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ANALYSIS OF FINITE ELEMENT METHODS ON BAKHVALOV-TYPE MESHES FOR LINEAR CONVECTION-DIFFUSION PROBLEMS IN 2D*

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Abstract. So far optimal error estimates on Bakhvalov-type meshes are only known for finite difference and finite element methods solving linear convection-diffusion problems in the one-dimensional case. We prove (almost) optimal error estimates for problems with exponential boundary layers in two dimensions.

Keywords: finite element method, singular perturbation, convection-diffusion problem, Bakhvalov-type meshes, layer-adapted meshes

MSC 2010: 65N30

1. Introduction

We shall examine the finite element method for the numerical solution of the singularly perturbed linear elliptic boundary value problem

$$Lu \equiv -\varepsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{in } \Omega = (0,1)\times(0,1), \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\varepsilon \ll 1$ is a small positive parameter, b, c and f are smooth. Assuming

$$-b = (-b_1, -b_2) > (\beta_1, \beta_2) > 0 \quad \text{on } \bar{\Omega} \tag{1.2}$$

with constants β_1, β_2, the solution of (1.1) typically has exponential boundary layers at $x = 0$ and $y = 0$ and a corner layer at $(0,0)$. Additionally weak corner singularities

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could exist but we assume still some compatibility such that the problem has a classical solution with $u \in C^{3,\alpha}(\bar{\Omega})$. Without loss of generality we can as well assume

$$c - \frac{1}{2}\operatorname{div} b \geqslant c_0 > 0,$$

where c_0 is a constant.

We want to solve (1.1) with linear or bilinear finite elements on a layer-adapted mesh. Let us introduce the ε-weighted $H^1(\Omega)$ norm by

$$\|v\|_\varepsilon^2 := \varepsilon |v|_1^2 + \|v\|_0^2.$$

First, for Shishkin meshes (see Section 2 for a detailed discussion of various meshes) it was proved that

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}\ln N, \tag{1.3}$$

for bilinear elements in [11], for linear elements in [1]. Here $N+1$ is the number of mesh points used in every coordinate direction to define a tensor-product mesh, thus the degrees of freedom are of order $O(N^2)$. We remark that throughout the paper C will denote a generic positive constant that is independent of ε and of the mesh.

Later in [8] it was proved for Shishkin-type meshes (generalizing a result from [3]) that

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}\max|\psi'|, \tag{1.4}$$

here ψ denotes the mesh characterizing function (see Section 2). For a Bakhvalov-Shishkin mesh with $\psi(t) = 1 - 2(1 - N^{-1})t$ and a modified Bakhvalov-Shishkin mesh due to Vulanovic with $\psi(t) = \exp(-t/(q-t))$ and $q = 1/2 + 1/(2\ln N)$ the factor $|\psi'|$ is uniformly bounded for $t \in [0, 1/2]$. Consequently, these meshes are examples of optimal meshes with

$$\|u - u^N\|_\varepsilon \leqslant CN^{-1}. \tag{1.5}$$

For Bakhvalov-type meshes, however, it is an open question whether or not (1.5) holds in 2D. For the one-dimensional case, see [9]. The ingredients used in the proof in [9] cannot be used in 2D, therefore in this paper we present a new approach for verifying (1.5).

Throughout the paper we shall assume $|\varepsilon\ln\varepsilon| \leqslant CN^{-1}$ as in general is satisfied for discretizations of convection-dominated problems.

2. Layer-adapted meshes and solution decomposition

Let N, our discretization parameter, be an even positive integer. We introduce the mesh points

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1, \quad 0 = y_0 < y_1 < \ldots < y_{N-1} < y_N = 1$$

and consider a tensor-product mesh with mesh points (x_i, y_j). Because both meshes have the same structure we only describe the meshes in the x-direction (for the mesh in the y-direction take $\beta_1 := \beta_2$).

The mesh is graded in $[0, x_{N/2}]$ but equidistant in $[x_{N/2}, 1]$. The graded part of the mesh is based on a mesh generating function φ with $\varphi(0) = 0$, $\varphi(1/2) = \ln(1/\varepsilon)$, moreover we assume φ to be continuous, monotonically increasing and differentiable. Set

$$x_i = \begin{cases} \dfrac{\sigma\varepsilon}{\beta_1}\varphi(t_i) \quad \text{with } t_i = i/N & \text{for } i = 0, 1, \ldots, N/2, \\ 1 - (1 - x_{N/2})2(N-i)/N & \text{for } i = N/2+1, \ldots, N. \end{cases} \tag{2.1}$$

Here σ is some positive constant which characterizes the order of the smallness of the layer term in $x_{N/2}$. A Bakhvalov-type mesh (B-mesh) is given by

$$\varphi(t) := -\ln[1 - 2(1-\varepsilon)t]. \tag{2.2}$$

Remark 1. For Shishkin-type meshes (S-type meshes), introduced and analyzed in [8], we require $\varphi(1/2) = \ln N$. Especially, a Bakhvalov-Shishkin mesh (B-S-mesh) is given by $\varphi(t) := -\ln[1 - 2(1 - N^{-1})t]$; Shishkin's original mesh, however, by the definition $\varphi(t) := 2(\ln N)t$. For a survey concerning layer-adapted meshes see [6].

Following [8] the mesh characterizing function ψ is defined by

$$\psi := \exp(-\varphi). \tag{2.3}$$

Consequently, we have for $t \in [0, 1/2]$

$$\psi(t) = \begin{cases} 1 - 2(1 - N^{-1})t & \text{for a B-S-mesh,} \\ 1 - 2(1-\varepsilon)t & \text{for a B-mesh.} \end{cases}$$

Now in [8] the following result is proved:

If $\sigma = 2$ and ψ satisfies

$$\frac{\max|\psi'|}{\psi} \leqslant CN, \tag{2.4}$$

then for linear or bilinear elements the finite element error can be estimated by

$$\|u-u^N\|_\varepsilon \leqslant CN^{-1}\max|\psi'|.$$

On a B-S-mesh the condition (2.4) is satisfied but on a Bakhvalov-type mesh it is not. Therefore we shall present in Section 3 a modification of the analysis of [8] which allows to handle Bakhvalov-type meshes.

We assume that the exact solution of (1.1) can be decomposed as follows:

$$u = S + E_1 + E_2 + E_{12} = S + E. \tag{2.5}$$

The smooth part S is characterized by bounds uniform with respect to ε for certain derivatives, while E_1, E_2, E_{12} describe the layers at $x=0$, $y=0$ and the corner layer at $(0,0)$, respectively. Precisely we suppose

$$\Big|\frac{\partial^{i+j}S}{\partial x^i\partial y^j}(x,y)\Big| \leqslant C, \tag{2.6a}$$

$$\Big|\frac{\partial^{i+j}E_1}{\partial x^i\partial y^j}(x,y)\Big| \leqslant C\varepsilon^{-i}\mathrm{e}^{-\beta_1 x/\varepsilon}, \tag{2.6b}$$

$$\Big|\frac{\partial^{i+j}E_2}{\partial x^i\partial y^j}(x,y)\Big| \leqslant C\varepsilon^{-j}\mathrm{e}^{-\beta_2 y/\varepsilon}, \tag{2.6c}$$

$$\Big|\frac{\partial^{i+j}E_{12}}{\partial x^i\partial y^j}(x,y)\Big| \leqslant C\varepsilon^{-(i+j)}\mathrm{e}^{-\beta_1 x/\varepsilon}\mathrm{e}^{-\beta_2 y/\varepsilon}, \tag{2.6d}$$

for all $(x,y)\in\overline{\Omega}$ and $0\leqslant i+j\leqslant 2$. See [10, Theorem III.1.26] for conditions that guarantee the existence of such a decomposition.

These pointwise estimates bounds are stronger than needed, see Remark 3.108 in [10].

3. Error estimation on a Bakhvalov-type mesh

Let us study the discretization of the problem (1.1) with bilinear or linear (draw additionally diagonals to decompose rectangles into triangles) finite elements on the Bakhvalov-type mesh characterized by (2.1) with (2.2).

First let us notice

$$x_{N/2-1} = -\frac{\sigma\varepsilon}{\beta_1}\ln\Big(\varepsilon+\frac{2(1-\varepsilon)}{N}\Big)$$

and draw the important conclusion

$$|E_1(x_{N/2-1},\cdot)| \leqslant c\Big(\varepsilon+\frac{2(1-\varepsilon)}{N}\Big)^\sigma \leqslant cN^{-\sigma}. \tag{3.1}$$

Moreover, the condition (2.4) is satisfied on the interval $[0, x_{N/2-1}]$ because ψ' is uniformly bounded and $1.5N^{-1}$ is a lower bound for the minimal value of ψ.

Next we observe that the largest mesh size of the graded part of the Bakhvalov-type mesh is $h_{N/2} = x_{N/2} - x_{N/2-1}$ which satisfies

$$h_{N/2} = \frac{\sigma\varepsilon}{\beta_1} \ln\Big(1 + \frac{2(1-\varepsilon)}{\varepsilon N}\Big) \geqslant \kappa\varepsilon \tag{3.2}$$

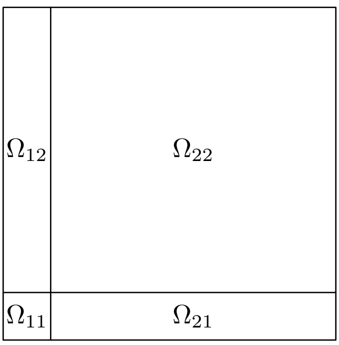
(the positive constant κ depends on σ, β_1 and the constant C in $\varepsilon \leqslant CN^{-1}$).

Remark 2. Instead of the Bakhvalov-type mesh given by (2.1) with (2.2) we could also study generalized Bakhvalov-type meshes given by (2.1) with (2.3). The mesh characterizing function ψ should be monotonically decreasing and differentiable with $\psi(0) = 1$ and $\psi(1/2) = \varepsilon$. In the analysis which follows we need the following three ingredients: the smallness of E_1 in the sense of (3.1) for $x \geqslant x_{N/2-1}$, condition (2.4) and $h_{N/2} \geqslant \kappa\varepsilon$. These three properties are guaranteed if there exist positive constants μ_1, μ_2 such that

$$\mu_1 N^{-1} \leqslant \psi(1/2 - N^{-1}) \leqslant \mu_2 N^{-1}$$

for $\varepsilon \leqslant CN^{-1}$.

Next we decompose Ω into 4 subdomains as illustrated in Fig. 1: $\bar{\Omega} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{21} \cup \Omega_{22}$.



$$\Omega_{11} := [0, x_{N/2-1}] \times [0, y_{N/2-1}]$$
$$\Omega_{12} := [0, x_{N/2-1}] \times [y_{N/2-1}, 1]$$
$$\Omega_{21} := [x_{N/2-1}, 1] \times [0, y_{N/2-1}]$$
$$\Omega_{22} := [x_{N/2-1}, 1] \times [y_{N/2-1}, 1]$$

Figure 1. Subregions of Ω.

Additionally we introduce

$$\Omega_{22}^* := [x_{N/2}, 1] \times [y_{N/2}, 1],$$

in Ω_{22}^* our mesh is uniform with mesh size of order $O(N^{-1})$.

Let us denote the piecewise linear or bilinear interpolant of a given continuous function w by w^I. Then our convergence analysis is based on the following results for the interpolation error:

Lemma 1. *Let us assume that u allows the decomposition* (2.6) *and a Bakhvalov-type mesh is given with $\sigma \geqslant 2$. Then*

$$\|E - E^I\|_{\infty,\Omega_{22}} \leqslant CN^{-2}, \quad \|E - E^I\|_{0,\Omega\setminus\Omega_{22}} \leqslant C\varepsilon^{1/2}N^{-1}, \tag{3.3a}$$

$$\varepsilon^{1/2}|E - E^I|_1 \leqslant CN^{-1}, \tag{3.3b}$$

$$N^{-1}|S - S^I|_1 + \|S - S^I\|_0 \leqslant CN^{-2}. \tag{3.3c}$$

Proof. For the smooth part S the estimates are standard applications of the known anisotropic interpolation estimates.

Concerning the layer components let us, for instance, consider E_1 and start to estimate the interpolation error. While in Ω_{11} and Ω_{12} the result is well known (see [8] or [2]), in Ω_{22} and Ω_{21} the pointwise smallness of E_1 is used in combination with $\|E^I\|_\infty \leqslant C\|E\|_\infty$ and $|\mathrm{meas}(\Omega_{21})| \leqslant C\varepsilon \ln N$.

Next consider $\|(E_1 - E_1^I)_x\|_0$, for instance. We have

$$\int_\Omega (E_1 - E_1^I)_x^2 \leqslant \int_{x\leqslant x_{N/2-1}} (E_1 - E_1^I)_x^2 + 2T$$

with

$$T = \int_{x_{N/2-1}\leqslant x\leqslant x_{N/2}} (E_1^I)_x^2 + \int_{x\geqslant x_{N/2}} (E_1^I)_x^2 + \int_{x\geqslant x_{N/2-1}} (E_1)_x^2.$$

Again, the estimate for $x \leqslant x_{N/2-1}$ is well known (see [8] or [2]). To estimate the integral with E_1 we just use the estimate in the solution decomposition, in $x \geqslant x_{N/2}$ we apply an inverse inequality combined with the smallness of E_1 to estimate the second term of T.

The first term of T is bounded by

$$\left|\int_{x_{N/2-1}\leqslant x\leqslant x_{N/2}} (E_1^I)_x^2\right| \leqslant C\frac{1}{h_{N/2}^2} h_{N/2}\|E_1^I\|^2_{\infty,x\geqslant x_{N/2-1}} \leqslant C\frac{N^{-4}}{\varepsilon},$$

because $h_{N/2} \geqslant \kappa\varepsilon$. Summarizing, the result follows. □

With linear or bilinear finite elements on our given Bakhvalov-type mesh and the corresponding finite element space $V^N \subset H_0^1(\Omega)$, the finite element method reads:

Find $u^N \in V^N$ with

$$a(u^N, v) = (f, v) \quad \forall\, v \in V^N. \tag{3.4}$$

The bilinear form $a(\cdot,\cdot)$ is given by

$$a(w,v) := \varepsilon(\nabla w, \nabla v) + (b\cdot\nabla w + cw, v);$$

due to our assumptions the bilinear form is uniformly V-elliptic with respect to the ε-weighted H^1 norm: one has

$$a(w,w) \geqslant \alpha\|w\|_\varepsilon^2 \quad \text{for all } w \in H_0^1(\Omega)$$

with some positive α independent of ε.

With the Lagrange interpolant $u^I \in V^N$ of u studied already in the previous lemma we introduce the splitting of the error into the components

$$\eta := u^I - u, \quad v^N := u^I - u^N \tag{3.5}$$

and start the error estimate from

$$\alpha\|u^I - u^N\|_\varepsilon^2 \leqslant a(u^I - u^N, u^I - u^N) = a(u^I - u, u^I - u^N) = a(\eta, v^N). \tag{3.6}$$

Write

$$a(\eta, v^N) = \varepsilon(\nabla\eta, \nabla v^N) + (b\cdot\nabla\eta, v^N) + (c\eta, v^N).$$

Based on Lemma 1 it is easy to estimate the first and the third term of that representation just using the Cauchy-Schwarz inequality.

It is important that for the crucial convection term we can split the interpolation error into $(S-S^I)+(E-E^I)$ and, using integration by parts based on $v^N \in H_0^1(\Omega)$, estimate

$$(b\cdot\nabla(S-S^I), v^N) \quad \text{and} \quad (E-E^I, b\cdot\nabla v^N).$$

Again Cauchy-Schwarz and Lemma 1 yield immediately

$$|(b\cdot\nabla(S-S^I), v^N)| \leqslant CN^{-1}\|v^N\|_0 \leqslant CN^{-1}\|v^N\|_\varepsilon.$$

Finally we have to estimate the convective term for a layer part E. Set

$$T = \int_\Omega (E-E^I)b\cdot\nabla v^N.$$

Next we split T in several parts:

$$T = \int_{\Omega_{22}^*} (E-E^I)b\cdot\nabla v^N + \int_{\Omega\setminus\Omega_{22}} (E-E^I)b\cdot\nabla v^N + \int_{\Omega_{22}\setminus\Omega_{22}^*} (E-E^I)b\cdot\nabla v^N.$$

Then we get applying an inverse inequality on Ω_{22}^*

$$|T_1| = \left| \int_{\Omega_{22}^*} (E - E^I) b \cdot \nabla v^N \right| \leqslant C N^{-1} \|v^N\|_0.$$

For the integral on $\Omega \setminus \Omega_{22}$ we use (3.3a):

$$|T_2| = \left| \int_{\Omega \setminus \Omega_{22}} (E - E^I) b \cdot \nabla v^N \right| \leqslant C \varepsilon^{1/2} N^{-1} |v^N|_1.$$

Finally we get

$$\begin{aligned} |T_3| &= \left| \int_{\Omega_{22} \setminus \Omega_{22}^*} (E - E^I) b \cdot \nabla v^N \right| \\ &\leqslant C \|E - E^I\|_{\infty, \Omega_{22}} (\operatorname{meas}(\Omega_{22} \setminus \Omega_{22}^*))^{1/2} |v^N|_1. \end{aligned}$$

Introducing

$$Q(N, \varepsilon) := \max\Big\{1, N^{-1} \Big(\ln \frac{1}{\varepsilon}\Big)^{1/2}\Big\},$$

we proved finally with

$$|T_3| \leqslant C Q(N, \varepsilon)\, N^{-1} \varepsilon^{1/2} |v^N|_1$$

the following result:

Theorem 1. *Let us assume that* u *allows the decomposition* (2.6) *and a Bakhvalov-type mesh is given with* $\sigma \geqslant 2$. *Suppose, additionally,* $|\varepsilon \ln \varepsilon| \leqslant C N^{-1}$. *Then the finite element error on the given Bakhvalov-type mesh satisfies the almost optimal estimate*

$$\|u - u^N\|_\varepsilon \leqslant C Q(N, \varepsilon) N^{-1}. \tag{3.7}$$

Remark that practically $Q(N, \varepsilon)$ is bounded: If we assume $N \geqslant 10$ and $\varepsilon \geqslant 10^{-100}$, then

$$Q(N, \varepsilon) \leqslant \sqrt{\ln 10}.$$

For bilinear elements on Shishkin-type meshes one can prove for $\sigma \geqslant 2.5$ and a decomposition of the solution with bounded derivatives up to order three the supercloseness result

$$\|u^I - u^N\|_\varepsilon \leqslant CN^{-2}(\max|\psi'| + \ln^{1/4} N)^2, \tag{4.1}$$

see [4] and [2]. For our Bakhvalov-type mesh we can derive a similar result:

$$\|u^I - u^N\|_\varepsilon \leqslant CQ(N,\varepsilon)N^{-2}\ln^{1/2} N. \tag{4.2}$$

The following ingredients are used to prove that estimate:

- The use of the Lin identities for improving the estimates for

$$(b\cdot\nabla(S - S^I), v^N) \quad \text{and} \quad (\nabla(S - S^I), \nabla v^N)$$

  analogously as in [4] and [2].
- The use of the Lin identities to estimate, e.g., the expression $\varepsilon(\nabla(E_1 - E_1^I), \nabla v^N)$ in $\Omega_{11} \cup \Omega_{12}$ analogously as in [4] and [2].
- The use of the smallness of E_1 for $x \geqslant x_{N/2-1}$ to improve the order with respect to N^{-1} in the estimate for $\varepsilon^{1/2}|E_1 - E_1^I|_1$ in that part of the domain.
- For the convective term and the layer components we use:
  (i) in Ω_{22}^* simple the smallness with respect to N^{-1} and an inverse inequality as before,
  (ii) in $\Omega \setminus \Omega_{22}$ an improved version with respect to the order of N^{-1} of (3.3a),
  (iii) in $\Omega_{22} \setminus \Omega_{22}^*$ the same technique as before (but E_1 is smaller due to the choice of σ).

Based on the supercloseness result for the Galerkin method it is possible to analyze streamline diffusion stabilization (see [12] for Shishkin meshes and [2] for general S-type meshes). Let us add to the Galerkin bilinear form the stabilization term

$$a_{\text{stab}}(w,v) := N^{-1}\sum_{T\subset\Omega_{22}^*}(-\varepsilon\Delta w + b\cdot\nabla w + cw, b\cdot\nabla v)_T$$

(T denotes some element in Ω_{22}^*). Then the error analysis requires additionally to the Galerkin terms to estimate

$$a_{\text{stab}}(u - u^I, v^N) = N^{-1}\sum_{T\subset\Omega_{22}^*}(-\varepsilon\Delta u + b\cdot\nabla(u - u^I) + c(u - u^I), b\cdot\nabla v^N)_T.$$

For the smooth part S again Lin identities can be used, moreover for the layer part its smallness in Ω_{22}^*.

We renounce to present details, because the analysis is very similar to the analysis on Shishkin-type meshes.

5. A numerical comparison

For our numerical experiments we consider the following test problem.

$$(5.1)\qquad \begin{aligned} -\varepsilon\Delta u - 2u_x - 3u_y + u &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where we choose the right-hand side f in such a way that

$$u(x,y) = 2\sin(1-x)(1-\mathrm{e}^{-2x/\varepsilon})(1-y)^2(1-\mathrm{e}^{-3y/\varepsilon})$$

is the exact solution of (5.1), which exhibits typical boundary and corner layer behavior. We take $\varepsilon = 10^{-4}, 10^{-8}, 10^{-12}$, which is sufficiently small to arise the phenomena of singular perturbation.

We observe for this test problem convergence of order one in the ε-weighted H^1-norm on sequences of B-meshes (cf. Tab. 1) as on B-S-meshes (cf. Tab. 2), as well. As predicted by (3.7) the last two columns of each table suggest that these results are uniform in ε. Moreover, even the constant in $\|u^I - u^N\|_\varepsilon \leqslant CN^{-1}$ for B-meshes and for B-S-meshes has almost the same value.

With respect to supercloseness we observe practically $\|u^I - u^N\|_\varepsilon \leqslant CN^{-2}$ on both mesh types, here (cf. Tabs. 3 and 4).

Our numerical experiments indicate: B-meshes as well as B-S-meshes are particularly suitable for solving singularly perturbed problems, the differences between these two kinds of meshes being extremely small.

N	$\varepsilon = 10^{-8}$			$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-12}$
	error	rate	$\|u - u^N\|_\varepsilon N$	error	error
8	3.0985e−1		2.4788	3.0828e−1	3.1035e−1
		1.0242			
16	1.5235e−1		2.4376	1.5212e−1	1.5242e−1
		1.0062			
32	7.5847e−2		2.4271	7.5807e−2	7.5856e−2
		1.0016			
64	3.7882e−2		2.4244	3.7872e−2	3.7883e−2
		1.0004			
128	1.8935e−2		2.4237	1.8932e−2	1.8936e−2
		1.0001			
256	9.4670e−3		2.4236	9.4657e−3	9.4671e−3

Table 1. Error on B-meshes in ε-weighted H^1-norm, $\sigma = 2$.

N	$\varepsilon = 10^{-8}$			$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-12}$
	error	rate	$\|u - u^N\|_\varepsilon\, N$	error	error
8	2.6679e−1	0.9078	2.1343	2.6678e−1	2.6679e−1
16	1.4220e−1	0.9542	2.2752	1.4219e−1	1.4220e−1
32	7.3393e−2	0.9773	2.3486	7.3390e−2	7.3393e−2
64	3.7279e−2	0.9886	2.3859	3.7278e−2	3.7279e−2
128	1.8788e−2	0.9934	2.4048	1.8787e−2	1.8788e−2
256	9.4367e−3		2.4158	9.4298e−3	9.4367e−3

Table 2. Error on B-S-meshes in ε-weighted H^1-norm, $\sigma = 2$.

N	$\varepsilon = 10^{-8}$				$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-12}$
	$\|\cdot\|_0$		$\|\cdot\|_\varepsilon$			
	$\|v^N\|_0$	rate	$\|v^N\|_\varepsilon$	rate	$\|v^N\|_\varepsilon$	$\|v^N\|_\varepsilon$
8	5.2819e−3	1.5466	2.7778e−2	1.8760	2.8330e−2	2.7654e−2
16	1.8080e−3	1.7693	7.5677e−3	1.9127	7.7037e−3	7.5316e−3
32	5.3038e−4	1.8641	2.0100e−3	1.9433	2.0356e−3	2.0022e−3
64	1.4569e−4	1.9133	5.2262e−4	1.9632	5.2532e−4	5.2107e−4
128	3.8679e−5	1.9426	1.3403e−4	1.9754	1.3262e−4	1.3372e−4
256	1.0062e−5	1.9617	3.4084e−5	1.9833	3.2857e−5	3.4025e−5
512	2.5831e−6		8.6204e−6		8.2202e−6	8.6091e−6

Table 3. Supercloseness on B-meshes, $v^N = u^I - u^N$, $\sigma = 2.5$.

N	$\varepsilon = 10^{-8}$				$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-12}$
	$\|\cdot\|_0$		$\|\cdot\|_\varepsilon$			
	$\|v^N\|_0$	rate	$\|v^N\|_\varepsilon$	rate	$\|v^N\|_\varepsilon$	$\|v^N\|_\varepsilon$
8	7.2833e−3	1.8750	2.3936e−2	1.7774	2.3934e−2	2.3936e−2
16	1.9857e−3	1.8641	6.9827e−3	1.8615	6.9811e−3	6.9827e−3
32	5.4546e−4	1.8933	1.9216e−3	1.9140	1.9192e−3	1.9216e−3
64	1.4683e−4	1.9226	5.0990e−4	1.9469	5.0728e−4	5.0990e−4
128	3.8730e−5	1.9456	1.3225e−4	1.9666	1.2992e−4	1.3226e−4
256	1.0055e−5	1.9625	3.3839e−5	1.9786	3.2488e−5	3.3839e−5
512	2.5798e−6		8.5861e−6		8.1918e−6	8.5864e−6

Table 4. Supercloseness on B-S-meshes, $v^N = u^I - u^N$, $\sigma = 2.5$.

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