v_t are then allowed to take any positive value and $v_t = u_t$.

b) $v = u + 1$, $-1 < v < 1$, v is decreasing. The values of v_t are then allowed to take any negative value and $v_t = u_t$.

c) $u - 1 < v < u + 1$, then $v_t = 0$ because v is constant.

d) $v = 1$, then $v_t = 0$ because v is constant.

e) $v = -1$, then $v_t = 0$ because v is constant.

The initial condition is increasing for $x \in (-\infty, \infty)$.

A: We are here inside the hysteresis loop. This means $v = 0$, $v_t = 0$ and we have the equation $u_t + u_x = 0$, i.e., $\kappa = 1$ in the above computations. Therefore, $t = x + k$ are characteristics and the solution is constant on them. The solution is determined by the initial condition $u(x,t) = u_0(x - t) = x - t$. This will be our solution as long as $-1 < u < 1$, because then we hit the right or left hysteresis boundary curve and therefore the equation will be changed. Thus $x - 1 < t < x + 1$. The characteristics are $t = x + k$.

B: The same situation as above, but now we hit the left boundary curve of the play operator and stay there ($u = -1$, $v = 0$). We are above the line $t = x + 1$ and the solution is determined by the continuity of the solution. The boundaries are determined by $x + 1 < t$.

In our cases A, B, κ is equal to one, v remains constant. In these cases the play operator does not play any role yet.

C: Now we start considering the play operator. This means $v = u + 1$, $v_t = u_t$ and we have the equation $u_t + \frac{1}{2}u_x = 0$, i.e., $\kappa = \frac{1}{2}$ in the above computations. Thus $t = 2x + k$ are characteristics. The solution is determined by the initial condition: $u(x,t) = x - \frac{1}{2}t$ and it must satisfy $u < -1$. So $x - \frac{1}{2}t < -1$ and $2x + 2 < t$. We move on the left hysteresis boundary curve of the play operator and so $v(x,t) = u(x,t) + 1 = x - \frac{1}{2}t + 1$. But v can be maximally equal to -1. So if we set $-1 = x - \frac{1}{2}t + 1$, then $t = 4 + 2x$ is the time when v reaches the value -1.

We first assume the equation with $\kappa = \frac{1}{2}$, so $u = x - \frac{1}{2}t$. Secondly, we consider the equation with $\kappa = 1$. So we have $u = x - t$, the characteristics are $t = x + k$. For $x \in (-\infty, \infty)$ some values of $t = 2x + 4$ belong to the interval $[0, \infty)$. If we try to find a line where both solutions coincide we find out $x = 0$, i.e., such a line does not exist. Thus $v(x,t)$ is equal to -1 in this interval. When we sketch the characteristics of our two equations ($t = 2x + k$, $t = x + k$), we find out that the latter ones assume higher values of the solution than the former and that they cross. Thus the solution must be discontinuous (see Fig. 5).

The consequence of the nonconvexity of this type of the play operator is its discontinuity.

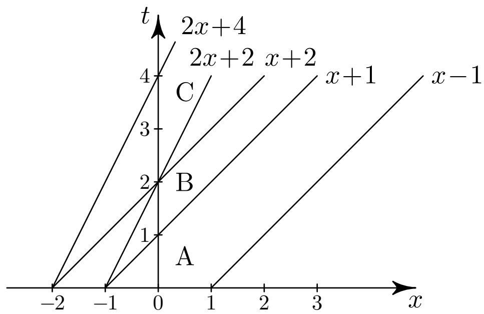

Figure 5. Characteristics intersect.

Now we can compare the results achieved. Assuming a hysteresis operator with nonconvex hysteresis curves we get the discontinuous solution. So the convexity of the hysteresis loop is broken down and a shock arises. Hence, it is necessary to consider the convex hysteresis model to get a continuous solution.

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