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SEVERAL COMMENTS ON THE HENSTOCK-KURZWEIL AND MCSHANE INTEGRALS OF VECTOR-VALUED FUNCTIONS

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Abstract. We make some comments on the problem of how the Henstock-Kurzweil integral extends the McShane integral for vector-valued functions from the descriptive point of view.

Keywords: Henstock-Kurzweil integral, McShane integral, Pettis integral, AC, AC_*, and AC_δ functions, Alexiewicz norm

MSC 2010: 26A39, 28B05

1. Introduction

The Henstock-Kurzweil integral and the McShane integral have definitions similar to that of the Riemann integral. The basic distinction between these integrals involves the class of partitions. Since a Henstock partition is clearly a McShane partition, it is more difficult for a function to be McShane integrable. More precisely, it can be shown that for real-valued functions the McShane integral is an absolute integral, which is equivalent to the Lebesgue integral, and the Henstock-Kurzweil integral is a non-absolute integral, which is equivalent to the restricted Denjoy integral (referred to as $\mathscr{D}_*$ in Saks [14]). Since an AC function is necessarily an ACG_* function, this is the descriptive way to see how the Henstock-Kurzweil integral generalizes the McShane integral. It is important to point out that the Henstock-Kurzweil and McShane integrals are equivalent for the class of bounded real-valued functions (both are equivalent to the Lebesgue integral in this case). As a result of this fact, if a real-valued function f is Henstock-Kurzweil integrable on $[a,b]$, then $[a,b]$ can be written as the union of an increasing sequence $\{F_n\}$ of closed sets on each of which f is McShane integrable (cf. Theorem 9.18 of [6]). A more involved proof shows that the sequence $\{F_n\}$ can be chosen so that the sequence $\{f\chi_{F_n}\}$ converges to f in the Alexiewicz norm [10].

The Henstock-Kurzweil and McShane integrals admit obvious extensions to the case of vector-valued functions [5], [4], [2], [3]. The main focus in this paper will be the difference between the Henstock-Kurzweil and McShane integrals of vector-valued functions from the descriptive point of view. We first consider three notions of absolute continuity of a vector-valued function on a set (AC, AC_*, AC_δ) and clarify some of the relationships between these function classes. We further obtain necessary and sufficient conditions to distinguish McShane integrable functions among the Henstock-Kurzweil integrable functions in terms of the three function classes. The Pettis integral (see [16] for the general theory of this integral) is descriptively the widest of the AC integrals of vector-valued functions [13]. Nevertheless, the Pettis and McShane integrals are equivalent for functions with values in some classes of Banach spaces; in particular, these integrals coincide in subspaces of Hilbert generated spaces [1]. In the last section this fact is employed to show that if a Henstock-Kurzweil integrable function f defined on $[a,b]$ assumes values in a Banach space that contains no isomorphic copy of c_0 and is a subspace of a Hilbert generated space, then $[a,b]$ can be written as the union of an increasing sequence $\{F_n\}$ of closed sets on each of which f is McShane integrable and the sequence $\{f\chi_{F_n}\}$ converges to f in the Alexiewicz norm.

2. Notation and terminology

First of all we set our notation and recall basic definitions. Throughout this paper $[a,b]$ will denote a fixed nondegenerate interval of the real line and I its closed nondegenerate subinterval. X denotes a real Banach space and X^* its dual. Given $F\colon [a,b] \to X$, $\Delta F(I)$ denotes the *increment* of F on I. Let E be a set and let t be a point, then $\operatorname{dist}(t,E)$ is the *distance* from t to E, $\operatorname{int} E$, $\overline{E}$, ∂E, χ_E, and $\lambda(E)$ will denote the *interior* of E, the *closure* of E, the *boundary* of E, the *characteristic function* of E, and the *Lebesgue measure* of E, respectively. For ease of notation, we will drop the adjective Lebesgue and refer to measurable sets. Finally, a *gauge* on E is any positive function defined on E.

In what follows we will need some standard notions related to the integration and differentiation of vector-valued functions. They are summarized below for the reader's convenience.

Definition 1. Let $F\colon [a,b] \to X$.

(a) Let $E \subset [a,b]$. A function $f\colon E \to X$ is a *scalar derivative* of F on E if for each x^* in X^* the function x^*F is differentiable almost everywhere on E and $(x^*F)' = x^*f$ almost everywhere on E (the exceptional set may vary with x^*).

(b) A function $f\colon [a,b] \to X$ is *Pettis integrable* on $[a,b]$ if for each measurable set E in $[a,b]$ there is a vector $\nu_f(E) \in X$ such that the Lebesgue integral $\int_E x^* \circ f \,\mathrm{d}\lambda$ exists and is equal to $x^*(\nu_f(E))$ for all x^* in X^*.

(c) A *partial McShane partition* of $[a,b]$ is a finite collection $\mathscr{P} = \{(I_k, t_k)\}_{k=1}^K$ such that $\{I_k\}_{k=1}^K$ is a family of mutually non-overlapping intervals and $t_k \in [a,b]$ for each k. $\mathscr{P}$ is *subordinate* to a gauge δ on $[a,b]$ if $I_k \subset (t_k - \delta(t_k), t_k + \delta(t_k))$ for each k. $\mathscr{P}$ is said to be a *McShane partition* of $[a,b]$ provided $\{I_k\}_{k=1}^K$ covers $[a,b]$.

We say that a function $f\colon [a,b] \to X$ is *McShane integrable* on $[a,b]$, with *McShane integral* $w \in X$, if for each positive number ε there is a gauge δ on $[a,b]$ such that

$$\left\| \sum_{k=1}^K f(t_k)\lambda(I_k) - w \right\| < \varepsilon \tag{1}$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a McShane partition of $[a,b]$ subordinate to δ.

(d) A *partial Henstock partition* (*Henstock partition*) of $[a,b]$ is a partial McShane partition (McShane partition) $\{(I_k, t_k)\}_{k=1}^K$ of $[a,b]$ with $t_k \in I_k$ for each k. A function $f\colon [a,b] \to X$ is *Henstock-Kurzweil integrable* on $[a,b]$, with *Henstock-Kurzweil integral* $w \in X$, if for each positive number ε there is a gauge δ on $[a,b]$ such that (1) holds for each Henstock partition $\{(I_k, t_k)\}_{k=1}^K$ of $[a,b]$ subordinate to δ.

As usual, we say that a function f is Pettis (McShane, Henstock-Kurzweil) integrable on a set $E \subset [a,b]$ if the function $f\chi_E$ is Pettis (McShane, Henstock-Kurzweil) integrable on $[a,b]$ and $\int_E f = \int_a^b f\chi_E$. Standard arguments show that a McShane (Henstock-Kurzweil) integrable on $[a,b]$ function is McShane (Henstock-Kurzweil) integrable on any subinterval I of $[a,b]$. Moreover, a McShane integrable on $[a,b]$ function is McShane integrable on any measurable subset of $[a,b]$ (see, for example, Theorem 9 of [8]). If f is Pettis (McShane, Henstock-Kurzweil) integrable on $[a,b]$, then it will be convenient to use the phrase *'indefinite integral'* to mean the function $F(t) = \int_a^t f$. In this case, it is easy to verify that the function f is a scalar derivative of its indefinite integral on $[a,b]$ and $\int_I f = \Delta F(I)$ for any subinterval I of $[a,b]$. At last, $\|f\|_A = \sup\limits_{a<t\leqslant b} \left\|\int_a^t f\right\|$ is the *Alexiewicz norm* of the function f.

3. Classes of absolutely continuous vector-valued functions

We begin by introducing different notions of absolute continuity on a set. Let $F\colon [a,b] \to X$ and let E be a non-empty subset of $[a,b]$.

Definition 2. F is said to be AC (AC_*) on E if for each positive number ε there exists a positive number η such that

$$\left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| < \varepsilon \tag{2}$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^{K}$ with $\partial I_k \subset E$ ($\partial I_k \cap E \neq \emptyset$) and

$$\sum_{k=1}^{K} \lambda(I_k) < \eta. \tag{3}$$

The third type of absolute continuity related to partial Henstock partitions arises naturally in the context of the Henstock-Kurzweil integral.

Definition 3. F is said to be AC_δ on E if for each positive number ε there exist a positive number η and a gauge δ on $[a,b]$ such that (2) holds for each partial Henstock partition $\{(I_k,t_k)\}_{k=1}^{K}$ of $[a,b]$ with $t_k \in \partial I_k \cap E$ and (3) subordinate to δ.

Remark 1. In the case where E is equal to $[a,b]$ straightforward arguments can be given to show that all the three function classes AC, AC_*, and AC_δ coincide. However, it is far from obvious how these three types of absolute continuity compare on an arbitrary set.

Remark 2. Our function classes AC, AC_*, and AC_δ are different from the analogous function classes in which the norm is inside of the sum (2) (see for example [12]).

Further, we say that F is ACG (ACG_*, ACG_δ) on E if E can be written as a countable union of sets on each of which F is AC (AC_*, AC_δ). If these sets can be chosen *closed*, then F is said to be $[ACG]$ ($[ACG_*]$, $[ACG_\delta]$) on E.

Theorem 1. *Let $F\colon [a,b] \to X$ and let E be a non-empty closed subset of $[a,b]$. Suppose that $F|_E$ is continuous on E. If F is ACG (ACG_*) on E, then F is $[ACG]$ ($[ACG_*]$) on E.*

Proof. Let $E = \bigcup\limits_{k=1}^{\infty} E_k$ so that F is AC (AC_*) on E_k for each k. Since $\overline{E}_k \subset E$ and $F|_E$ is continuous on E, the function $F|_{\overline{E}_k}$ is continuous on $\overline{E}_k$ for each k. Now it follows from part (e) of Theorem 3.1 of [13] that F is AC (AC_*) on $\overline{E}_k$ for each k. $\square$

In the case where E is closed the next theorem relates the AC_* functions on E with the AC functions on E.

Theorem 2. *Let $F\colon [a,b] \to X$ be continuous on $[a,b]$. Suppose that E is a non-empty closed subset of $[a,b]$ and let $\{(a_k,b_k)\}_{k=1}^{\infty}$ be a sequence of intervals in $[a,b]$ contiguous to $E \cup \{a,b\}$. The following two statements are equivalent:*

(i) *F is AC_* on E.*

(ii) *F is AC on E and for each positive ε there is a positive integer N such that*

$$\Big\|\sum_{k\in\pi}\{F(d_k)-F(c_k)\}\Big\| < \varepsilon$$

whenever π is a finite subset of $\{N+1,\dots\}$ and $a_k \leqslant c_k < d_k \leqslant b_k$ for each k.

Proof. (i) $\Rightarrow$ (ii) Fix a positive number ε and let $\eta > 0$ correspond to ε in Definition 2. As $\sum_k (b_k - a_k) < \infty$, there is a positive integer N such that $\sum_{k>N}(b_k - a_k) < \eta$ and $a_k, b_k \in E$ for each $k > N$. Let π be a finite subset of $\{N+1,\dots\}$ with $a_k \leqslant c_k < d_k \leqslant b_k$ for each $k \in \pi$. Then

$$\begin{aligned}&\Big\|\sum_{k\in\pi}\{F(d_k)-F(c_k)\}\Big\|\\ &\quad= \Big\|\sum_{k\in\pi}\{F(b_k)-F(a_k)\} - \{F(c_k)-F(a_k)\} - \{F(b_k)-F(d_k)\}\Big\| < 3\varepsilon,\end{aligned}$$

which is what we desired. In passing we point out that this direction of the theorem is valid even if the function F is not continuous on $[a,b]$.

(ii) $\Rightarrow$ (i) Fix a positive number ε and let $\eta > 0$ correspond to ε in Definition 2 with $\|\Delta F(I)\| < \varepsilon/N$ whenever $\lambda(I) < \eta$. Let $\{I_k\}_{k=1}^{K}$ be a finite collection of mutually non-overlapping intervals with $\partial I_k \cap E \neq \emptyset$ and $\sum_{k=1}^{K}\lambda(I_k) < \eta$. By partitioning each interval if necessary, we may assume that for each k either $k \in \pi_1 = \{k\colon \partial I_k \subset E\}$ or $k \in \pi_2 = \{k\colon \operatorname{int} I_k \cap E = \emptyset \text{ and } \partial I_k \cap E = \{\tau_k\}\}$. Then we have, using (ii),

$$\begin{aligned}\Big\|\sum_{k=1}^{K}\Delta F(I_k)\Big\| &\leqslant \Big\|\sum_{k\in\pi_1}\Delta F(I_k)\Big\| + \Big\|\sum_{k\in\pi_2}\Delta F(I_k)\Big\|\\ &< \varepsilon + \Big\|\sum_{k\in\pi_2}\Delta F(I_k)\Big\| < \varepsilon + \frac{\varepsilon}{N}\cdot 2N + 2\varepsilon = 5\varepsilon.\end{aligned}$$

The proof is complete. □

Corollary 1. *Suppose that E is a non-empty closed subset of $[a,b]$ and let $\{(a_k,b_k)\}_{k=1}^{\infty}$ be a sequence of intervals in $[a,b]$ contiguous to $E\cup\{a,b\}$. Let $F\colon [a,b]\to X$ be AC_* on E and let $\{k_n\}_{n=1}^{\infty}$ be an increasing sequence of positive integers. Then the series $\sum_n \{F(d_{k_n})-F(c_{k_n})\}$ is unconditionally convergent whenever $[c_{k_n},d_{k_n}]\subset[a_{k_n},b_{k_n}]$ for each n.*

P r o o f. The unconditional convergence of the series $\sum_n \{F(d_{k_n})-F(c_{k_n})\}$ results from Proposition 1.c.1 of [11]. □

4. Indefinite Henstock-Kurzweil integrals

The properties of the indefinite Henstock-Kurzweil integral will be considered in this section. We begin with the Saks-Henstock Lemma. As an easy consequence, this lemma guarantees the continuity of the indefinite Henstock-Kurzweil integral.

Lemma 1 (Saks-Henstock Lemma)**.** *Let $f\colon [a,b]\to X$ be Henstock-Kurzweil (McShane) integrable on $[a,b]$, let $F\colon [a,b]\to X$ be the indefinite Henstock-Kurzweil (McShane) integral of f, and let $\varepsilon>0$. Suppose that a gauge δ on $[a,b]$ corresponds to ε in the definition of the Henstock-Kurzweil (McShane) integral of f on $[a,b]$. If $\{(I_k,t_k)\}_{k=1}^{K}$ is a partial Henstock (McShane) partition of $[a,b]$ subordinate to δ, then*

$$\left\|\sum_{k=1}^{K} f(t_k)\lambda(I_k)-\sum_{k=1}^{K}\Delta F(I_k)\right\|\leqslant\varepsilon.$$

P r o o f. We omit the proof patterned after that of Lemma 9.11 of [6]. □

For the reader's convenience we give a proof of the Uniform Henstock Lemma established in [10], Lemma 3. The lemma is stated differently in [10], but we will need it in a weaker form concerning vector-valued functions defined on a compact interval of the real line (cf. Lemma 1 of [15]).

Lemma 2 (Uniform Henstock Lemma)**.** *Let $f\colon [a,b]\to X$ be Henstock-Kurzweil integrable on $[a,b]$ and let $\varepsilon>0$. Suppose that a gauge δ on $[a,b]$ corresponds to ε in the definition of the Henstock-Kurzweil integral of f on $[a,b]$. If $\{(I_k,t_k)\}_{k=1}^{K}$ is a partial Henstock partition of $[a,b]$ subordinate to δ, then*

$$\left\|\sum_{k=1}^{K} f(t_k)\chi_{I_k}-\sum_{k=1}^{K} f\chi_{I_k}\right\|_A\leqslant 2\varepsilon.$$

P r o o f. Fix a partial Henstock partition $\{(I_k, t_k)\}_{k=1}^K$ of $[a, b]$ with $\max I_k \leqslant \min I_{k+1}$ for each $k < K$. If $t \in \operatorname{int} I_K$, then we have, using the Saks-Henstock Lemma,

$$\left\|(\mathrm{HK})\int_a^t \left(\sum_{k=1}^K f(t_k)\chi_{I_k} - \sum_{k=1}^K f\chi_{I_k}\right)\right\| \leqslant \begin{cases} \varepsilon, & \text{if } t_K \leqslant t,\\ 2\varepsilon, & \text{if } t_K > t.\end{cases}$$

It follows that

$$\left\|(\mathrm{HK})\int_a^t \left(\sum_{k=1}^K f(t_k)\chi_{I_k} - \sum_{k=1}^K f\chi_{I_k}\right)\right\| \leqslant 2\varepsilon$$

for each $t \in (a, b]$ and the proof is complete. □

The next auxiliary result is an immediate consequence of the Uniform Henstock Lemma (cf. Lemma 4 of [10]).

Lemma 3. *Let $f\colon [a, b] \to X$ be Henstock-Kurzweil integrable on $[a, b]$, let f be Henstock-Kurzweil integrable on a measurable subset E of $[a, b]$, and let $\varepsilon > 0$. Then there is a gauge δ on $[a, b]$ such that*

$$\left\|\sum_{k=1}^K f\chi_{I_k\cap E} - \sum_{k=1}^K f\chi_{I_k}\right\|_A \leqslant \varepsilon$$

whenever $\{(I_k, t_k)\}_{k=1}^K$ is a partial Henstock partition of $[a, b]$ subordinate to δ with $t_k \in E$ for each k.

P r o o f. Let gauges δ_1 and δ_2 correspond to $\varepsilon/4$ in the definition of the Henstock-Kurzweil integral of f on $[a, b]$ and on E, respectively. Define a gauge δ on $[a, b]$ by $\delta = \min(\delta_1, \delta_2)$. Let $\{(I_k, t_k)\}_{k=1}^K$ be a partial Henstock partition of $[a, b]$ subordinate to δ with $t_k \in E$ for each k. By Lemma 2, we obtain

$$\begin{aligned}&\left\|\sum_{k=1}^K f\chi_{I_k\cap E} - \sum_{k=1}^K f\chi_{I_k}\right\|_A \\ &\qquad\leqslant \left\|\sum_{k=1}^K f\chi_E\chi_{I_k} - \sum_{k=1}^K f\chi_E(t_k)\chi_{I_k}\right\|_A + \left\|\sum_{k=1}^K f(t_k)\chi_{I_k} - \sum_{k=1}^K f\chi_{I_k}\right\|_A \leqslant \varepsilon.\end{aligned}$$

□

Theorem A provides the descriptive characterization of the Pettis integral [13].

Theorem A. *Let $f\colon [a,b] \to X$. Then f is Pettis integrable on $[a,b]$ if and only if there is a function $F\colon [a,b] \to X$ such that F is AC on $[a,b]$ and f is a scalar derivative of F on $[a,b]$.*

On the other hand, Fremlin's criterion, Theorem B, can be used to determine whether or not a given Henstock-Kurzweil integrable function is McShane integrable [2].

Theorem B. *Let $f\colon [a,b] \to X$. Then f is McShane integrable on $[a,b]$ if and only if f is both Henstock-Kurzweil and Pettis integrable on $[a,b]$.*

Combining Theorems A and B, we obtain the following descriptive version of Theorem C.

Theorem 3. *Let $f\colon [a,b] \to X$. Then f is McShane integrable on $[a,b]$ if and only if f is Henstock-Kurzweil integrable on $[a,b]$ and its indefinite Henstock-Kurzweil integral is AC on $[a,b]$.*

Proof. The necessity part of the theorem results from Lemma 6 of [8].

Suppose that f is Henstock-Kurzweil integrable on $[a,b]$ and let F be its indefinite Henstock-Kurzweil integral. Since F is AC on $[a,b]$, by Theorem A, f is Pettis integrable on $[a,b]$. Now Theorem B yields McShane integrability of f on $[a,b]$. □

Remark 3. Our Theorem 3 is similar in spirit to Fremlin's Corollary 9 of [2] that provides two other characterizations of McShane integrable functions (see also Theorem 14.55 of [12] for another modification of Fremlin's result).

Remark 4. Let $\{I_n\}_{n=1}^{\infty}$ be a fixed sequence of intervals in $[a,b]$ such that $b_n = \max I_n < \min I_{n+1}$ for each n, $\lim\limits_n b_n = b$. We write φ_n to represent the function

$$\frac{\chi_{I_{2n-1}}}{2\lambda(I_{2n-1})} - \frac{\chi_{I_{2n}}}{2\lambda(I_{2n})}.$$

Let $\{e_n\}_{n=1}^{\infty}$ denote the standard unit vector basis of c_0. Define a sequence $\{x_n\}$ in c_0 by

$$e_1, \frac{e_2}{2}, \frac{e_2}{2}, \frac{e_3}{3}, \frac{e_3}{3}, \frac{e_3}{3}, \frac{e_4}{4}, \dots$$

and a function $g\colon [a,b] \to c_0$ by $g = \sum\limits_n \varphi_n x_n$.

In Example 4.2 of [13], the following three properties of g were established.

(a) g is Dunford integrable on $[a,b]$.

(b) If $G(t) = (\mathrm{D}) \int_a^t g$ for each t in $[a,b]$, then G is continuous on $[a,b]$.

(c) G is not AC on $[a,b]$.

Since g is a bounded step function (a *step function* is a linear combination of characteristic functions of intervals) on $[a,t]$ for all t in (a,b), (b) shows that g is Henstock-Kurzweil integrable on $[a,b]$. By (c), g is *not* McShane integrable on $[a,b]$. Consequently, in the sufficiency part of Theorem 3, the AC property of the indefinite integral cannot be replaced with the weaker condition that the integrand is Dunford integrable.

Here is another remark related to the preceding.

Remark 5. A function $F\colon [a,b] \to X$ is said to be VB on $[a,b]$ if there is a positive number M such that

$$\left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| \leqslant M$$

for each finite collection of pairwise non-overlapping intervals $\{I_k\}_{k=1}^{K}$. Note that, by Lemma 3.1 of [13], G is a VB function on $[a,b]$. This means that a result of Lee Tuo-Yeong, Theorem 3.1 of [9], essentially stating that, in the sufficiency part of Theorem 3, the AC property can be replaced with the VB property is false. As a consequence the main result of [9], Theorem 3.3, fails to be proved. However, both these results are still valid in Banach spaces containing no isomorphic copy of c_0 [17]. For the sake of completeness, we include proofs of these two facts below (see Corollaries 2 and 6).

Corollary 2. *Suppose that X does not contain an isomorphic copy of c_0. If $f\colon [a,b] \to X$ is Henstock-Kurzweil integrable on $[a,b]$ and its indefinite integral is VB on $[a,b]$, then f is McShane integrable on $[a,b]$.*

P r o o f. The corollary results from Theorem 3 and a Banach-Zarecki type theorem, Theorem 4.2 of [13]. □

Next we seek to explore the relationships between various types of absolute continuity of the indefinite Henstock-Kurzweil integral and McShane integrability of the integrand on a set. The following two theorems provide a set of sufficient conditions for McShane integrability on a closed set.

Theorem 4. *Let $f\colon [a,b] \to X$ be Henstock-Kurzweil integrable on $[a,b]$ and let $F\colon [a,b] \to X$ be the indefinite Henstock-Kurzweil integral of f. Suppose that E is a non-empty closed subset of $[a,b]$. If f is Henstock-Kurzweil integrable on E and F is AC on E, then f is McShane integrable on E.*

Proof. Since F is AC on E and $f\chi_E$ is a scalar derivative of F on E, by Corollary 5.1 of [13], $f\chi_E$ is Pettis integrable on $[a,b]$. Now Theorem B yields McShane integrability of $f\chi_E$ on $[a,b]$. □

Theorem 5. *Let $f\colon [a,b]\to X$ be Henstock-Kurzweil integrable on $[a,b]$ and let $F\colon [a,b]\to X$ be the indefinite Henstock-Kurzweil integral of f. Suppose that E is a non-empty closed subset of $[a,b]$. If F is AC_* on E, then f is McShane integrable on E.*

Proof. We will first prove that f is Henstock-Kurzweil integrable on E. Fix a positive number ε and let a gauge δ_0 on $[a,b]$ correspond to ε in the definition of the Henstock-Kurzweil integral of f on $[a,b]$. Suppose that F is the indefinite Henstock-Kurzweil integral of f and $\{(a_k,b_k)\}_{k=1}^{\infty}$ is a sequence of intervals in $[a,b]$ contiguous to $E\cup\{a,b\}$. Fix a positive integer N that corresponds to ε in (ii) of Theorem 2 with $a_k,b_k\in E$ for each $k>N$. We make note of the fact that, by Corollary 1, the series $\sum\limits_{k=1}^{\infty}\{F(b_k)-F(a_k)\}$ is unconditionally convergent to a vector, w say, in X. Further, as F is continuous on $[a,b]$, there is a positive number η such that $\|\Delta F(I)\|<\varepsilon/N$ for any subinterval I of $[a,b]$ with $\lambda(I)<\eta$. We will show that (HK) $\int_E f = F(b)-w$. Let $E_0=\bigcup\limits_{k=1}^{N}\{a_k,b_k\}$. Define a gauge δ on $[a,b]$ by

$$\delta(t)=\begin{cases}\min(\delta_0(t),\operatorname{dist}(t,\{a_k,b_k\}), & \text{if } t\in(a_k,b_k) \text{ for some } k,\\ \min(\delta_0(t),\operatorname{dist}(t,E_0)), & \text{if } t\in E\setminus E_0,\\ \min(\delta_0(t),\eta), & \text{if } t\in E_0.\end{cases}$$

Let $\{(I_k,t_k)\}_{k=1}^{K}$ be a Henstock partition of $[a,b]$ subordinate to δ. It follows that

$$\begin{aligned}&\Bigl\|\sum_{k=1}^{K} f\chi_E(t_k)\lambda(I_k)-F(b)+w\Bigr\|\\ &\quad\leqslant\Bigl\|\sum_{k\colon t_k\in E} f(t_k)\lambda(I_k)-\sum_{k\colon t_k\in E}\Delta F(I_k)\Bigr\|+\Bigl\|\sum_{k\colon t_k\in E}\Delta F(I_k)-F(b)+w\}\Bigr\|\\ &\quad\leqslant\varepsilon+\Bigl\|w-\sum_{k\colon t_k\notin E}\Delta F(I_k)\Bigr\|\\ &\quad\leqslant\varepsilon+\Bigl\|\sum_{k>N}\{F(b_k)-F(a_k)\}\Bigr\|+\Bigl\|\sum_{k=1}^{N}\{F(b_k)-F(a_k)\}-\sum_{k\colon t_k\notin E}\Delta F(I_k)\Bigr\|\\ &\quad<\varepsilon+\varepsilon+\frac{\varepsilon}{N}\cdot 2N+\varepsilon=5\varepsilon.\end{aligned}$$

This means that the function $f\chi_E$ is Henstock-Kurzweil integrable on $[a,b]$. Now Theorem 4 applies to f. The proof is complete. □

In addition to the above two sufficient conditions, we have the following necessary condition for McShane integrability on a measurable set.

Theorem 6. *Let $f\colon [a,b] \to X$ be Henstock-Kurzweil integrable on $[a,b]$ and let $F\colon [a,b] \to X$ be the indefinite Henstock-Kurzweil integral of f. Suppose that E is a non-empty measurable subset of $[a,b]$. If f is McShane integrable on E, then F is AC_δ on E.*

Proof. By Theorem 9 of [8], f is McShane integrable on each measurable subset of E. Given a measurable subset A of $[a,b]$, let

$$\Phi(A) = (\mathrm{M})\int_{A\cap E} f.$$

Fix a positive number ε and let $\eta > 0$ be such that $\|\Phi(A)\| < \varepsilon$ whenever $\lambda(A) < \eta$ (see Theorem 11 of [8]). Let gauges δ_1 and δ_2 on $[a,b]$ correspond to ε in the definitions of the Henstock-Kurzweil integral of f on $[a,b]$ and on E, respectively. Define a gauge δ on $[a,b]$ by $\delta = \min(\delta_1,\delta_2)$. Let $\{(I_k,t_k)\}_{k=1}^K$ be a partial Henstock partition of $[a,b]$ with $t_k \in \partial I_k \cap E$ and (3) subordinate to δ. Then it follows that

$$\begin{aligned}\Big\|\sum_{k=1}^K \Delta F(I_k)\Big\| &\leqslant \Big\|\sum_{k=1}^K \Phi(I_k)\Big\| + \Big\|\sum_{k=1}^K \Delta F(I_k) - \sum_{k=1}^K \Phi(I_k)\Big\| \\ &\leqslant \Big\|\Phi\Big(\bigcup_{k=1}^K I_k\Big)\Big\| + \Big\|\sum_{k=1}^K \Delta F(I_k) - \sum_{k=1}^K f(t_k)\lambda(I_k)\Big\| \\ &\quad + \Big\|\sum_{k=1}^K f\chi_E(t_k)\lambda(I_k) - \sum_{k=1}^K (\mathrm{HK})\int_{I_k} f\chi_E\Big\| < \varepsilon+\varepsilon+\varepsilon = 3\varepsilon.\end{aligned}$$

Thus F is AC_δ on E. □

In the case in which an integrand f is bounded on a set E, it can easily be seen that the indefinite Henstock-Kurzweil integral of f is necessarily AC_δ on E.

Theorem 7. *Let $f\colon [a,b] \to X$ be Henstock-Kurzweil integrable on $[a,b]$ and let $F\colon [a,b] \to X$ be the indefinite Henstock-Kurzweil integral of f. Suppose that E is a non-empty subset of $[a,b]$. If f is bounded on E, then F is AC_δ on E.*

Proof. Let $M > 0$ be such that $\|f(t)\| \leqslant M$ for each t in E. Fix a positive number ε and let a gauge δ on $[a,b]$ correspond to ε in the definition of the Henstock-Kurzweil integral of f on $[a,b]$. Let $\eta = \varepsilon/M$. Let $\{(I_k,t_k)\}_{k=1}^K$ be a partial Henstock partition of $[a,b]$ with $t_k \in \partial I_k \cap E$ and (3) subordinate to δ. Then

$$\Big\|\sum_{k=1}^{K} \Delta F(I_k)\Big\| \leqslant \Big\|\sum_{k=1}^{K} f(t_k)\lambda(I_k)\Big\| + \Big\|\sum_{k=1}^{K} f(t_k)\lambda(I_k) - \sum_{k=1}^{K} \Delta F(I_k)\Big\| < M\eta + \varepsilon = 2\varepsilon.$$

□

Corollary 3. *If $f\colon [a,b] \to X$ is both Henstock-Kurzweil integrable and bounded on $[a,b]$, then f is McShane integrable on $[a,b]$.*

Proof. Since the indefinite Henstock-Kurzweil integral of f is AC_δ on $[a,b]$, it is AC on $[a,b]$. Now Theorem 3 applies to f. □

Corollary 4. *If $f\colon [a,b] \to X$ is Henstock-Kurzweil integrable on $[a,b]$, then the indefinite Henstock-Kurzweil integral of f is ACG_δ on $[a,b]$.*

Proof. The corollary results from the fact that

$$[a,b] = \bigcup_{n=1}^{\infty} \{t \in [a,b]\colon\ n-1 \leqslant \|f(t)\| < n\}.$$

□

We conclude this section with a more involved approximation property of the indefinite Henstock-Kurzweil integral (cf. Theorem 6 of [10]).

Theorem 8. *Let $f\colon [a,b] \to X$ be Henstock-Kurzweil integrable on $[a,b]$. Suppose that $[a,b]$ can be written as a countable union of closed sets on each of which f is McShane integrable. Then $[a,b]$ is the union of an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets such that f is McShane integrable on F_n and $\|f\chi_{F_n} - f\|_A < n^{-1}$ for each n.*

Proof. With no loss of generality, we may assume that $[a,b]$ is the union of an *increasing* sequence $\{E_i\}_{i=1}^{\infty}$ of closed sets such that f is McShane integrable on E_i for each i.

Fix a positive number ε and a positive integer n. We will prove that there exists a closed set F such that $E_n \subset F \subset E_{i(n,\varepsilon)}$ for some $i(n,\varepsilon) > n$ and $\|f\chi_F - f\|_A < \varepsilon$. For each i, let a gauge δ_i on $[a,b]$ correspond to E_i and $\varepsilon/2^i$ in Lemma 3. Define a gauge δ on $[a,b]$ by

$$\delta(t) = \begin{cases} \delta_n(t), & \text{if } t \in E_n,\\ \min(\delta_i(t), \operatorname{dist}(t, E_{i-1})), & \text{if } t \in E_i \setminus E_{i-1} \text{ for some } i > n. \end{cases}$$

Fix a Henstock partition $\{(I_k, t_k)\}_{k=1}^{K}$ of $[a,b]$ subordinate to δ. Let $D_n = E_n$, $D_i = E_i \setminus E_{i-1}$ for each $i > n$, and

$$F = \bigcup_{i=n}^{\infty} \bigcup_{k\colon t_k \in D_i} I_k \cap E_i.$$

It is easy to check that F is closed, $E_n \subset F$, and $F \subset E_{i(n,\varepsilon)}$ for some $i(n,\varepsilon) > n$. By Lemma 3, we obtain

$$\begin{aligned}\|f\chi_F - f\|_A &\leqslant \sum_{i=n}^{\infty} \Big\| \sum_{k\colon t_k \in D_i} (f\chi_{F\cap I_k} - f\chi_{I_k}) \Big\|_A \\ &= \sum_{i=n}^{\infty} \Big\| \sum_{k\colon t_k \in D_i} (f\chi_{E_i\cap I_k} - f\chi_{I_k}) \Big\|_A < \sum_{i=n}^{\infty} \frac{\varepsilon}{2^i} \leqslant \varepsilon.\end{aligned}$$

Now define inductively a sequence $\{F_n\}_{n=1}^{\infty}$ of sets and a sequence $\{i_n\}_{n=1}^{\infty}$ of positive integers as follows. Let F_1 be a closed set such that $E_1 \subset F_1 \subset E_{i_1}$ for some $i_1 > 1$ and $\|f\chi_{F_1} - f\|_A < 1$. For each $n > 1$, let F_n be a closed set such that $E_{i_{n-1}} \subset F_n \subset E_{i_n}$ for some $i_n > i_{n-1}$ and $\|f\chi_{F_n} - f\|_A < n^{-1}$. Evidently, the sequence $\{F_n\}_{n=1}^{\infty}$ has all the desired properties. The proof is complete. □

5. Henstock-Kurzweil integration in some classes of Banach spaces

In this section we will refine the relationships between the Henstock-Kurzweil integral and the McShane integral in the situation in which some restrictions are placed on the Banach space involved. It should be noted that we approach our results in this section by means of the Pettis integral. For this reason, it is unclear whether results similar to ours are valid in a more general context.

We begin by showing that in Banach spaces that do not contain an isomorphic copy of c_0 an indefinite Henstock-Kurzweil integral is necessarily $[ACG]$.

Theorem 9. *Suppose that X does not contain an isomorphic copy of c_0. Let $F\colon [a,b] \to X$ be an indefinite Henstock-Kurzweil integral. Then F is $[ACG]$ on $[a,b]$.*

P r o o f. Note that F has a scalar derivative on $[a,b]$. Since the function x^*F is ACG_* on $[a,b]$ for each x^* in X^*, by a Banach-Zarecki type theorem, Corollary 4.4 of [13], the function F is $[ACG]$ on $[a,b]$. □

Corollary 5. *Suppose that X does not contain an isomorphic copy of c_0. Let $f\colon [a,b] \to X$ be Henstock-Kurzweil integrable on $[a,b]$. Then $[a,b]$ can be written as a countable union of closed sets on each of which f is Pettis integrable.*

Proof. Let $F\colon [a,b] \to X$ be the indefinite Henstock-Kurzweil integral of f on $[a,b]$. By the preceding theorem, $[a,b]$ is the union of $\{E_k\}_{k=1}^{\infty}$ so that E_k is closed and F is AC on E_k for each k. As f is a scalar derivative of F on E_k, Corollary 5.1 of [13] yields Pettis integrability of f on E_k. □

Corollary 6. *Suppose that X does not contain an isomorphic copy of c_0. If $f\colon [a,b] \to X$ is Henstock-Kurzweil integrable on $[a,b]$, then f is McShane integrable on some nondegenerate subinterval of $[a,b]$.*

Proof. Let $F\colon [a,b] \to X$ be the indefinite Henstock-Kurzweil integral of f. Since F is both continuous and ACG on $[a,b]$, by part (c) of Theorem 3.2 of [13], there is a nondegenerate interval $[c,d] \subset [a,b]$ such that F is AC on $[c,d]$. Note that

$$(\mathrm{HK})\int_c^t f = F(t) - F(c)$$

for each t in $[c,d]$. Now Theorem 3 applies to f on the interval $[c,d]$. The proof is complete. □

A Banach space X is said to be *Hilbert generated* if there exist a Hilbert space H and a bounded linear operator $T\colon H \to X$ such that $T(H)$ is dense in X. In particular, Hilbert generated spaces form a subclass of *weakly compactly generated* spaces. Some facts about Hilbert generated spaces are gathered in [7, § 6.3]. The importance of Hilbert generated spaces for the integration of vector-valued functions stems from the following theorem [1].

Theorem C. *Suppose that X is a subspace of a Hilbert generated space. Let $f\colon [a,b] \to X$ be Pettis integrable on $[a,b]$. Then f is McShane integrable on $[a,b]$.*

As an illustration, the McShane integral and the Pettis integral coincide for functions with values in separable spaces, $c_0(\Gamma)$, super-reflexive spaces (for example, $L^p(\mu)$, where $1 < p < \infty$), $L^1(\mu)$, where μ is a finite measure (see Corollary 3.8 of [1]).

Theorem 10. *Suppose that X does not contain an isomorphic copy of c_0 and is a subspace of a Hilbert generated space. Let $f\colon [a,b] \to X$ be Henstock-Kurzweil integrable on $[a,b]$. Then $[a,b]$ is the union of an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets such that f is McShane integrable on F_n and $\|f\chi_{F_n} - f\|_A < n^{-1}$ for each n.*

P r o o f. By Corollary 5 and Theorem C, $[a, b]$ is the union of a sequence of closed sets on each of which f is McShane integrable. Now Theorem 8 applies to f. The proof is complete. □

Remark 6. For example, X satisfies the assumptions of Theorem 10 if X is separable and does not contain an isomorphic copy of c_0, or if X is super-reflexive, or if X equals $L^1(\mu)$, where μ is a finite measure.

Remark 7. Theorem 10 provides a partial solution to Problem 3.4 of [9].

Corollary 7. *Suppose that X does not contain an isomorphic copy of c_0 and is a subspace of a Hilbert generated space. Let $F\colon [a, b] \to X$ be an indefinite Henstock-Kurzweil integral. Then F is $[ACG_\delta]$ on $[a, b]$.*

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