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EXISTENCE OF SOLUTIONS FOR A NONLINEAR DISCRETE SYSTEM INVOLVING THE p-LAPLACIAN*

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Abstract. The existence of solutions for boundary value problems for a nonlinear discrete system involving the p-Laplacian is investigated. The approach is based on critical point theory.

Keywords: critical point theory, boundary value problems, discrete systems, p-Laplacian

MSC 2010: 39A10, 58E50, 70H05, 37J45

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ be the set of all natural numbers, integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \ldots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \ldots, b\}$ when $a \leqslant b$. Consider the boundary value problem for nonlinear discrete systems involving the p-Laplacian

$$\Delta(\phi_p(\Delta u(t-1))) + \lambda \nabla F(t, u(t)) = 0, \quad t \in \mathbb{Z}(1, M), \tag{1.1}$$

$$u(0) = u(M+1) = 0, \tag{1.2}$$

where $\lambda > 0$ is a parameter, $M > 1$ is a fixed positive integer, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\phi_p(s) = |s|^{p-2}s$, $1 < p < \infty$ and $F\colon \mathbb{Z}(0, M) \times \mathbb{R}^m \to \mathbb{R}$ is continuously differential in x for every $t \in \mathbb{Z}(0, M)$.

As is known, the critical-point theory is an important tool when dealing with the existence of solutions of differential equations (see [8]–[14], [18]). For difference

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equations, there have also been some results (see [1]–[6], [15], [16], [19]). In particular, by using the Linking Theorem, Guo and Yu have successfully proved the existence of periodic solutions for the difference equation

$$\Delta^2 x(t-1) + f(t, x(t)) = 0, \quad t \in \mathbb{Z}(1, M), \tag{1.3}$$

when either $f(t, y)$ is superlinear in the second variable y or $f(t, y)$ is sublinear in the second variable in [5] and [6], respectively. In [19], Zhou, Yu and Guo generalized such results to discrete systems. In [15], by the Local Linking Theorem, and in [16], by the Saddle Point Theorem, Xue and Tang proved the existence of periodic solutions for discrete systems. Especially, in [1], by a suitable version of Clark's Theorem, in the case $p = 2$ and $m = 1$, Bai and Xu have established

Theorem A. *Assume that the following conditions hold:*

(B_1) $f\colon [0, M+1] \times \mathbb{R} \to \mathbb{R}$ *is continuous;*
(B_2) *there exists an* $\alpha > 0$ *such that* $f(t, \alpha) = 0$ *and* $f(t, x) > 0$ *for* $x \in (0, \alpha)$*;*
(B_3) $f(t, x)$ *is odd in* x.

Then there exists a $\lambda^* > 0$ *such that if* $\lambda > \lambda^*$, (1.1)–(1.2) *with* $p = 2$ *has at least* M *distinct pairs of nontrivial solutions. Furthermore, each solution* u *satisfies* $|u(t)| \leqslant \alpha$, $t \in \mathbb{Z}(0, M+1)$.

Put

$$F(t, x) = (t - M)|x|^2 + M|x|^{3/2}, \quad t \in \mathbb{Z}(1, M), \ x \in \mathbb{R}^m. \tag{1.4}$$

We verify that F does not satisfy condition (B_2). In fact, when $m = 1$, then

$$f(t, x) = \frac{\partial F(t, x)}{\partial x} = 2(t - M)x + \frac{3}{2}\, M x^{1/2}, \quad t \in \mathbb{Z}(1, M), \ x \in \mathbb{R}^+.$$

Furthermore, when $x \in (0, \infty)$ and $t = M$, we have

$$f(M, x) = \frac{3M}{2}\, x^{1/2} > 0,$$

which shows that there is no $\alpha > 0$ such that $f(M, \alpha) = 0$. Therefore, it is worth while to further study the existence of multiple solutions to system (1.1)–(1.2).

Moreover, in [17], we have also treated system (1.1)–(1.2) with $p = 2$ by using Clark's Theorem. However, now we find that the results in [17] can be done better.

In this paper, by using the critical point theorems, we obtain, also for the one dimensional case, some solvability conditions for system (1.1)–(1.2). To be precise, for λ large enough and under suitable growth conditions on F, we establish the existence of at least mM solutions to system (1.1)–(1.2) (Theorem 3.1). On the other hand, for $\lambda > 0$, under coercivity conditions, at least one solution can be guaranteed (Theorem 3.2). Moreover, some useful consequences of our main results are pointed out (Corollaries 3.1–3.4). Finally, some examples of applications of our results, involving functions like F in (1.4), are given.

2. Preliminaries

In this section we recall some basic notation and lemmas, which come from [1], [11], and [19].

In the following statement, for any $m \in \mathbb{N}$, $(\cdot,\cdot)$ will denote the inner product in $\mathbb{R}^m$ defined by

$$(u,v) = \sum_{i=1}^{m} u_i \cdot v_i, \quad \forall\, u = (u_1, u_2, \ldots, u_m),\ v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m,$$

and $|\cdot|$ will denote the corresponding norm in $\mathbb{R}^m$, i.e.

$$|u| = \left(\sum_{i=1}^{m} u_i^2\right)^{1/2}, \quad \forall\, u = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m.$$

Let S be the set of sequences

$$u = (u(0), u(1), \ldots, u(M+1)) = \{u(t)\}_{t=0}^{M+1},$$

where $u(t) = (u_{t1}, \ldots, u_{tm})^\top \in \mathbb{R}^m$. For any $u, v \in S$, $a, b \in \mathbb{R}$, $au + bv$ is defined by

$$au + bv := \{au(t) + bv(t)\}_{t=0}^{M+1}.$$

Then S is a vector space.

For any given positive integer M, E_M is defined as a subspace of S by

$$E_M = \{u = \{u(t)\} \in S\colon\ u(0) = u(M+1) = 0\}$$

equipped with the norm

$$\|u\| := \left(\sum_{t=1}^{M} |u(t)|^p\right)^{1/p}, \quad \forall\, u \in E_M.$$

It is easy to verify that $(E_M, \|\cdot\|)$ is a Banach space and $\dim E_M = mM$.

Let X be a real Banach space. For $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the Palais-Smale condition (henceforth denoted by (PS)) if any sequence $\{u_m\} \subset X$ for which $\varphi(u_m)$ is bounded and $\varphi'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

Denote by θ the zero element of X. Σ indicates the family of sets $A \subset X \setminus \{\theta\}$ where A is closed in X and symmetric with respect to θ, i.e. $u \in A$ implies $-u \in A$.

Now, we state the main tools used to investigate system (1.1)–(1.2).

Lemma 2.1 (see [8]). *Assume that $\varphi \in C^1(E, \mathbb{R})$ is bounded from below (above) and satisfies the* (PS) *condition. Then*

$$c = \inf_{u \in E} \varphi(u) \quad (c = \sup_{u \in E} \varphi(u))$$

is a critical value of φ.

Lemma 2.2 (see [11, Theorem 9.1]). *Let X be a real Banach space and φ an even function belonging to $C^1(X, \mathbb{R})$ with $\varphi(\theta) = 0$, bounded from below and satisfying the* (PS) *condition. Suppose that there is a set $K \in \Sigma$ such that K is homeomorphic to S^{j-1} ($j-1$ dimension unit sphere) by an odd map and $\sup_K \varphi < 0$. Then φ has at least j distinct pairs of nonzero critical points.*

3. Main results

Lemma 3.1. *For any $u \in E_M$,*

$$\frac{1}{(2pM)^p} \|u\|^p \leqslant \sum_{t=0}^{M} |\Delta u(t)|^p \leqslant 2^p \|u\|^p.$$

P r o o f. It follows from $u(0) = u(M+1) = 0$ and the Hölder inequality that

$$\begin{aligned} \sum_{t=0}^{M} |\Delta u(t)|^p &= \sum_{t=0}^{M} |u(t+1) - u(t)|^p \leqslant \sum_{t=0}^{M} (|u(t+1)| + |u(t)|)^p \\ &\leqslant 2^{p-1} \sum_{t=0}^{M} |u(t+1)|^p + 2^{p-1} \sum_{t=0}^{M} |u(t)|^p \\ &= 2^{p-1} \sum_{t=1}^{M} |u(t)|^p + 2^{p-1} \sum_{t=1}^{M} |u(t)|^p = 2^p \|u\|^p. \end{aligned}$$

Thus the right-hand side has been proved. In order to prove the left-hand side, we need the inequality

$$|x^p - y^p| \leqslant p|x-y|(x^{p-1} + y^{p-1}) \quad \text{for all } x \geqslant 0,\ \ y \geqslant 0,$$

which is an immediate consequence of the Lagrange differential mean value theorem (or see [7]).

Since for any $s \in \mathbb{Z}(0, M)$,

$$\begin{aligned}
\Delta|u(s)|^p &= |u(s+1)|^p - |u(s)|^p \\
&\leqslant p\big||u(s+1)| - |u(s)|\big|(|u(s+1)|^{p-1} + |u(s)|^{p-1}) \\
&\leqslant p|u(s+1) - u(s)|(|u(s+1)|^{p-1} + |u(s)|^{p-1}) \\
&= p|\Delta u(s)|(|u(s+1)|^{p-1} + |u(s)|^{p-1}),
\end{aligned}$$

thus by the Hölder inequality and $u(0) = u(M+1) = 0$ we obtain that for any $t \in \mathbb{Z}(1, M)$,

$$\begin{aligned}
|u(t)|^p = \sum_{s=0}^{t-1} \Delta|u(s)|^p &= \sum_{s=0}^{t-1} \big[p|\Delta u(s)||u(s+1)|^{p-1} + p|\Delta u(s)||u(s)|^{p-1}\big] \\
&\leqslant \sum_{s=0}^{M} p|\Delta u(s)||u(s+1)|^{p-1} + \sum_{s=0}^{M} p|\Delta u(s)||u(s)|^{p-1} \\
&\leqslant p\Bigg(\sum_{s=0}^{M} |\Delta u(s)|^p\Bigg)^{1/p} \cdot \Bigg(\sum_{s=0}^{M} |u(s+1)|^p\Bigg)^{(p-1)/p} \\
&\quad + p\Bigg(\sum_{s=0}^{M} |\Delta u(s)|^p\Bigg)^{1/p} \cdot \Bigg(\sum_{s=0}^{M} |u(s)|^p\Bigg)^{(p-1)/p} \\
&= 2p\Bigg(\sum_{s=0}^{M} |\Delta u(s)|^p\Bigg)^{1/p} \cdot \Bigg(\sum_{s=1}^{M} |u(s)|^p\Bigg)^{(p-1)/p}.
\end{aligned}$$

Furthermore, we get

$$\sum_{t=1}^{M} |u(t)|^p \leqslant 2pM\Bigg(\sum_{s=0}^{M} |\Delta u(s)|^p\Bigg)^{1/p} \cdot \Bigg(\sum_{s=1}^{M} |u(s)|^p\Bigg)^{(p-1)/p},$$

that is,

$$\Bigg(\sum_{t=1}^{M} |u(t)|^p\Bigg)^{1/p} \leqslant 2pM\Bigg(\sum_{s=0}^{M} |\Delta u(s)|^p\Bigg)^{1/p}$$

from which, one has

$$\frac{1}{(2pM)^p}\|u\|^p \leqslant \sum_{t=0}^{M} |\Delta u(t)|^p,$$

and our conclusion is proved. □

Lemma 3.2. *For any $u, v \in E_M$, the following useful equality holds:*

$$-\sum_{t=1}^{M}\big(\Delta(\phi_p(\Delta u(t-1))), v(t)\big) = \sum_{t=0}^{M}\big(\phi_p(\Delta u(t)), \Delta v(t)\big).$$

P r o o f. In fact, it follows from $u(0) = u(M+1) = v(0) = v(M+1) = 0$ that

$$\begin{aligned}
-\sum_{t=1}^{M}(\Delta(\phi_p(\Delta u(t-1))), v(t)) &= -\sum_{t=1}^{M}\Delta(|\Delta u(t-1)|^{p-2}\Delta u(t-1), v(t)) \\
&= -\sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}\Delta u(t) - |\Delta u(t-1)|^{p-2}\Delta u(t-1), v(t)\big) \\
&= -\sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), v(t)\big) \\
&\quad + \sum_{t=1}^{M}\big(|\Delta u(t-1)|^{p-2}(u(t) - u(t-1)), v(t)\big) \\
&= -\sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), v(t)\big) + (|u(1)|^{p-2}u(1), v(1)) \\
&\quad + \sum_{t=2}^{M}\big(|\Delta u(t-1)|^{p-2}(u(t) - u(t-1)), v(t)\big) \\
&= -\sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), v(t)\big) + (|u(1)|^{p-2}u(1), v(1)) \\
&\quad + \sum_{t=1}^{M-1}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), v(t+1)\big) \\
&= -\sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), v(t)\big) + (|u(1)|^{p-2}u(1), v(1)) \\
&\quad + \sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), v(t+1)\big) \quad \text{(since } v(M+1) = 0\text{)} \\
&= \sum_{t=1}^{M}\big(|\Delta u(t)|^{p-2}(u(t+1) - u(t)), \Delta v(t) \\
&\quad + (|u(1) - u(0)|^{p-2}(u(1) - u(0)), v(1) - v(0)\big) \\
&= \sum_{t=0}^{M}(|\Delta u(t)|^{p-2}\Delta u(t), \Delta v(t)) = \sum_{t=0}^{M}(\phi_p(\Delta u(t)), \Delta v(t)).
\end{aligned}$$

The proof is complete. $\square$

In Theorem 3.1 below, we will assume that $F(t,x)$ satisfies the following conditions:

(I_1) for all $t \in \mathbb{Z}(0,M)$, $F(t,0)=0$ and for all $t \in \mathbb{Z}(1,M)$, $F(t,x)$ is even in x;

(I_2) there exists $r>0$ such that, for all $u \in E_M$ with $\|u\|=r$, $\sum_{t=1}^{M} F(t,u(t))>0$.

Consider the functional φ defined on E_M by

$$\varphi(u)=\sum_{t=0}^{M}\Big[\frac{1}{p}|\Delta u(t)|^p-\lambda F(t,u(t))\Big]=\sum_{t=0}^{M}\frac{1}{p}|\Delta u(t)|^p-\lambda\sum_{t=1}^{M}F(t,u(t)). \tag{3.1}$$

It is well known that the functional φ on E_M is continuously differentiable. Moreover, since for any $u,v\in E_M$, $v(0)=0$, we have

$$\begin{aligned}\langle\varphi'(u),v\rangle &= \sum_{t=0}^{M}\big[(\phi_p(\Delta u(t)),\Delta v(t))-\lambda(\nabla F(t,u(t)),v(t))\big]\\ &= \sum_{t=0}^{M}(\phi_p(\Delta u(t)),\Delta v(t))-\lambda\sum_{t=1}^{M}(\nabla F(t,u(t)),v(t)),\end{aligned}$$

for any $u,v\in E_M$ (see [9]). Then $u\in E_M$ is a critical point of φ if and only if

$$\sum_{t=0}^{M}(\phi_p(\Delta u(t)),\Delta v(t))=\lambda\sum_{t=1}^{M}(\nabla F(t,u(t)),v(t)). \tag{3.2}$$

By the arbitrariness of v, we conclude that

$$\Delta(\phi_p(\Delta u(t-1)))+\lambda\nabla F(t,u(t))=0,\quad \forall\, t\in\mathbb{Z}(1,M).$$

Since $u\in E_M$, we have $u(0)=u(M+1)=0$ and hence $u\in E_M$ is a critical point of φ if and only if u satisfies system (1.1)–(1.2). Thus the problem of finding the solutions to system (1.1)–(1.2) is reducing to that of seeking the critical points of the functional φ on E_M.

Theorem 3.1. *Suppose that $F(t,x)$ satisfies (I_1), (I_2) and the condition*

(I_3)
$$\limsup_{|x|\to\infty}\frac{F(t,x)}{|x|^p}\leqslant 0\quad \textit{for all } t\in\mathbb{Z}(1,M).$$

Then, if $\lambda>2^p r^p/(p\delta)$ with $\delta=\inf_{\|u\|=r}\sum_{t=1}^{M}F(t,u(t))$, system (1.1)–(1.2) has at least mM distinct pairs of nontrivial solutions.

P r o o f. By virtue of (I_1), it is easy to verify that $\varphi(0)=0$ and $\varphi(\cdot)$ is even. Next, we show that φ is coercive, that is $\lim\limits_{\|u\|\to\infty}\varphi(u)=\infty$. For any $\varepsilon>0$ with $\varepsilon<1/(\lambda p(2pM)^p)$, by ($\mathrm{I}_3$), there is a $\varrho_1>0$ such that $F(t,x)\leqslant\varepsilon|x|^p$ for all $x\in\mathbb{R}^N$ with $|x|>\varrho_1$ and all $t\in\mathbb{Z}(1,M)$. Let $a_{\varrho_1}=\max\{|F(t,x)|\colon t\in\mathbb{Z}(1,M),\ x\in\mathbb{R}^m,\ |x|\leqslant\varrho_1\}$. Hence,

$$F(t,x)\leqslant\varepsilon|x|^p+a_{\varrho_1} \tag{3.3}$$

for all $x\in\mathbb{R}^N$ and all $t\in\mathbb{Z}(1,M)$. If there is a sequence $\{u_n\}\subset E_M$ and a constant c such that $\|u_n\|\to\infty$, $n\to\infty$ and $\varphi(u_n)\leqslant c$, $n=1,2,\ldots$, then by Lemma 3.1 and (3.3) we have

$$\begin{aligned}\frac{c}{\|u_n\|^p}\geqslant\frac{\varphi(u_n)}{\|u_n\|^p}&=\frac{1}{p}\frac{\sum\limits_{t=0}^{M}|\Delta u_n(t)|^p}{\|u_n\|^p}-\frac{\lambda\sum\limits_{t=1}^{M}F(t,u_n(t))}{\|u_n\|^p}\\&\geqslant\frac{1}{p(2pM)^p}-\frac{\lambda}{\|u_n\|^p}\sum_{t=1}^{M}(\varepsilon|u_n(t)|^p+a_{\varrho_1})\\&=\frac{1}{p(2pM)^p}-\lambda\varepsilon-\frac{\lambda Ma_{\varrho_1}}{\|u_n\|^p}.\end{aligned}$$

Let $n\to\infty$. Then we have $1/(p(2pM)^p)-\lambda\varepsilon\leqslant0$, which contradicts $\varepsilon<1/(\lambda p(2pM)^p)$. Therefore, φ is coercive. Furthermore, it is easy to observe that φ is bounded from below and the (PS) condition follows at once from the coercivity of I, as the space E_M has finite dimension.

Define

$$K=\{u\in E_M\colon\ \|u\|=r\}.$$

We can find that $0\notin K$ and K is closed in E_M and symmetric with respect to 0. It is clear that K is homeomorphic to S^{mM-1} by an odd map.

Clearly, by (I_2), we get $\delta=\inf\limits_{\|u\|=r}\sum\limits_{t=1}^{M}F(t,u(t))>0$. Then it follows from Lemma 3.1 and $\lambda>2^pr^p/(p\delta)$ that, for any $u\in K$,

$$\varphi(u)=\frac{1}{p}\sum_{t=0}^{M}|\Delta u(t)|^p-\lambda\sum_{t=1}^{M}F(t,u(t))\leqslant\frac{2^p}{p}\|u\|^p-\lambda\sum_{t=1}^{M}F(t,u(t))\leqslant\frac{2^pr^p}{p}-\lambda\delta<0.$$

Thus all the conditions of Lemma 2.2 are satisfied and then φ has at least mM distinct pairs of nonzero critical points. Consequently, (1.1)–(1.2) has at least mM distinct pairs of nontrivial solutions. Thus we have completed the proof. $\square$

R e m a r k 3.1. By the proof of Theorem 3.1, it is easy to know that if $F(t,x)$ satisfies only the condition (I_3), then φ is bounded from below and satisfies the (PS) condition. Hence, by Lemma 2.1, if $\lambda > 0$, then (1.1)–(1.2) has at least one solution.

Corollary 3.1. *Suppose that $F(t,x)$ satisfies (I_1), (I_2) and the following condition:*

(I_4) *there exists $\alpha \in [0,p)$ such that*

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^\alpha} < \infty \quad \textit{for all } t \in \mathbb{Z}(1,M).$$

Then, if $\lambda > 2^p r^p/(p\delta)$ with $\delta = \inf\limits_{\|u\|=r} \sum\limits_{t=1}^{M} F(t,u(t))$, system (1.1)–(1.2) has at least mM distinct pairs of nontrivial solutions.

P r o o f. For every $t \in \mathbb{Z}(1,M)$, put

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^\alpha} = A(t).$$

Then by (I_4), the inequality $A(t) < \infty$ holds for all $t \in \mathbb{Z}(1,M)$.

Now, we distinguish two cases.

Case (i): If for every $t \in \mathbb{Z}(1,M)$, $A(t) > -\infty$, then for any $\varepsilon > 0$ there exists $\varrho_2(t) > 0$ such that $F(t,x) \leqslant (A(t)+\varepsilon)|x|^\alpha$ for all $x \in \mathbb{R}^N$ with $|x| > \varrho_2(t)$. Moreover, let

$$a_{\varrho_2}(t) = \max\{|F(t,x)|\colon\ x \in \mathbb{R}^m,\ |x| \leqslant \varrho_2(t)\}.$$

Then, for every $t \in \mathbb{Z}(1,M)$ and all $x \in \mathbb{R}^m$, we have

$$F(t,x) \leqslant |A(t)+\varepsilon||x|^\alpha + a_{\varrho_2}(t) \leqslant (A_1+\varepsilon)|x|^\alpha + a_{\varrho_2},$$

where

$$A_1 = \max_{t\in\mathbb{Z}(1,M)} |A(t)|, \quad a_{\varrho_2} = \max_{t\in\mathbb{Z}(1,M)} a_{\varrho_2}(t).$$

Since $\alpha < p$, we have

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^p} \leqslant 0,$$

which shows that (I_3) in Theorem 3.1 holds.

Case (ii): If there exist $t_1,\ldots,t_k \in \mathbb{Z}(1,M)$ $(1 \leqslant k \leqslant M)$ such that $A(t_i) = -\infty$ $(1 \leqslant i \leqslant k)$, then for any $G_1 > 0$ there exists $\varrho_3(t_i) > 0$ such that $F(t_i,x) \leqslant -G_1|x|^\alpha$ for all $x \in \mathbb{R}^m$ with $|x| \geqslant \varrho_3(t_i)$. Let

$$a_{\varrho_3}(t_i) = \max\{|F(t_i,x)|\colon\ x \in \mathbb{R}^m,\ |x| \leqslant \varrho_3(t_i)\}.$$

Then we have, for all $x \in \mathbb{R}^m$,

$$F(t_i, x) \leqslant G_1|x|^\alpha + a_{\varrho 3}(t_i) \leqslant G_1|x|^\alpha + a_{\varrho 3}, \quad \forall\, i \in \mathbb{Z}(1, k), \tag{3.4}$$

where $a_{\varrho 3} = \max\limits_{1\leqslant i\leqslant k} a_{\varrho 3}(t_i)$.

For $t \in \mathbb{Z}(1, M)/\{t_1, \ldots, t_k\}$, since $A(t) > -\infty$, similarly to the argument in case (i) we can get, for all $x \in \mathbb{R}^m$,

$$F(t, x) \leqslant (A_2 + \varepsilon)|x|^\alpha + a_{\varrho 4}, \tag{3.5}$$

where

$$\begin{aligned} A_2 &= \max\{|A(t)|\colon\ t \in \mathbb{Z}(1, M)/\{t_1, \ldots, t_k\}\},\\ a_{\varrho 4} &= \max\{a_{\varrho 4}(t)\colon\ t \in \mathbb{Z}(1, M)/\{t_1, \ldots, t_k\}\}. \end{aligned}$$

By (3.4) and (3.5), it is easy to obtain that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$F(t, x) \leqslant C_1|x|^\alpha + C_2, \quad \forall\, t \in \mathbb{Z}(1, M),\ \forall\, x \in \mathbb{R}^m. \tag{3.6}$$

Then we also get

$$\limsup_{|x|\to\infty} \frac{F(t, x)}{|x|^p} \leqslant 0,$$

which shows that (I_3) holds. By Theorem 3.1, we complete the proof. □

R e m a r k 3.2. Corollary 3.1 shows that when $p = 2$, Theorem 3.1 in [17] coincides with our Corollary 3.1.

Corollary 3.2. *Suppose that $F(t, x)$ satisfies (I_1), (I_2) and the following condition:*

(I_5) *there exist $\mu \in (0, p)$ and $R > 0$ such that*

$$(\nabla F(t, x), x) \leqslant \mu F(t, x)$$

for all $x \in \mathbb{R}^m$ with $|x| > R$ and all $t \in \mathbb{Z}(1, M)$.

Then, if $\lambda > 2^p r^p/(p\delta)$ with $\delta = \inf\limits_{\|u\|=r} \sum\limits_{t=1}^{M} F(t, u(t))$, (1.1)–(1.2) has at least mM distinct pairs of nontrivial solutions.

P r o o f. Choose R_1 such that $R_1 > R$. Similarly to the argument in [12], for all $x \in \mathbb{R}^m/\{0\}$ and all $t \in \mathbb{Z}(1, M)$, define

$$y(s) = F(t, sx), \quad Q(s) = y'(s) - \frac{\mu}{s}\, y(s),\ \forall\, s \geqslant \frac{R_1}{|x|}. \tag{3.7}$$

Then, by (I_5), we have

$$Q(s) = \frac{1}{s}\,[(\nabla F(t,sx),sx) - \mu F(t,sx)] \leqslant 0 \tag{3.8}$$

for all $s \geqslant R_1/|x|$. It follows from (3.7) that $y(s) = F(t,sx)$ is a solution of the first order linear ordinary differential equation

$$y'(s) = \frac{\mu}{s}\,y(s) + Q(s),$$

which implies that

$$F(t,sx) = s^{\mu}\left(\int_1^s r^{-\mu}Q(r)\,\mathrm{d}r + F(t,x)\right)$$

for $s \geqslant R_1/|x|$. Moreover, by the continuity of $F(t,x)$ and (3.8), we have

$$C_3 \geqslant F(t, R_1 x/|x|) \geqslant (R_1/|x|)^{\mu} F(t,x)$$

for all $x \in \mathbb{R}^m$ with $|x| \geqslant R_1$ and all $t \in \mathbb{Z}(1,M)$, where

$$C_3 = \max\{|F(t,x)|\colon\ t \in \mathbb{Z}(1,M),\ |x| \leqslant R_1\}.$$

Hence,

$$F(t,x) \leqslant \frac{C_3}{R_1^{\mu}}\,|x|^{\mu} + C_3$$

for all $x \in \mathbb{R}^m$ and all $t \in \mathbb{Z}(1,M)$. Since $\mu < p$, this implies

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^p} \leqslant 0,$$

which shows that (I_3) holds. By Theorem 3.1, we complete the proof. □

Remark 3.3. If $F(t,x)$ satisfies either condition (I_4) or (I_5), then φ is bounded from below and satisfies the (PS) condition. Hence, by Lemma 2.1, if $\lambda > 0$, (1.1)–(1.2) has at least one solution.

Theorem 3.2. *Assume that F satisfies $F(0,0)=0$ and the following condition:*

(I_6) *there exists a constant $\beta > p$ such that*

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^\beta} > 0 \quad \textit{for all } t \in \mathbb{Z}(1,M).$$

Then, if $\lambda > 0$, (1.1)–(1.2) has at least one solution.

P r o o f. For every $t \in \mathbb{Z}(1,M)$, put

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^\beta} = B(t).$$

By (I_6), $B(t) > 0$ for all $t \in \mathbb{Z}(1,M)$ and $B_1 = \min_{t\in\mathbb{Z}(1,M)} B(t) > 0$.

Now, we distinguish two cases.

Case (i): If $B(t) < \infty$ for all $t \in \mathbb{Z}(1,M)$, then for any $0 < \varepsilon < B_1$ there exists $\varrho_5(t) > 0$ such that $F(t,x) \geqslant (B(t)-\varepsilon)|x|^\beta$ for all $x \in \mathbb{R}^m$ with $|x| \geqslant \varrho_5(t)$. Let

$$a_{\varrho_5}(t) = \max\{|F(t,x)|\colon\ |x| \leqslant \varrho_5(t)\}.$$

Then, for all $x \in \mathbb{R}^m$,

$$F(t,x) \geqslant (B(t)-\varepsilon)|x|^\beta - (B(t)-\varepsilon)\varrho_5^\beta(t) - a_{\varrho_5}(t) \geqslant (B_1-\varepsilon)|x|^\beta - a_{\varrho_5},$$

where

$$a_{\varrho_5} = \max_{t\in\mathbb{Z}(1,M)} \{(B(t)-\varepsilon)\varrho_5^\beta(t) + a_{\varrho_5}(t)\}.$$

Case (ii): If there exist $t_1,\dots,t_k \in \mathbb{Z}(1,M)$ $(1 \leqslant k \leqslant M)$ such that $B(t_i) = \infty$ $(1 \leqslant i \leqslant k)$, then for any $G_2 > 0$ there exists $\varrho_6(t_i) > 0$ such that $F(t_i,x) \geqslant G_2|x|^\beta$ for all $x \in \mathbb{R}^m$ with $|x| > \varrho_6(t_i)$. Let

$$a_{\varrho_6}(t_i) = \max\{|F(t_i,x)|\colon\ x \in \mathbb{R}^m,\ |x| \leqslant \varrho_6(t_i)\}.$$

Then we obtain that, for all $x \in \mathbb{R}^m$,

$$F(t_i,x) \geqslant G_2|x|^\beta - G_2\varrho_6^\beta(t_i) - a_{\varrho_6}(t_i) \geqslant G_2|x|^\beta - a_{\varrho_6}, \quad \forall\, i \in \mathbb{Z}(1,k), \tag{3.9}$$

where $a_{\varrho_6} = \max_{i\in\mathbb{Z}(1,k)} \{G_2\varrho_6^\beta(t_i) + a_{\varrho_6}(t_i)\}$.

For $t \in \mathbb{Z}(1,M)/\{t_1,\dots,t_k\}$, since $B(t) < \infty$, similarly to the argument in case (i), for all $x \in \mathbb{R}^m$ and $0 < \varepsilon < B_2$ we get

$$F(t,x) \geqslant (B_2-\varepsilon)|x|^\beta - a_{\varrho_7}, \tag{3.10}$$

where

$$B_2 = \min\{|B(t)|\colon\ t \in \mathbb{Z}(1, M)/\{t_1, \dots, t_k\}\} > 0,$$

and

$$a_{\varrho_7} = \max\{(B(t) - \varepsilon)\varrho_7^{\beta}(t) + a_{\varrho_7}(t)\colon\ t \in \mathbb{Z}(1, M)/\{t_1, \dots, t_k\}\}.$$

By (3.9) and (3.10), it is easy to obtain that there exist $C_4 > 0$ and $C_5 > 0$ such that

$$F(t, x) \geqslant C_4|x|^{\beta} - C_5, \quad \forall\, t \in \mathbb{Z}(1, M),\ \forall\, x \in \mathbb{R}^m. \tag{3.11}$$

Both the cases (i) and (ii) imply that there exist $C_6 > 0$ and $C_7 > 0$ such that

$$F(t, x) \geqslant C_6|x|^{\beta} - C_7 \tag{3.12}$$

for all $x \in \mathbb{R}^m$ and all $t \in \mathbb{Z}(1, M)$. By Hölder's inequality we have

$$\sum_{t=1}^{M} |u(t)|^p \leqslant M^{1-p/\beta}\left(\sum_{t=1}^{M} |u(t)|^{\beta}\right)^{p/\beta}.$$

Consequently, since $\beta > p$, $\lambda > 0$, and bearing in mind (3.12) and Lemma 3.1, for all $u \in E_M$, one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\sum_{t=0}^{M} |\Delta u(t)|^p - \lambda \sum_{t=1}^{M} F(t, u(t)) \\
&\leqslant \frac{2^p}{p}\|u\|^p - \lambda C_6 \sum_{t=1}^{M} |u(t)|^{\beta} + \lambda M C_7 \\
&\leqslant \frac{2^p}{p}\|u\|^p - \lambda C_6 M^{1-\beta/p}\left(\sum_{t=1}^{M} |u(t)|^p\right)^{\beta/p} + \lambda M C_7 \\
&= \frac{2^p}{p}\|u\|^p - \lambda C_6 M^{1-\beta/p}\|u\|^{\beta} + \lambda M C_7 \to -\infty \quad (\text{as } \|u\| \to \infty),
\end{aligned}$$

which implies that I is bounded from above and the (PS) sequence must be bounded in E_M. Then the (PS) sequence has a convergent subsequence, since E_M has finite dimension. Hence, φ satisfies the (PS) condition. By Lemma 2.1, (1.1)–(1.2) has at least one solution. We have completed the proof. □

Corollary 3.3. *Suppose that $F(t,x)$ satisfies $F(0,0)=0$ and the following assumptions:*

(I_7) *there exist $\gamma > 0$ and $L > 0$ such that*

$$F(t,x) \geqslant \gamma |x|^p$$

for all $x \in \mathbb{R}^m$ with $|x| > L$ and all $t \in \mathbb{Z}(1,M)$;

(I_8) *there exists $\mu > p$ such that*

$$\limsup_{|x|\to\infty} \frac{\mu F(t,x) - (\nabla F(t,x), x)}{|x|^p} \leqslant 0$$

for all $t \in \mathbb{Z}(1,M)$.

Then, if $\lambda > 0$, (1.1)–(1.2) has at least one solution.

Proof. The proof is similar to Lemma 1 in [13]. By (I_8), there is a constant $R_2 > L$ such that

$$\mu F(t,x) - (\nabla F(t,x), x) \leqslant D_1 |x|^p \tag{3.13}$$

for all $x \in \mathbb{R}^m$ with $|x| > R_2$ and all $t \in \mathbb{Z}(1,M)$, where $D_1 = \frac{1}{2}(\mu - p)\gamma$. By the continuity of F and ∇F, there exists a constant D_2 such that

$$\mu F(t,x) - (\nabla F(t,x), x) \leqslant D_2$$

for all $x \in \mathbb{R}^m$ with $|x| \leqslant R_2$ and all $t \in \mathbb{Z}(1,M)$. Hence, we obtain that

$$(\nabla F(t,x), x) \geqslant \mu F(t,x) - D_1 |x|^p - D_2 \tag{3.14}$$

for all $x \in \mathbb{R}^m$ and all $t \in \mathbb{Z}(1,M)$. Define

$$f(s) = F(t, sx), \quad \forall\, s \geqslant \frac{R_2}{|x|}$$

for all $x \in \mathbb{R}^m/\{0\}$ and all $t \in \mathbb{Z}(1,M)$. Then we deduce from (3.13)

$$f'(s) = \frac{1}{s}\,(\nabla F(t,sx), sx) \geqslant \frac{\mu}{s} F(t,sx) - D_1 s^{p-1}|x|^p = \frac{\mu}{s} f(s) - D_1 s^{p-1} |x|^p,$$

which implies that

$$g(s) = f'(s) - \frac{\mu}{s}\, f(s) + D_1 s^{p-1} |x|^p \geqslant 0.$$

By solving the above equation, we obtain

$$f(s) = \Big(\int_{R_2/|x|}^{s} \frac{g(r) - D_1 r^{p-1}|x|^p}{r^\mu}\, \mathrm{d}r + D_3 \Big) s^\mu \tag{3.15}$$

for $s \geqslant R_2/|x|$, and

$$f\Big(\frac{R_2}{|x|}\Big) = D_3 \Big(\frac{R_2}{|x|}\Big)^\mu.$$

Then, we get

$$D_3 = \Big(\frac{|x|}{R_2}\Big)^\mu f\Big(\frac{R_2}{|x|}\Big).$$

By (3.15), we have

$$\begin{aligned}
f(s) &= \Big[\int_{R_2/|x|}^{s} \frac{g(r) - D_1 r^{p-1}|x|^p}{r^\mu}\, \mathrm{d}r + D_3 \Big] s^\mu \\
&= \Big[\int_{R_2/|x|}^{s} \frac{g(r)}{r^\mu}\, \mathrm{d}r - D_1 |x|^p \int_{R_2/|x|}^{s} r^{p-1-\mu}\, \mathrm{d}r + D_3 \Big] s^\mu \\
&\geqslant D_3 s^\mu + \Big[\frac{D_1 |x|^p}{\mu - p} s^{p-\mu} - \frac{D_1}{(\mu-p) R_2^{\mu-p}} |x|^\mu \Big] s^\mu \\
&\geqslant \Big[R_2^{-\mu} f\Big(\frac{R_2}{|x|}\Big) - \frac{D_1}{(\mu-p) R_2^{\mu-p}} \Big] |x|^\mu s^\mu \\
&= \Big[R_2^{-\mu} F\Big(t, \frac{R_2}{|x|} x\Big) \frac{D_1}{(\mu-p) R_2^{\mu-p}} \Big] |x|^\mu s^\mu.
\end{aligned}$$

So, we obtain

$$F(t,x) = f(1) \geqslant \Big[R_2^{-\mu} F\Big(t, \frac{R_2}{|x|} x\Big) - \frac{D_1}{(\mu-p) R_2^{\mu-p}} \Big] |x|^\mu$$

for all $x \in \mathbb{R}^m$ with $|x| > R_2$ and all $t \in \mathbb{Z}(1, M)$. By (I_7), the above inequlity and $D_1 = \frac{1}{2}(\mu - p)\gamma$, we have

$$F(t,x) \geqslant D_4 |x|^\mu$$

for all $x \in \mathbb{R}^m$ with $|x| > R_2$ and all $t \in \mathbb{Z}(1, M)$, where $D_4 = \big(\gamma - (D_1/(\mu - p)\big) \times R_2^{p-\mu} = \frac{1}{2}\gamma R_2^{p-\mu} > 0$. Because of the continuity of $F(t,x)$, there is a positive constant D_5 such that

$$|F(t,x)| \leqslant D_5$$

for $x \in \mathbb{R}^m$ with $|x| \leqslant R_2$ and all $t \in \mathbb{Z}(1, M)$. Let

$$D_6 = D_4 R_2^\mu + D_5.$$

Then we have

$$F(t,x) \geqslant D_4|x|^\mu - D_6.$$

Consequently,

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^\mu} \geqslant D_4 > 0.$$

Hence, by Theorem 3.2 with $\beta = \mu$, we complete the proof. □

The next result involves the well-known Ambrosetti-Rabinowitz condition.

Corollary 3.4. *Suppose that $F(t,x)$ satisfies $F(0,0) = 0$ and the (AR) condition, that is*

(I_9) *there exists $\xi > p$ and $K > 0$ such that*

$$0 < \xi F(t,x) \leqslant (\nabla F(t,x), x)$$

for all $x \in \mathbb{R}^m$ with $|x| > K$ and all $t \in \mathbb{Z}(1,M)$.

Then, if $\lambda > 0$, (1.1)–(1.2) has at least one solution.

Proof. Choose R_3 such that $R_3 > K$. Define

$$f(s) = F(t,sx), \quad \forall\, s \geqslant \frac{R_3}{|x|}$$

for all $x \in \mathbb{R}^m/\{0\}$ and all $t \in \mathbb{Z}(1,M)$. Then we deduce from (I_9)

$$f'(s) = \frac{1}{s}\,(\nabla F(t,sx), sx) \geqslant \frac{\xi}{s}\,F(t,sx) = \frac{\xi}{s}\,f(s),$$

which implies that

$$g(s) = f'(s) - \frac{\xi}{s}\,f(s) \geqslant 0.$$

By solving the above equation, we obtain

$$f(s) = \Big(\int_{R_3/|x|}^{s} \frac{g(r)}{r^\mu}\,\mathrm{d}r + D_7\Big) s^\xi \tag{3.16}$$

for $s \geqslant R_3/|x|$, and

$$f\Big(\frac{R_3}{|x|}\Big) = D_7\Big(\frac{R_3}{|x|}\Big)^\xi.$$

Then, we get

$$D_7 = \Big(\frac{|x|}{R_3}\Big)^\xi f\Big(\frac{R_3}{|x|}\Big).$$

By (3.16), we have

$$
\begin{aligned}
f(s) &= \left[\int_{R_3/|x|}^{s} \frac{g(r)}{r^{\xi}}\,\mathrm{d}r + D_7\right] s^{\xi} \\
&\geqslant D_7 s^{\xi} \geqslant R_3^{-\xi} f\Big(\frac{R_3}{|x|}\Big)|x|^{\xi} s^{\xi} = R_3^{-\xi} F\Big(t, \frac{R_3}{|x|}x\Big)|x|^{\xi} s^{\xi}.
\end{aligned}
$$

So, we obtain

$$
F(t,x) = f(1) \geqslant R_3^{-\xi} F\Big(t, \frac{R_3}{|x|}\, x\Big)|x|^{\xi}
$$

for all $x \in \mathbb{R}^m$ with $|x| > R_3$ and all $t \in \mathbb{Z}(1,M)$. By (I_9), we know that $F(t,y) > 0$ for all $y \in \mathbb{R}^m$ with $|y| = R_3$ and all $t \in \mathbb{Z}(1,M)$. Hence, the above inequality implies that

$$
F(t,x) \geqslant D_8 |x|^{\xi}
$$

for all $x \in \mathbb{R}^m$ with $|x| > R_3$ and all $t \in \mathbb{Z}(1,M)$, where $D_8 = R_3^{-\xi} \min\limits_{|y|=R_3} F\,(t,y) > 0$. By the continuity of $F(t,x)$, there is a constant $D_9 > 0$ such that

$$
|F(t,x)| \leqslant D_9
$$

for $x \in \mathbb{R}^m$ with $|x| \leqslant R_3$ and all $t \in \mathbb{Z}(1,M)$. Let

$$
D_{10} = D_8 R_3^{\xi} + D_9.
$$

Then we have

$$
F(t,x) \geqslant D_8 |x|^{\xi} - D_{10}.
$$

Consequently,

$$
\liminf \frac{F(t,x)}{|x|^{\xi}} \geqslant D_8 > 0.
$$

Hence, by Theorem 3.2 with $\beta = \xi$, we complete the proof. $\square$

4. Examples

In this section, some examples will be given to illustrate our results.

Example 4.1. Consider the system

$$\Delta^2 u(t-1) + \lambda \nabla F(t, u(t)) = 0, \quad t \in \mathbb{Z}(1, M), \tag{4.1}$$

$$u(0) = u(M+1) = 0, \tag{4.2}$$

where F is defined by (1.4). Then $p = 2$. It is easy to see that F satisfies (I_1) of Theorem 3.1. Moreover, since

$$\Big(\sum_{t=1}^{M} |u(t)|^2\Big)^{3/4} \leqslant \sum_{t=1}^{M} |u(t)|^{3/2},$$

we have

$$\begin{aligned}\sum_{t=1}^{M} F(t, u(t)) &= \sum_{t=1}^{M} (t-M)|u(t)|^2 + M \sum_{t=1}^{M} |u(t)|^{3/2} \\ &\geqslant (1-M) \sum_{t=1}^{M} |u(t)|^2 + M \Big(\sum_{t=1}^{M} |u(t)|^2\Big)^{3/4} \\ &= (1-M)\|u\|^2 + M\|u\|^{3/2}.\end{aligned} \tag{4.3}$$

Therefore, there exists $0 < r < M^2/(M-1)^2$ such that for all $u \in E_M$ with $\|u\| = r$, $\sum\limits_{t=1}^{M} F(t, u(t)) > 0$. Thus ($\mathrm{I}_2$) is verified. It is easy to obtain that

$$\limsup_{|x| \to \infty} \frac{F(t, x)}{|x|^2} = t - M \leqslant 0 \quad \text{for all } t \in \mathbb{Z}(1, M),$$

which implies (I_3) holds. By (4.3), we have

$$\delta = \inf_{\|u\|=r} \sum_{t=1}^{M} F(t, u(t)) \geqslant \inf_{\|u\|=r} \{(1-M)\|u\|^2 + M\|u\|^{3/2}\} = (1-M)r^2 + Mr^{3/2}.$$

Then, by Theorem 3.1, we know that if $\lambda > 2r^2/((1-M)r^2 + Mr^{3/2}) = 2\sqrt{r} \times ((1-M)\sqrt{r} + M)^{-1}$, system (4.1)–(4.2) has at least mM distinct pairs of solutions.

Example 4.2. Consider the system

$$\Delta(\phi_4(\Delta u(t-1))) + \lambda \nabla F(t, u(t)) = 0, \quad t \in \mathbb{Z}(1, M), \tag{4.4}$$

$$u(0) = u(M+1) = 0, \tag{4.5}$$

where

$$F(t,x) = (t-M)|x|^4 + M|x|^{7/2}.$$

It is easy to see that F satisfies (I_1) of Theorem 3.1. Moreover, since

$$\Bigl(\sum_{t=1}^{M} |u(t)|^4\Bigr)^{7/8} \leqslant \sum_{t=1}^{M} |u(t)|^{7/2},$$

we have

$$\begin{aligned} \sum_{t=1}^{M} F(t,u(t)) &\geqslant \sum_{t=1}^{M} (t-M)|u(t)|^4 + M\sum_{t=1}^{M} |u(t)|^{7/2} \\ &\geqslant (1-M)\|u\|^4 + M\Bigl(\sum_{t=1}^{M} |u(t)|^4\Bigr)^{7/8} \\ &= (1-M)\|u\|^4 + M\|u\|^{7/2}. \end{aligned} \tag{4.6}$$

Therefore, there exists $0 < r < M^2/(M-1)^2$ such that for all $u \in E_M$ with $\|u\| = r$, $\sum\limits_{t=1}^{M} F(t,u(t)) > 0$. Thus (I_2) is verified. It is easy to obtain that

$$\liminf \frac{F(t,x)}{|x|^4} = t - M \leqslant 0 \quad \text{for all } \ t \in \mathbb{Z}(1,M),$$

which implies (I_3) holds. By (4.6), we have

$$\delta = \inf_{\|u\|=r} \sum_{t=1}^{M} F(t,u(t)) \geqslant \inf_{\|u\|=r} \{(1-M)\|u\|^4 + M\|u\|^{7/2}\} = (1-M)r^4 + Mr^{7/2}.$$

Then, by Theorem 3.1, we know that if $\lambda > 4r^4/((1-M)r^4 + Mr^{7/2}) = 4\sqrt{r} \times ((1-M)\sqrt{r} + M)^{-1}$, system (4.5)–(4.6) has at least mM distinct pairs of solutions.

Example 4.3. Consider the system

$$\Delta(\phi_4(\Delta u(t-1))) + \lambda \nabla F(t,u(t)) = 0, \quad t \in \mathbb{Z}(1,M), \tag{4.7}$$

$$u(0) = u(M+1) = 0, \tag{4.8}$$

where

$$F(t,x) = (M+1-t)|x|^5 + g(t)|x|^3 + (h(t),x),$$

and $h\colon\ \mathbb{Z}(0,M) \to \mathbb{R}^m$. It is easy to see that

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^5} = M + 1 - t > 0 \quad \text{for all } \ t \in \mathbb{Z}(1,M).$$

Let $\beta = 5$ in Theorem 3.2. Then, if $\lambda > 0$, system (4.7)–(4.8) has at least one solution.

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