Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 4, 941–974

Persistent URL: http://dml.cz/dmlcz/141799

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UNCERTAINTY PRINCIPLES FOR THE WEINSTEIN TRANSFORM

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(Received June 29, 2010)

Dedicated to Khalifa Trimèche for his 65 birthday

Abstract. The Weinstein transform satisfies some uncertainty principles similar to the Euclidean Fourier transform. A generalization and a variant of Cowling-Price theorem, Miyachi's theorem, Beurling's theorem, and Donoho-Stark's uncertainty principle are obtained for the Weinstein transform.

Keywords: Weinstein transform, Hardy's type theorem, Cowling-Price's theorem, Beurling's theorem, Miyachi's theorem, Donoho-Stark's uncertainty principle

MSC 2010: 35B53, 43A32, 44A05, 44A20

1. Introduction

We consider the Weinstein operator defined on $\mathbb{R}^d \times]0,+\infty[$ by:

$$\Delta_\beta := \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\beta+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + \mathscr{L}_\beta, \quad \beta > -\frac{1}{2}$$

where Δ_d is the Laplacian for the d-first variables and $\mathscr{L}_\beta$ the Bessel operator for the last variable, given by

$$\mathscr{L}_\beta = \frac{\partial^2}{\partial x_{d+1}^2} + \frac{2\beta+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}}, \quad \beta > -\frac{1}{2}.$$

For $d > 2$, the operator Δ_β is the Laplace-Beltrami operator on the Riemanian space $\mathbb{R}^d \times]0,+\infty[$ equipped with the metric (cf. [1])

$$\mathrm{d}s^2 = x_{d+1}^{2(2\beta+1)/(d-1)} \sum_{i=1}^{d+1} \mathrm{d}x_i^2.$$

The Weinstein operator Δ_β has several applications in pure and applied Mathematics especially in Fluid Mechanics (cf. [5]).

The harmonic analysis associated with the Weinstein operator is studied by Ben Nahia and Ben Salem (cf. [1], [2]). In particular the authors have introduced and studied the generalized Fourier transform associated with the Weinstein operator. This transform is called the Weinstein transform. In this work we are interested in the principles of uncertainty associated with the transformation of Weinstein.

There are many theorems known which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. Here the concept of the smallness has taken different interpretations in different contexts. Hardy [10], Morgan [18], Cowling and Price [7], Beurling [4], Miyachi [17] and [8] for example interpreted the smallness as sharp pointwise estimates or integrable decay of functions. Slepian and Pollak [22], Slepian [23], Benedicks [3] and Donoho and Stark [9] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms.

Hardy's theorem [10] for the usual Fourier transform $\mathscr{F}$ on $\mathbb{R}$ asserts that f and its Fourier transform $\hat{f} = \mathscr{F}(f)$ can not both be very small. More precisely, let a and b be positive constants and assume that f is a measurable function on $\mathbb{R}$ such that

$$|f(x)| \leqslant C\mathrm{e}^{-ax^2} \text{ a.e.} \quad \text{and} \quad |\hat{f}(y)| \leqslant C\mathrm{e}^{-by^2}$$

for some positive constant C. Then $f = 0$ a.e. if $ab > \frac{1}{4}$, f is a constant multiple of e^{-ax^2} if $ab = \frac{1}{4}$, and there are infinitely many nonzero functions satisfying the assumptions if $ab < \frac{1}{4}$. Considerable attention has been devoted to discovering generalizations to new contexts for the Hardy's theorem. In particular, Cowling and Price [7] have studied an L^p version of Hardy's theorem which states that for p, q in $[1, +\infty]$, at least one of them is finite, if $\|\mathrm{e}^{ax^2} f\|_{L^p(\mathbb{R})} < +\infty$ and $\|\mathrm{e}^{by^2} \hat{f}\|_{L^q(\mathbb{R})} < +\infty$, then $f = 0$ a.e. if $ab \geqslant \frac{1}{4}$. Another generalization of Hardy's theorem is given by Miyachi [17] where it is proved that, if f is a measurable function on $\mathbb{R}$ such that

$$\mathrm{e}^{ax^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \log^+ \frac{|\hat{f}(\xi)\mathrm{e}^{\frac{1}{4}a^{-1}\xi^2}|}{\lambda} \, \mathrm{d}\xi < \infty$$

for some positive constants a and λ, then f is a constant multiple of e^{-ax^2}. Beurling's theorem for the classical Fourier transform on $\mathbb{R}$, which was recovered by Hörmander [11], says that for any non trivial function f in $L^2(\mathbb{R})$, the product $f(x)\mathscr{F}(f)(y)$

is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{|x||y|}\,dx\,dy$. A far reaching generalization of this result has been recently proved by Bonami, Demange and Jaming [6]. They proved that a square-integrable function f on $\mathbb{R}^d$ satisfying for an integer N,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)||\mathscr{F}(f)(y)|}{(1+\|x\|+\|y\|)^N}\, e^{\|x\|\|y\|}\,dx\,dy < +\infty,$$

has the form $f(x) = P(x)e^{-\beta\|x\|^2}$ where P is a polynomial of degree strictly lower than $\frac{1}{2}(N-d)$ and β is a positive constant.

As a generalization of these Euclidean uncertainty principles for $\mathscr{F}$, in this paper we want to prove Hardy's theorem, Cowling-Price's theorem, Beurling's theorem, Miyachi's theorem, and Donoho-Stark's uncertainty principles for the Weinstein transform $\mathscr{F}_W$.

The structure of this paper is the following. In § 2 we recall some results associated with the Weinstein operator which we need in the sequel. § 3 is devoted to generalized Cowling-Price's theorem for $\mathscr{F}_W$. In § 4 and § 5 we give variants of the theorem. In § 6 we generalize Miyachi's theorem and in § 7 Beurling's theorem for $\mathscr{F}_W$. § 8 is devoted to Donoho-Stark's uncertainty principle for $\mathscr{F}_W$. Finally in the last section we give some applications.

Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence.

2. Preliminaries

In order to set up basic and standard notation we briefly overview the Weinstein operator and related harmonic analysis. Main references are [1], [2].

2.1. Harmonic analysis associated with the Weinstein operator

In this subsection we collect some notation and results on the Weinstein kernel, the Weinstein intertwining operator and its dual, the Weinstein transform, and the Weinstein convolution.

In the following we denote by

- $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times [0,+\infty[$.
- $x = (x_1,\dots,x_d,x_{d+1}) = (x', x_{d+1}) \in \mathbb{R}^{d+1}_+$.
- $S^d_+ = \{x \in \mathbb{R}^{d+1}_+ : \|x\| = 1\}$.
- $C_*(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $C^p_*(\mathbb{R}^{d+1})$ the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.

- $\mathscr{E}_*(\mathbb{R}^{d+1})$ the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathscr{S}_*(\mathbb{R}^{d+1})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $D_*(\mathbb{R}^{d+1})$ the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $\mathscr{E}'_*(\mathbb{R}^{d+1})$ the space of distributions with compact support on $\mathbb{R}^{d+1}$, even with respect to the last variable. It is the topological dual of $\mathscr{E}_*(\mathbb{R}^{d+1})$.
- $\mathscr{S}'_*(\mathbb{R}^{d+1})$ the space of temperate distributions on $\mathbb{R}^{d+1}$, even with respect to the last variable. It is the topological dual of $\mathscr{S}_*(\mathbb{R}^{d+1})$.
- $\mathscr{P}_*^{d+1}$ the set of polynomials on $\mathbb{R}^{d+1}$ even with respect to the last variable.
- $\mathscr{P}_{*,m}^{d+1}$ the set of homogeneous polynomials on $\mathbb{R}^{d+1}$ of degree m, even with respect to the last variable.

We consider the Weinstein operator Δ_β defined by

$$\forall\, x = (x', x_{d+1}) \in \mathbb{R}^d \times]0,+\infty[, \qquad (2.1)$$
$$\Delta_\beta f(x) = \Delta_{x'} f(x', x_{d+1}) + \mathscr{L}_{\beta,x_{d+1}} f(x', x_{d+1}), \quad f \in C_*^2(\mathbb{R}^{d+1}),$$

where $\Delta_{x'}$ is the Laplace operator on $\mathbb{R}^d$, and $\mathscr{L}_{\beta,x_{d+1}}$ the Bessel operator on $]0,+\infty[$ given by

$$\mathscr{L}_{\beta,x_{d+1}} := \frac{\mathrm{d}^2}{\mathrm{d}x_{d+1}^2} + \frac{2\beta+1}{x_{d+1}}\frac{\mathrm{d}}{\mathrm{d}x_{d+1}}, \quad \beta > -\frac{1}{2}. \qquad (2.2)$$

The Weinstein kernel Λ is given by

$$\Lambda(x,z) := \mathrm{e}^{\mathrm{i}\langle x', z'\rangle} j_\beta(x_{d+1} z_{d+1}), \quad \text{for all } (x,z) \in \mathbb{R}^{d+1} \times \mathbb{C}^{d+1}, \qquad (2.3)$$

where $j_\beta(x_{d+1}z_{d+1})$ is the normalized Bessel function. The Weinstein kernel satisfies the following properties:

i) For all $z, t \in \mathbb{C}^{d+1}$, we have

$$\Lambda(z,t) = \Lambda(t,z); \quad \Lambda(z,0) = 1 \quad \text{and} \quad \Lambda(\lambda z, t) = \Lambda(z, \lambda t), \quad \text{for all } \lambda \in \mathbb{C}. \qquad (2.4)$$

ii) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}^{d+1}$ and $z \in \mathbb{C}^{d+1}$, we have

$$|D_z^\nu \Lambda(x,z)| \leqslant \|x\|^{|\nu|} \exp(\|x\|\|\mathrm{Im}\, z\|), \qquad (2.5)$$

where $D_z^\nu = \partial^\nu/(\partial z_1^{\nu_1} \ldots \partial z_{d+1}^{\nu_{d+1}})$ and $|\nu| = \nu_1 + \ldots + \nu_{d+1}$. In particular

$$|\Lambda(x,y)| \leqslant 1, \quad \text{for all } x, y \in \mathbb{R}^{d+1}. \qquad (2.6)$$

The Weinstein intertwining operator is the operator $\mathscr{R}_\beta$ defined on $C_*(\mathbb{R}^{d+1})$ by

$$\mathscr{R}_\beta f(x', x_{d+1}) = \begin{cases} \dfrac{2\Gamma(\beta+1)}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})}\, x_{d+1}^{-2\beta} \displaystyle\int_0^{x_{d+1}} (x_{d+1}^2 - t^2)^{\beta-\frac{1}{2}} f(x', t)\,\mathrm{d}t, & x_{d+1} > 0, \\ f(x', 0), & x_{d+1} = 0. \end{cases}$$

$\mathscr{R}_\beta$ is a topological isomorphism from $\mathscr{E}_*(\mathbb{R}^{d+1})$ onto itself satisfying the following transmutation relation

$$\Delta_\beta(\mathscr{R}_\beta f) = \mathscr{R}_\beta(\Delta_{d+1} f), \quad \text{for all } f \in \mathscr{E}_*(\mathbb{R}^{d+1}), \tag{2.7}$$

where $\Delta_{d+1} = \sum\limits_{j=1}^{d+1} \partial_j^2$ is the Laplacian on $\mathbb{R}^{d+1}$.

We put

$$b_j(r) = \frac{r^{2j}}{d_j(\beta)}, \quad \text{for all } r \geqslant 0 \tag{2.8}$$

with

$$d_j(\beta) = \frac{2^{2j} j!\,\Gamma(\beta+j+1)}{\Gamma(\beta+1)}. \tag{2.9}$$

Proposition 1 ([2]). *Let f be in $\mathscr{E}_*(\mathbb{R}^{d+1})$. Suppose that for all compact K of $\mathbb{R}^{d+1}$ there is $C > 0$ such that:*

$$\sup_{x\in K} |D_{x'}^{\alpha} \mathscr{L}_{\beta, x_{d+1}}^{j} f(x)| \leqslant C^{|\alpha|+2j} \alpha!(2j)!,$$

where D^α is the operator $D^\alpha = \partial_1^{\alpha_1} \circ \partial_2^{\alpha_2} \circ \ldots \circ \partial_d^{\alpha_d}$, with ∂_i, $i = 1, 2, \ldots, d$, the partial derivatives operators. Then

$$\forall\, x \in \mathbb{R}^{d+1}, \quad f(x) = \sum_{n=0}^{\infty} \sum_{|\nu|=|\alpha|+2j=n} m_\nu(x) D_{x'}^{\alpha} \mathscr{L}_{\beta, x_{d+1}}^{j} f(0), \tag{2.10}$$

where m_ν is the moment function defined by

$$\forall\, x \in \mathbb{R}^{d+1}, \quad m_\nu(x) := \mathscr{R}_\beta\Big(\frac{x^\nu}{\nu!}\Big) = b_j(x_{d+1}) \frac{(x')^\alpha}{\alpha!}, \quad \nu = (\alpha, 2j). \tag{2.11}$$

The dual of the Weinstein intertwining operator $\mathscr{R}_\beta$ is the operator ${}^t\mathscr{R}_\beta$ defined on $D_*(\mathbb{R}^{d+1})$ by

$${}^t\mathscr{R}_{k,\beta}(f)(y) = \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})} \int_{y_{d+1}}^{\infty} (s^2 - y_{d+1}^2)^{\beta-\frac{1}{2}} f(y', s) s\,\mathrm{d}s. \tag{2.12}$$

${}^t\mathscr{R}_\beta$ is a topological isomorphism from $\mathscr{S}_*(\mathbb{R}^{d+1})$ onto itself satisfying the following transmutation relation

$$ {}^t\mathscr{R}_\beta(\Delta_\beta f) = \Delta_{d+1}({}^t\mathscr{R}_\beta f), \quad \text{for all } f \in \mathscr{S}_*(\mathbb{R}^{d+1}). \tag{2.13}$$

It satisfies for f in $D_*(\mathbb{R}^{d+1})$ and g in $\mathscr{E}_*(\mathbb{R}^{d+1})$ the following relation

$$ \int_{\mathbb{R}^{d+1}_+} {}^t\mathscr{R}_\beta(f)(y) g(y)\,\mathrm{d}y = \int_{\mathbb{R}^{d+1}_+} f(y)\mathscr{R}_\beta(g)(y)\,\mathrm{d}\mu_\beta(y), \tag{2.14}$$

where $\mathrm{d}\mu_\beta$ is the measure on $\mathbb{R}^{d+1}_+$ given by

$$ \mathrm{d}\mu_\beta(x', x_{d+1}) := x_{d+1}^{2\beta+1}\,\mathrm{d}x'\,\mathrm{d}x_{d+1}. $$

We denote by $L^p_\beta(\mathbb{R}^{d+1}_+)$ the space of measurable functions on $\mathbb{R}^{d+1}_+$ such that

$$ \begin{aligned} \|f\|_{L^p_\beta(\mathbb{R}^{d+1}_+)} &= \left(\int_{\mathbb{R}^{d+1}_+} |f(x)|^p\,\mathrm{d}\mu_\beta(x)\,\mathrm{d}x\right)^{1/p} < +\infty, \quad \text{if } 1 \leqslant p < +\infty, \\ \|f\|_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^{d+1}_+} |f(x)| < +\infty. \end{aligned} $$

Proposition 2 ([12])**.** *Let f in $L^1_\beta(\mathbb{R}^{d+1}_+)$. Then for almost all y, the function*

$$ y \mapsto {}^t\mathscr{R}_{k,\beta}(f)(y) = \frac{2\Gamma(\beta+1)}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})} \int_{y_{d+1}}^{\infty} (s^2 - y_{d+1}^2)^{\beta-\frac{1}{2}} f(y', s) s\,\mathrm{d}s $$

is defined almost everywhere on $\mathbb{R}^{d+1}_+$ and belongs to $L^1(\mathbb{R}^{d+1}_+)$. Moreover for all bounded function g in $C_*(\mathbb{R}^{d+1})$ we have the formula

$$ \int_{\mathbb{R}^{d+1}_+} {}^t\mathscr{R}_\beta(f)(y) g(y)\,\mathrm{d}y = \int_{\mathbb{R}^{d+1}_+} f(x)\mathscr{R}_\beta(g)(x)\,\mathrm{d}\mu_\beta(x). \tag{2.15}$$

Remark 1. Let f be in $L^1_\beta(\mathbb{R}^{d+1}_+)$. By taking $g \equiv 1$ in the relation (2.15) we deduce that

$$ \int_{\mathbb{R}^{d+1}_+} {}^t\mathscr{R}_\beta(f)(y)\,\mathrm{d}y = \int_{\mathbb{R}^{d+1}_+} f(x)\,\mathrm{d}\mu_\beta(x). \tag{2.16}$$

The Weinstein transform is given for f in $L^1_\beta(\mathbb{R}^{d+1}_+)$ by

$$ \mathscr{F}_W(f)(y) = \int_{\mathbb{R}^{d+1}_+} f(x)\Lambda(-x, y)\,\mathrm{d}\mu_\beta(x), \quad \text{for all } y \in \mathbb{R}^{d+1}_+. \tag{2.17}$$

Some basic properties of this transform are the following:

i) For f in $L^1_\beta(\mathbb{R}^{d+1}_+)$,

$$\|\mathscr{F}_W(f)\|_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} \leqslant \|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}. \tag{2.18}$$

ii) For f in $\mathscr{S}_*(\mathbb{R}^{d+1})$ we have

$$\mathscr{F}_W(\Delta_\beta f)(y) = -\|y\|^2 \mathscr{F}_W(f)(y), \quad \text{for all } y \in \mathbb{R}^{d+1}_+. \tag{2.19}$$

iii) For all $f \in \mathscr{S}(\mathbb{R}^{d+1}_*)$, we have

$$\mathscr{F}_W(f)(y) = \mathscr{F}_0 \circ {}^t\mathscr{R}_\beta(f)(y), \quad \text{for all } y \in \mathbb{R}^{d+1}_+, \tag{2.20}$$

where $\mathscr{F}_0$ is the transform defined by: $\forall\, y \in \mathbb{R}^{d+1}_+$,

$$\mathscr{F}_0(f)(y) = \int_{\mathbb{R}^{d+1}_+} f(x) \mathrm{e}^{-\mathrm{i}\langle y', x'\rangle} \cos(x_{d+1} y_{d+1})\, \mathrm{d}x, \quad f \in D_*(\mathbb{R}^{d+1}). \tag{2.21}$$

iv) For all f in $L^1_\beta(\mathbb{R}^{d+1}_+)$, if $\mathscr{F}_W(f)$ belongs to $L^1_\beta(\mathbb{R}^{d+1}_+)$, then

$$f(y) = C(\beta) \int_{\mathbb{R}^{d+1}_+} \mathscr{F}_W(f)(x) \Lambda(x, y)\, \mathrm{d}\mu_\beta(x), \quad \text{a.e.} \tag{2.22}$$

where

$$C(\beta) := \frac{1}{\pi^d 4^{\beta + d/2} (\Gamma(\beta+1))^2}. \tag{2.23}$$

v) For $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$, if we define

$$\overline{\mathscr{F}_W}(f)(y) = \mathscr{F}_W(f)(-y),$$

then

$$\mathscr{F}_W \overline{\mathscr{F}_W} = \overline{\mathscr{F}_W} \mathscr{F}_W = C(\beta)\, \mathrm{Id}\,. \tag{2.24}$$

Proposition 3.

i) *The Weinstein transform $\mathscr{F}_W$ is a topological isomorphism from $\mathscr{S}_*(\mathbb{R}^{d+1})$ onto itself and for all f in $\mathscr{S}_*(\mathbb{R}^{d+1})$,*

$$\int_{\mathbb{R}^{d+1}_+} |f(x)|^2\, \mathrm{d}\mu_\beta(x) = C(\beta) \int_{\mathbb{R}^{d+1}_+} |\mathscr{F}_W(f)(\xi)|^2\, \mathrm{d}\mu_\beta(\xi). \tag{2.25}$$

ii) *In particular, the renormalized Weinstein transform $f \to C(\beta)^{1/2} \mathscr{F}_W(f)$ can be uniquely extended to an isometric isomorphism from $L^2_\beta(\mathbb{R}^{d+1}_+)$ onto itself.*

The generalized translation operator τ_x, $x \in \mathbb{R}^{d+1}_+$, associated with the operator Δ_β is defined by

$$\forall\, y \in \mathbb{R}^{d+1}_+,$$
$$\tau_x f(y) = \frac{\Gamma(\beta+1)}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})}\int_0^{\pi} f\Big(x'+y', \sqrt{x_{d+1}^2+y_{d+1}^2+2x_{d+1}y_{d+1}\cos\theta}\Big)(\sin\theta)^{2\beta}\,\mathrm{d}\theta$$

where $f \in C_*(\mathbb{R}^{d+1})$.

By using the Weinstein kernel, we can also define a generalized translation. For a function $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}^{d+1}_+$ the generalized translation $\tau_y f$ is defined by the following relation:

$$\mathscr{F}_W(\tau_y f)(x) = \Lambda(x,y)\mathscr{F}_W(f)(x).$$

For example, for $t>0$, we see that

$$\tau_y(\mathrm{e}^{-t\|\xi\|^2})(x) = \mathrm{e}^{-t(\|x\|^2+\|y\|^2)}\Lambda(-2\mathrm{i}ty, x). \tag{2.26}$$

By using the generalized translation, we define the generalized convolution product $f *_W g$ of functions $f, g \in L^1_\beta(\mathbb{R}^{d+1}_+)$ as follows:

$$f *_W g(x) = \int_{\mathbb{R}^{d+1}_+} \tau_x f(-y', y_{d+1})g(y)\,\mathrm{d}\mu_\beta(y). \tag{2.27}$$

This convolution is commutative and associative and satisfies the following:

i) For all $f, g \in L^1_\beta(\mathbb{R}^{d+1}_+)$, $f *_W g$ belongs to $L^1_\beta(\mathbb{R}^{d+1}_+)$ and

$$\mathscr{F}_W(f *_W g) = \mathscr{F}_W(f)\mathscr{F}_W(g). \tag{2.28}$$

ii) Let $1 \leqslant p, q, r \leqslant \infty$ such that $1/p + 1/q - 1/r = 1$. If $f \in L^p_\beta(\mathbb{R}^{d+1}_+)$ and $g \in L^q_\beta(\mathbb{R}^{d+1}_+)$, then $f *_W g \in L^r_\beta(\mathbb{R}^{d+1}_+)$ and

$$\|f *_W g\|_{L^r_\beta(\mathbb{R}^{d+1}_+)} \leqslant \|f\|_{L^p_\beta(\mathbb{R}^{d+1}_+)}\|g\|_{L^q_\beta(\mathbb{R}^{d+1}_+)}. \tag{2.29}$$

2.2. Heat functions related to the Weinstein operator

The generalized heat kernel $N_\beta(s,x)$, $x \in \mathbb{R}^{d+1}_+$, $s>0$, associated with the Weinstein operator Δ_β is given by

$$N_\beta(s,x) := \frac{2}{\pi^{d/2}\Gamma(\beta+1)(4s)^{\beta+1+d/2}}\,\mathrm{e}^{-\|x\|^2/(4s)}, \tag{2.30}$$

which is a solution of the generalized heat equation:

$$\frac{\partial}{\partial s}N_\beta(s,x) - \Delta_\beta N_\beta(s,x) = 0.$$

Some basic properties of $N_\beta(s,x)$ are the following:

i) For all $x \in \mathbb{R}^{d+1}_+$, $s > 0$,

$$\mathscr{F}_W(N_\beta(s,\cdot))(x) = \mathrm{e}^{-s\|x\|^2}. \tag{2.31}$$

ii) For all $\lambda > 0$,

$$N_\beta(\lambda s, \lambda^{1/2}x) = \lambda^{-(\beta+1+d/2)}N_\beta(s,x).$$

iii) For $s > 0$,

$$\|N_\beta(s,\cdot)\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} = 1. \tag{2.32}$$

iv) For all $t, s > 0$,

$$N_\beta(t,\cdot) *_W N_\beta(s,\cdot)(x) = N_\beta(t+s,x).$$

For $r > 0$, $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$, we define the generalized heat functions $W^\beta_{\alpha,j}(r,\cdot)$ related to the Weinstein operator Δ_β by:

$$W^\beta_{\alpha,j}(r,x) := (D^\alpha_{x'}\mathscr{L}^j_{\beta,x_{d+1}}N_\beta(r,\cdot))(x), \quad x \in \mathbb{R}^{d+1}_+, \tag{2.33}$$

where D^α is the operator $D^\alpha = \partial_1^{\alpha_1} \circ \partial_2^{\alpha_2} \circ \ldots \circ \partial_d^{\alpha_d}$, with ∂_i, $i = 1, 2, \ldots, d$, the partial derivatives operators.

For $j \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$ we have

$$\forall\, y \in \mathbb{R}^{d+1}_+, \quad \mathscr{F}_W(W^\beta_{\alpha,j}(r,\cdot))(y) = \mathrm{i}^{|\alpha|}(-1)^j {y_1}^{\alpha_1}\ldots y_d^{\alpha_d}y_{d+1}^{2j}\mathrm{e}^{-r\|y\|^2}. \tag{2.34}$$

Proposition 4 ([12])**.** *Let ψ be in $\mathscr{P}^{d+1}_{*,m}$. Then for all $\delta > 0$, there exists a polynomial $Q \in \mathscr{P}^{d+1}_{*,m}$ such that*

$$\forall\, y \in \mathbb{R}^{d+1}_+, \quad \mathscr{F}_W(\psi \mathrm{e}^{-\delta\|x\|^2})(y) = Q(y)\mathrm{e}^{-\frac{1}{4}\delta^{-1}\|y\|^2}. \tag{2.35}$$

3. Cowling-Price's theorem for the Weinstein transform

We shall prove a generalization of the Cowling-Price theorem for the Weinstein transform.

Theorem 1. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that*

$$\int_{\mathbb{R}_+^{d+1}} \frac{\mathrm{e}^{ap\|x\|^2}|f(x)|^p}{(1+\|x\|)^n}\,\mathrm{d}\mu_\beta(x) < \infty \tag{3.1}$$

and

$$\int_{\mathbb{R}_+^{d+1}} \frac{\mathrm{e}^{bq\|\xi\|^2}|\mathscr{F}_W(f)(\xi)|^q}{(1+\|\xi\|)^m}\,\mathrm{d}\xi < \infty, \tag{3.2}$$

for some constants $a>0$, $b>0$, $1\leqslant p,q<+\infty$, and for any

$$n \in\,]d+2\beta+2, d+2\beta+2+p] \quad \text{and} \quad m \in\,]d+1, d+1+q].$$

i) *If $ab > \frac{1}{4}$, then $f=0$ almost everywhere.*
ii) *If $ab = \frac{1}{4}$, we have $f = CN_\beta(b,\cdot)$.*
iii) *If $ab < \frac{1}{4}$, then for all $\delta \in\,]b, \frac{1}{4}a^{-1}[$, all functions of the form $f(x) = P(x) \times N_\beta(\delta, x)$, $P \in \mathscr{P}_*$, satisfy (3.1) and (3.2).*

Proof. We shall show that $\mathscr{F}_W(f)(z)$ exists and is an entire function in $z \in \mathbb{C}^{d+1}$ and

$$|\mathscr{F}_W(f)(z)| \leqslant C\mathrm{e}^{\frac{1}{4}a^{-1}\|\mathrm{Im}\, z\|^2}(1+\|\mathrm{Im}\, z\|)^s, \quad \text{for all } z\in\mathbb{C}^{d+1}, \quad \text{for some } s>0. \tag{3.3}$$

The first assertion follows from the hypothesis on the function f and Hölder's inequality using (2.5) and the theorem on derivation under the integral sign. We want to prove (3.3). Actually, it follows from (2.17) and (2.5) that for all $z=\xi+\mathrm{i}\eta \in \mathbb{C}^{d+1}$,

$$\begin{aligned}
&|\mathscr{F}_W(f)(\xi+\mathrm{i}\eta)| \\
&\quad\leqslant \int_{\mathbb{R}_+^{d+1}} |f(x)\|\Lambda(x,\xi+\mathrm{i}\eta)|\,\mathrm{d}\mu_\beta(x) \\
&\quad\leqslant \mathrm{e}^{\|\eta\|^2/(4a)} \int_{\mathbb{R}_+^{d+1}} \frac{\mathrm{e}^{a\|x\|^2}|f(x)|}{(1+\|x\|)^{n/p}}(1+\|x\|)^{n/p}\mathrm{e}^{-a(\|x\|-\|\eta/2a\|)^2}\,\mathrm{d}\mu_\beta(x).
\end{aligned}$$

Then by using the Hölder's inequality and (3.1) we can obtain that

$$
\begin{aligned}
&|\mathscr{F}_W(f)(\xi+\mathrm{i}\eta)| \\
&\quad\leqslant C\mathrm{e}^{\|\eta\|^2/(4a)}\bigg(\int_{\mathbb{R}_+^{d+1}}(1+\|x\|)^{np'/p}\mathrm{e}^{-ap'(\|x\|-\|\eta/2a\|)^2}\,\mathrm{d}\mu_\beta(x)\bigg)^{1/p'} \\
&\quad\leqslant C\mathrm{e}^{\|\eta\|^2/(4a)}\bigg(\int_0^{\infty}(1+r)^{np'/p+2\beta+d+1}\mathrm{e}^{-ap'(r-\|\eta/2a\|)^2}\,\mathrm{d}r\bigg)^{1/p'} \\
&\quad\leqslant C\mathrm{e}^{\|\eta\|^2/(4a)}(1+\|\eta\|)^{n/p+(2\beta+d+1)/p'}.
\end{aligned}
$$

If $ab=\frac{1}{4}$, then

$$|\mathscr{F}_W(f)(\xi+\mathrm{i}\eta)|\leqslant C\mathrm{e}^{b\|\eta\|^2}(1+\|\eta\|)^{n/p+(2\beta+d+1)/p'}.$$

Therefore, if we let $g(z)=\mathrm{e}^{bz^2}\mathscr{F}_W(f)(z)$, then

$$|g(z)|\leqslant C\mathrm{e}^{b(\operatorname{Re}z)^2}(1+\|\operatorname{Im}z\|)^{n/p+(2\beta+d+1)/p'}.$$

On the other hand it follows from (3.2) that

$$\int_{\mathbb{R}_+^{d+1}}\frac{|g(\xi)|^q}{(1+\|\xi\|)^m}\,\mathrm{d}\xi<\infty.$$

Here we use the following lemma.

Lemma 1 ([21]). *Let h be an entire function on $\mathbb{C}^{d+1}$ such that*

$$|h(z)|\leqslant C\mathrm{e}^{a\|\operatorname{Re}z\|^2}(1+\|\operatorname{Im}z\|)^m$$

for some $m>0$, $a>0$ and

$$\int_{\mathbb{R}_+^{d+1}}\frac{|h(x)|^q}{(1+\|x\|)^s}|Q(x)|\,\mathrm{d}x<\infty$$

for some $q\geqslant 1$, $s>1$ and $Q\in\mathscr{P}_{*,M}^{d+1}$. Then h is a polynomial with $\deg h\leqslant\min\{m,(s-M-d-1)/q\}$ and, if $s\leqslant q+M+d+1$, then h is a constant.

Hence by this lemma, g is a polynomial, we say P_b, with

$$\deg P_b\leqslant\min\Big\{\frac{n}{p}+\frac{2\beta+d+1}{p'},\frac{m-d-1}{q}\Big\}.$$

Thus, $\mathscr{F}_W(f)(z)=P_b(z)\mathrm{e}^{-bz^2}$, for all $z\in\mathbb{C}^{d+1}$.

If $m \leqslant q+d+1$, then clearly P_b is constant. This proves ii).

If $ab > \frac{1}{4}$, then we can choose positive constants, $a_1, b_1\colon\ a > a_1 = \frac{1}{4}b_1^{-1} > \frac{1}{4}b^{-1}$. Then f and $\mathscr{F}_W(f)$ also satisfy (3.1) and (3.2) with a and b replaced by a_1 and b_1 respectively. Therefore, it follows that $\mathscr{F}_W(f)(x) = P_{b_1}(x)\mathrm{e}^{-b_1\|x\|^2}$. But then $\mathscr{F}_W(f)$ cannot satisfy (3.2) unless $P_{b_1} \equiv 0$, which implies $f \equiv 0$. This proves i).

If $ab < \frac{1}{4}$, then for all $\delta \in\,]b, \frac{1}{4}a^{-1}[$, the functions of the form $f(x) = P(x)N_\beta(\delta, x)$, where $P \in \mathscr{P}_*$, satisfy (3.1) and (3.2). This proves iii). $\square$

The following is an immediate consequence of Theorem 1.

Corollary 1. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that*

$$|f(x)| \leqslant M\mathrm{e}^{-a\|x\|^2}(1+\|x\|)^r \quad \text{a.e.} \tag{3.4}$$

and for all $\xi \in \mathbb{R}_+^{d+1}$,

$$|\mathscr{F}_W(f)(\xi)| \leqslant M\mathrm{e}^{-b\|\xi\|^2} \tag{3.5}$$

for some constants $a, b > 0$, $r \geqslant 0$ and $M > 0$.

i) *If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.*
ii) *If $ab = \frac{1}{4}$, then f is of the form $f(x) = CN_\beta(b, x)$.*
iii) *If $ab < \frac{1}{4}$, then there are infinity many nonzero f satisfying* (3.4) *and* (3.5).

Remark 2. When $r = 0$, we obtain Hardy's theorem for the Weinstein transform on $\mathbb{R}_+^{d+1}$.

4. Cowling-Price's theorem via the generalized spherical harmonics coefficients

We replace the assumption (3.2) by one involving the generalized spherical harmonics coefficients of f, which will be defined as follows. In this section we suppose that $d \geqslant 1$ and $\lambda > 0$. For a non-negative integer l, we put

$$\mathscr{H}_l^\beta := \{P \in \mathscr{P}_{*,l}\colon\ P \text{ is homogeneous and } \Delta_\beta P = 0\},$$

which is called the space of generalized spherical harmonics of degree l. We fix a $P_l \in \mathscr{H}_l^\beta$ and define the Weinstein coefficients of $f \in L_\beta^1(\mathbb{R}_+^{d+1})$ in the angular variable by

$$f_{l,\beta}(\lambda) = \int_{S^d_+} f(\lambda t) P_l(t)\,\mathrm{d}\sigma_\beta(t), \tag{4.1}$$

with $\mathrm{d}\sigma_\beta(t) := t_{d+1}^{2\beta+1}\mathrm{d}\sigma_{d+1}(t)$. Then the Weinstein spherical harmonic coefficients of $f \in L^1_\beta(\mathbb{R}^{d+1}_+)$ are given by

$$F_{l,\beta}(\lambda) = \lambda^{-l}\int_{S^d_+} \mathscr{F}_W(f)(\lambda,t) P_l(t)\,\mathrm{d}\sigma_\beta(t), \tag{4.2}$$

where

$$\mathscr{F}_W(f)(\lambda,t) = \int_{\mathbb{R}^{d+1}_+} \Lambda(\lambda x, -t) f(x)\,\mathrm{d}\mu_\beta(x) \tag{4.3}$$

for $t \in S^d_+$. The relation between $f_{l,\beta}$ and $F_{l,\beta}$ is given by the following.

Proposition 5. *Let notation be as above. Then for* $z \in S^{d+2l-1}_+$,

$$\begin{aligned} F_{l,\beta}(\lambda) &= C\int_{\mathbb{R}^{d+2l}_+} f_{l,\beta}(\|x\|)\|x\|^{-l}\Lambda_l(\lambda x, -z)\,\mathrm{d}\mu_\beta(x) \\ &= C\mathscr{F}_{W,l}(f_{l,\beta}(\|\cdot\|)\|\cdot\|^{-l})(\lambda z), \end{aligned} \tag{4.4}$$

where $\mathscr{F}_{W,l}$ *and* Λ_l *are the Weinstein transform and the Weinstein kernel on* $\mathbb{R}^{d+2l}_+$ *respectively.*

P r o o f. From (2.4), (4.3), and (4.2) it follows that

$$F_{l,\beta}(\lambda) = \lambda^{-l}\int_{\mathbb{R}^{d+1}_+}\left(\int_{S^d_+}\Lambda(t,-\lambda x)P_l(t)\,\mathrm{d}\sigma_\beta(t)\right)f(x)\,\mathrm{d}\mu_\beta(x).$$

Here we recall the generalized Funk-Hecke identity.

Lemma 2 ([12])**.** *Let* $H \in \mathscr{H}^\beta_l$. *Then for all* $x \in \mathbb{R}^{d+1}_+$,

$$\int_{S^d_+}\Lambda(t,x)H(t)\,\mathrm{d}\sigma_\beta(t) = C_{l,\beta}H(x)j_{\beta+l+d/2}(\|x\|). \tag{4.5}$$

Therefore, we see that

$$F_{l,\beta}(\lambda) = C_{l,\beta}\int_{\mathbb{R}^{d+1}_+} P_l(x)j_{\beta+l+d/2}(\lambda\|x\|)f(x)\,\mathrm{d}\mu_\beta(x).$$

Then by using (4.5) with d replaced by $d+2l$, we can obtain that for all $z \in S_+^{d+2l}$,

$$\begin{aligned} F_{l,\beta}(\lambda) &= C_{l,\beta} \int_0^\infty \int_{S_+^d} j_{\beta+l+d/2}(\lambda r) r^{2\beta+l+d+1} P_l(t) f(rt) \, \mathrm{d}\sigma_\beta(t) \, \mathrm{d}r \\ &= C_{l,\beta} \int_0^\infty f_{l,\beta}(r) j_{\beta+d/2+l}(\lambda r) r^{2\beta+l+d+1} \, \mathrm{d}r \\ &= C \int_0^\infty \Big(\int_{S_+^{d+2l}} \Lambda_l(t, -\lambda r z) t_{d+1}^{2\beta+1} \, \mathrm{d}\sigma_{d+2l+1}(t) \Big) f_{l,\beta}(r) r^{2\beta+l+d+1} \, \mathrm{d}r \\ &= C \int_{\mathbb{R}_+^{d+2l+1}} f_{l,\beta}(\|x\|) \|x\|^{-l} \Lambda_l(x, -\lambda z) \, \mathrm{d}\mu_\beta(x). \end{aligned}$$

This establishes the proposition. □

Theorem 2. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that for $p, q \in [1, \infty[$, $a, b > 0$ and for each non-negative integer l,*

$$\int_{\mathbb{R}_+^{d+1}} \frac{\mathrm{e}^{ap\|x\|^2} |f(x)|^p}{(1+\|x\|)^n} \, \mathrm{d}\mu_\beta(x) < \infty \tag{4.6}$$

and

$$\int_{\mathbb{R}+} \frac{\mathrm{e}^{bq\lambda^2} |F_{l,\beta}(\lambda)|^q}{(1+\lambda)^m} \, \mathrm{d}\lambda < \infty, \tag{4.7}$$

for any $n \in]d+2\beta+2, d+2\beta+2+p]$ and $m > 1$.

i) *If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.*
ii) *If $ab = \frac{1}{4}$, then $f = CN_\beta(b, \cdot)$.*
iii) *If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4}a^{-1}[$, all functions of the form $f(x) = P(x) \times N_\beta(\delta, x)$, where $P \in \mathscr{P}_*$, satisfy (4.6) and (4.7).*

Proof. (4.6) implies that $f \in L^1_\beta(\mathbb{R}_+^{d+1})$, and thus each $f_{l,\beta}$ is well-defined. Moreover, it follows from (4.1), (4.4) and the Hölder inequality that

$$I_l = \int_0^\infty \frac{\mathrm{e}^{apr^2} |f_{l,\beta}(r)|^p}{(1+r)^n} r^{2\beta+d+1} \, \mathrm{d}r \leqslant C \int_{\mathbb{R}_+^{d+1}} \frac{\mathrm{e}^{ap\|x\|^2} |f(x)|^p}{(1+\|x\|)^n} \, \mathrm{d}\mu_\beta(x) < \infty.$$

Here we used Hölder's inequality and the compactness of S_+^d to obtain the last inequality. Then, by applying this estimate in the polar coordinates in (4.4) and using the same argument as in the proof of Theorem 1, we see that $F_{l,\beta}(\lambda)$ has an entire holomorphic extension on $\mathbb{C}$ and there exists $N \geqslant 0$ such that

$$|F_{l,\beta}(u + \mathrm{i}v)| \leqslant C \mathrm{e}^{\frac{1}{4}a^{-1}v^2} (1 + |v|)^N.$$

If $ab \geqslant \frac{1}{4}$ then $|F_{l,\beta}(u+\mathrm{i}v)| \leqslant C\mathrm{e}^{bv^2}(1+|v|)^N$. Therefore, if we put $G_{l,\beta}(z) = F_{l,\beta}(z)\mathrm{e}^{bz^2}$, then

$$|G_{l,\beta}(z)| \leqslant C\mathrm{e}^{bu^2}(1+|v|)^N \quad \text{and} \quad \int_{\mathbb{R}} \frac{|G_{l,\beta}(x)|^q}{(1+|x|)^m}\,\mathrm{d}x < \infty$$

by (4.7). Applying Lemma 1 for $d = 0$ to $G_{l,\beta}(z)$, we see that $F_{l,\beta}(\lambda) = C_{l,\beta}\mathrm{e}^{-b\lambda^2}P(\lambda)$, where $\lambda \in \mathbb{R}$ and P is polynomial whose degree depends on N and l. By noting (4.4) and (2.35), the injectivity of the Weinstein transform on $\mathbb{R}_+^{d+2l}$ implies that for all $x \in \mathbb{R}_+^{d+2l}$, $f_{l,\beta}(\|x\|) = C_{l,\beta}\|x\|^l Q(x) N_{l,\beta}(b,x)$, where $N_{l,\beta}$ is the generalized heat kernel on $\mathbb{R}_+^{d+2l}$.

If $ab > \frac{1}{4}$, then I_l is finite provided $f_{l,\beta} = 0$ for all l. Therefore, $f = 0$ almost everywhere. If $ab = \frac{1}{4}$, then I_l is finite provided $n - lp - (2\beta + d + 1) > 1$, that is, $n > d+2\beta+2+lp$. Therefore, the assumption on n implies that $l = 0$ and $\deg Q = 0$. Clearly, $f = CN_\beta(b,x)$ satisfy (4.6) and (4.7). If $ab < \frac{1}{4}$, then for a given family of functions, we see that $\mathscr{F}_W(f)(y) = Q(y)\mathrm{e}^{-\delta\|y\|^2}$ for some $Q \in \mathscr{P}_*$. These functions clearly satisfy (4.6) and (4.7) for all $\delta \in]b, \frac{1}{4}a^{-1}[$. □

5. A variant of Cowling-Price's theorem for the Weinstein transform

The aim of this section is to give a variant of Cowling-Price's theorem for the Weinstein transform. Our approach is different from [14].

Theorem 3. *Let $a, b > 0$ and let $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$ satisfy for all $\xi \in \mathbb{R}_+^{d+1}$,*

$$|\mathscr{F}_W(f)(\xi)| \leqslant C\mathrm{e}^{-b\|\xi\|^2}$$

and for all $(\alpha, j) \in \mathbb{N}^d \times \mathbb{N}$,

$$\|D_{x'}^{\alpha}\mathscr{L}_{\beta,x_{d+1}}^{j}\mathscr{F}_W(f)\|_{L_\beta^2(\mathbb{R}_+^{d+1})}^2 \leqslant C\alpha!(2j)!(2a)^{-(|\alpha|+2j)}. \tag{5.1}$$

If $ab > \frac{1}{4}$ then $f = 0$.
If $ab = \frac{1}{4}$, then $\mathscr{F}_W(f)(\xi) = \varphi(\xi)\mathrm{e}^{-b\|\xi\|^2}$, where φ is a bounded function.

In order to prove Theorem 3 we need the following lemma.

Lemma 3. *Let $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$ and assume that for all $(\alpha, j) \in \mathbb{N}^d \times \mathbb{N}$,*

$$\|D_{x'}^{\alpha}\mathscr{L}_{\beta,x_{d+1}}^{j} f\|_{L_\beta^2(\mathbb{R}_+^{d+1})}^2 \leqslant C\alpha!(2j)!(2a)^{-(|\alpha|+2j)}.$$

Then for all $(\alpha, j) \in \mathbb{N}^d \times \mathbb{N}$,

$$|D_{x'}^{\alpha} \mathscr{L}_{\beta,x_{d+1}}^{j} f(x)|^2 \leqslant C \prod_{i=1}^{d} (\alpha_i + m)!(2j+m)!(2a)^{-(|\alpha|+2j)}$$

with $m = [(2\beta + d)/2] + 2$, *where* C *is independent of* (α, j).

P r o o f. Let $s \in \mathbb{R}$. We define the Weinstein-Sobolev space $H_\beta^s(\mathbb{R}_+^{d+1})$ as the set of distributions $u \in \mathscr{S}'(\mathbb{R}_+^{d+1})$ such that $(1 + \|\xi\|^2)^{s/2} \mathscr{F}_W(u)$ belongs to $L_\beta^2(\mathbb{R}_+^{d+1})$, equipped with the scalar product

$$\langle u, v \rangle_{H_\beta^s(\mathbb{R}_+^{d+1})} = \int_{\mathbb{R}_+^{d+1}} (1 + \|\xi\|^2)^s \mathscr{F}_W(u)(\xi) \overline{\mathscr{F}_W(v)(\xi)} \, \mathrm{d}\mu_\beta(\xi)$$

and the norm

$$\|u\|_{H_\beta^s(\mathbb{R}_+^{d+1})}^2 = \langle u, u \rangle_{H_\beta^s(\mathbb{R}_+^{d+1})}.$$

We proceed as in [16] to prove that if $n \in \mathbb{N}$ and $s \in \mathbb{R}$ satisfy $s > \frac{1}{2}d + \beta + n + 1$, then

$$H_\beta^s(\mathbb{R}_+^{d+1}) \hookrightarrow C_*^n(\mathbb{R}^{d+1}). \tag{5.2}$$

We note that $|D_{x'}^{\alpha} \mathscr{L}_{\beta,x_{d+1}}^{n} f(x)| \leqslant C_m \|D_{x'}^{\alpha} \mathscr{L}_{\beta,x_{d+1}}^{n} f\|_{H_\beta^m(\mathbb{R}_+^{d+1})}$ by (5.2) and

$$\|D_{x'}^{\alpha} \mathscr{L}_{\beta,x_{d+1}}^{n} f\|_{H_\beta^m(\mathbb{R}_+^{d+1})}^2 \leqslant C_m \sum_{|\beta|+j \leqslant m} \|D_{x'}^{\alpha+\beta} \mathscr{L}_{\beta,x_{d+1}}^{n+j} f\|_{L_\beta^2(\mathbb{R}_+^{d+1})}^2$$

by the definition of $H_\beta^m(\mathbb{R}_+^{d+1})$. Hence the desired result follows. □

Let $m = [(2\beta + d)/2] + 2$. Then it follows from Lemma 3 that (5.1) implies that for all $x \in \mathbb{R}_+^{d+1}$

$$|D_{x'}^{\alpha} \mathscr{L}_{\beta,x_{d+1}}^{n} \mathscr{F}_W(f)(x)|^2 \leqslant C \prod_{i=1}^{d} (\alpha_i + m)!(2n+m)!(2a)^{-(|\alpha|+2n)}.$$

Therefore, Theorem 3 follows from the following.

Theorem 4. *Let* $a, b > 0$ *and let* $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$ *satisfy for all* $\xi \in \mathbb{R}_+^{d+1}$,

$$|\mathscr{F}_W(f)(\xi)| \leqslant C\mathrm{e}^{-b\|\xi\|^2} \tag{5.3}$$

and for all $(\alpha, j) \in \mathbb{N}^d \times \mathbb{N}$,

$$(5.4)\qquad |D^{\alpha}_{x'} \mathscr{L}^{j}_{\beta,x_{d+1}} \mathscr{F}_W(f)(\xi)|^2 \leqslant C \prod_{i=1}^{d} (\alpha_i + m)!(2j+m)!(2a)^{-(|\alpha|+2j)}$$

with $m = [(2\beta + d)/2] + 2$.

If $ab > \frac{1}{4}$, *then* $f = 0$.

If $ab = \frac{1}{4}$, *then* $\mathscr{F}_W(f)(\xi) = \varphi(\xi)\mathrm{e}^{-b\|\xi\|^2}$, *where* φ *is a bounded function.*

In order to prove Theorem 4 we need the following lemmas.

Lemma 4. *Let* $a > 0$. *We consider* F *in* $\mathscr{S}_*(\mathbb{R}^{d+1})$ *satisfying for all* $(\alpha, j) \in \mathbb{N}^d \times \mathbb{N}$*:*

$$(5.5)\qquad \forall\, x \in \mathbb{R}^{d+1}, \quad |D^{\alpha}_{x'} \mathscr{L}^{j}_{\beta,x_{d+1}} F(x)|^2 \leqslant C \prod_{i=1}^{d} (\alpha_i + m)!(2j+m)!(2a)^{-(|\alpha|+2j)}.$$

Then the function F *extends to* $\mathbb{C}^{d+1}$ *as an entire function which satisfies for every* $b > \frac{1}{4}a^{-1}$ *the relation*

$$(5.6)\qquad \forall\, z \in \mathbb{C}^{d+1}, \quad |F(z)| \leqslant C\mathrm{e}^{b\|z\|^2}.$$

P r o o f. i) From Proposition 1, the function F satisfies the relation

$$(5.7)\qquad \forall\, x \in \mathbb{R}^{d+1}, \quad F(x) = \sum_{n=0}^{\infty} \sum_{|\nu|=|\alpha|+2j=n} m_{\nu}(x) D^{\alpha}_{x'} \mathscr{L}^{j}_{\beta,x_{d+1}} F(0).$$

Thus the function F can be extended to an entire function on $\mathbb{C}^{d+1}$, and we denote also by F the function given by

$$(5.8)\qquad \forall\, z \in \mathbb{C}^{d+1}, \quad F(z) = \sum_{n=0}^{\infty} \sum_{|\nu|=|\alpha|+2j=n} m_{\nu}(z) D^{\alpha}_{x'} \mathscr{L}^{j}_{\beta,x_{d+1}} F(0).$$

ii) For $b > \frac{1}{4}a^{-1}$, the relations (5.8), (2.8), (2.9), (2.11) and Cauchy-Schwarz's inequality give that

$$\begin{aligned} &\forall\, z \in \mathbb{C}^{d+1},\\ &\quad |F(z)| \leqslant \sum_{n=0}^{\infty} \sum_{|\alpha|+2j=n} \left\| \frac{(z')^{\alpha} z_{d+1}^{2j}}{\alpha!\, d_j(\beta)} \right\| |D^{\alpha}_{x'} \mathscr{L}^{j}_{\beta,x_{d+1}} F(0)|\\ &\qquad \leqslant \sum_{n=0}^{\infty} \sum_{|\alpha|+2j=n} \frac{\|z'\|^{|\alpha|} |z_{d+1}|^{2j}}{\alpha!\, d_j(\beta)} |D^{\alpha}_{x'} \mathscr{L}^{j}_{\beta,x_{d+1}} F(0)| \end{aligned}$$

$$\leqslant \Big(\sum_{n=0}^{\infty} \frac{(2b\|z\|^2)^n}{n!}\Big)^{1/2} \Big(\sum_{n=0}^{\infty} \sum_{|\alpha|+2j=n} |D_{x'}^{\alpha} \mathscr{L}_{\beta,x_{d+1}}^{j} F(0)|^2 \frac{(2b)^{-n}}{n!}\Big)^{1/2}$$

$$\leqslant C\Big(\sum_{n=0}^{\infty} \frac{(n+m)!}{(4ab)^n n!}\Big)^{1/2} \mathrm{e}^{b\|z\|^2}.$$

Thus there exists a positive constant $C(\beta, a, b)$ such that

$$\forall\, z \in \mathbb{C}^{d+1}, \quad |F(z)| \leqslant C(\beta, a, b)\mathrm{e}^{b\|z\|^2}.$$

This completes the proof of the lemma. □

Lemma 5 ([20]). *Let $c > 0$, $d > 0$. We consider F an entire function on $\mathbb{C}^{d+1}$ which satisfies*

$$\forall\, z \in \mathbb{C}^{d+1}, \quad |F(z)| \leqslant C\mathrm{e}^{c\|\mathrm{Im}\, z\|^2},$$

and

$$\forall\, x \in \mathbb{R}^{d+1}, \quad |F(x)| \leqslant C\mathrm{e}^{-d\|x\|^2}.$$

Then $F = 0$ whenever $c < d$ and $F(z) = C\mathrm{e}^{-cz^2}$ for $c = d$.

Lemma 6 ([20]). *Let F be an entire function on $\mathbb{C}$ of order ϱ and type β. Let*

$$h(\theta) = \limsup_{r\to\infty} \frac{\log|F(r\mathrm{e}^{\mathrm{i}\theta})|}{r^{\varrho}}, \quad \theta \in \mathbb{R}_+$$

be its indicator, and assume that

$$h\Big(\frac{2\pi j}{\varrho}\Big) \leqslant -\beta, \quad j = 0, 1, \ldots, \varrho - 1.$$

Then $F(z) = P(z)\mathrm{e}^{-\beta z^2}$, where P is an entire function at most of minimal type and of order ϱ.

P r o o f of Theorem 4. *First case*: $ab > \frac{1}{4}$. Choose b' such that $b > b' > \frac{1}{4}a^{-1}$. We consider the function F defined on $\mathbb{C}^{d+1}$ by

$$F(z) = \mathrm{e}^{-b'z^2} \mathscr{F}_W(f)(z).$$

By Lemma 4 with b', we have

$$\forall\, z \in \mathbb{C}^{d+1}, \quad |F(z)| \leqslant C\mathrm{e}^{2b'\|\mathrm{Im}\, z\|^2}.$$

But from (5.3) we have

$$\forall\, x \in \mathbb{R}^{d+1}, \quad |F(x)| \leqslant C\mathrm{e}^{-(b+b')\|x\|^2}$$

as $b' < b$, then by applying Lemma 5 we conclude that $\mathscr{F}_W(f) = 0$ and thus from Proposition 3 we obtain $f = 0$.

Second case: $ab = \frac{1}{4}$. From Lemma 4 the function $F(z_{d+1}) = \mathscr{F}_W(f)(z', z_{d+1})$ is an entire function on $\mathbb{C}$ of order at most 2. It can not decay on $\mathbb{R}$ faster than its order. So its order is 2. Since for all $b' > \frac{1}{4}a^{-1}$ we have the estimate

$$\forall\, \xi_{d+1} \in \mathbb{R}, \quad |F(\xi_{d+1})| \leqslant C\mathrm{e}^{-b'\xi_{d+1}^2},$$

then its type is $\frac{1}{4}a^{-1}$. Now we apply Lemma 6 to conclude that

$$F(\xi_{d+1}) = C(\xi', \xi_{d+1})\mathrm{e}^{-b'\xi_{d+1}^2}.$$

But now the function $C(\xi', \xi_{d+1})$ satisfies the same estimates as $\mathscr{F}_W(f)$ on $\mathbb{R}^d$. By using induction we can obtain $\mathscr{F}_W(f)(\xi) = \varphi(\xi)\mathrm{e}^{-b\|\xi\|^2}$, where φ is a bounded function. □

As an application of Theorem 3, we can obtain the following.

Corollary 2. *Let $a, b > 0$ and $p \in [1, +\infty[$. If $f \in \mathscr{S}_*(\mathbb{R}^{d+1})$ satisfies for all $\xi \in \mathbb{R}_+^{d+1}$,*

$$|\mathscr{F}_W(f)(\xi)| \leqslant C\mathrm{e}^{-b\|\xi\|^2} \tag{5.9}$$

and for all $(\alpha, j) \in \mathbb{N}^d \times \mathbb{N}$,

$$\|D_{x'}^{\alpha}\mathscr{L}_{\beta,x_{d+1}}^{j}\mathscr{F}_W(f)\|^2_{L^p_\beta(\mathbb{R}_+^{d+1})} \leqslant C\alpha!(2j)!(2a)^{-(|\alpha|+2j)} \tag{5.10}$$

with $m = [(2\beta + d)/2] + 2$, then $f = 0$ if $ab > \frac{1}{4}$.

P r o o f. We put $F(x) = (\mathscr{F}_W(f) *_W N_\beta(\frac{1}{4}b^{-1}, \cdot))(x)$ where $N_\beta(t, \cdot)$ is the generalized heat kernel given by (2.30). Then by (2.29), it follows that for all $x \in \mathbb{R}_+^{d+1}$,

$$|D_{x'}^{\alpha}\mathscr{L}_{\beta,x_{d+1}}^{j}F(x)| \leqslant \|D_{x'}^{\alpha}\mathscr{L}_{\beta,x_{d+1}}^{j}\mathscr{F}_W(f)\|_{L^p_\beta(\mathbb{R}_+^{d+1})}\|N_\beta(t, \cdot)\|_{L^{p'}_\beta(\mathbb{R}_+^{d+1})},$$

where p' is the conjugate exponent of p. (5.10) implies that

$$|D_{x'}^{\alpha}\mathscr{L}_{\beta,x_{d+1}}^{j}F(x)|^2 \leqslant C\alpha!\,(2j)!\,(2a)^{-(|\alpha|+2j)}.$$

On the other hand, it follows from (5.9) and (2.29) that for all $x \in \mathbb{R}_+^{d+1}$,

$$|F(x)| \leqslant C\mathrm{e}^{-b\|x\|^2}.$$

Therefore, by Theorem 4 $F(x) = 0$ and thus, $\overline{\mathscr{F}_W}(F) = 0$. (2.28) and (2.24) imply that $f = 0$ for $ab > \frac{1}{4}$. □

6. Miyachi's theorem for the Weinstein transform

Miyachi's theorem is generalized for the Weinstein transform as follows.

Theorem 5. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that*

$$\mathrm{e}^{a\|x\|^2} f \in L^p_\beta(\mathbb{R}_+^{d+1}) + L^q_\beta(\mathbb{R}_+^{d+1}) \tag{6.1}$$

and

$$\int_{\mathbb{R}_+^{d+1}} \log^+ \frac{|\mathscr{F}_W(f)(\xi)\mathrm{e}^{b\|\xi\|^2}|}{\lambda}\,\mathrm{d}\xi < \infty, \tag{6.2}$$

for some constants $a, b, \lambda > 0$ and $1 \leqslant p, q \leqslant +\infty$.

i) *If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.*
ii) *If $ab = \frac{1}{4}$, then $f = CN_\beta(b,\cdot)$ with $|C| \leqslant \lambda$.*
iii) *If $ab < \frac{1}{4}$, then for all $\delta \in \,]b, \frac{1}{4}a^{-1}[$, all functions of the form $f(x) = P(x)N_\beta(\delta, x)$, $P \in \mathscr{P}_*$, satisfy (6.1) and (6.2).*

To prove this result we need the following lemmas.

Lemma 7. *Let h be an entire function on $\mathbb{C}^{d+1}$ such that*

$$|h(z)| \leqslant A\mathrm{e}^{B\|\mathrm{Re}\, z\|^2} \quad \textit{and} \quad \int_{\mathbb{R}_+^{d+1}} \log^+ |h(y)|\,\mathrm{d}y < \infty, \tag{6.3}$$

for some positive constants A, B. Then h is a constant.

Proof. (6.3) and the Fubini theorem yield that there is a set $E \subset \mathbb{R}^{d+1}$ with E^c of Lebesgue measure zero such that for all $(x_2, \dots, x_{d+1}) \in E$,

$$\int_{\mathbb{R}} \log^+ |h(x, x_2 \dots, x_{d+1})|\,\mathrm{d}x < +\infty.$$

On the other hand, the function $z_1 \mapsto h(z_1, x_2, \dots, x_{d+1})$ is entire and $O(\mathrm{e}^{B(\mathrm{Re}\, z_1)^2})$ on $\mathbb{C}$. Then by Lemma 4 in [17] this function is bounded on $\mathbb{C}$. Therefore, by the Liouville theorem we see that for all $z_1 \in \mathbb{C}$ and all $(x_2, \dots, x_{d+1}) \in E$,

$$h(z_1, x_2, \dots, x_{d+1}) = h(0, x_2, \dots, x_{d+1}).$$

Since h is continuous, this relation holds for all $z_1, \dots, z_{d+1} \in \mathbb{C}$. Then, by induction, we can deduce that h is a constant. $\square$

Lemma 8. *Let $r \in [1, +\infty]$, $a > 0$. Then for $g \in L^r_\beta(\mathbb{R}^{d+1}_+)$, there exists $C > 0$ such that*

$$\|\mathrm{e}^{a\|x\|^2}\,{}^t\mathscr{R}_\beta(\mathrm{e}^{-a\|y\|^2} g)\|_{L^r(\mathbb{R}^{d+1}_+)} \leqslant C\|g\|_{L^r_\beta(\mathbb{R}^{d+1}_+)}.$$

Proof. From the hypothesis it follows that $\mathrm{e}^{-a\|y\|^2} g$ belongs to $L^1_\beta(\mathbb{R}^{d+1}_+)$. Then by Proposition 2, ${}^t\mathscr{R}_\beta(\mathrm{e}^{-a\|y\|^2} g)$ is defined almost everywhere on $\mathbb{R}^{d+1}_+$. Here we consider two cases.

i) If $r \in [1, \infty[$, then

$$\begin{aligned}
&\|\mathrm{e}^{a\|x\|^2}\,{}^t\mathscr{R}_\beta(\mathrm{e}^{-a\|y\|^2} g)\|^r_{L^r(\mathbb{R}^{d+1}_+)} \\
&\quad\leqslant \int_{\mathbb{R}^{d+1}_+} \mathrm{e}^{ar\|x\|^2}\Big(\int_{x_{d+1}}^{\infty} (s^2 - x_{d+1}^2)^{\beta-1/2}\mathrm{e}^{-a(\|x'\|^2+s^2)}|g(x', s)|s\,\mathrm{d}s\Big)^r\,\mathrm{d}x \\
&\quad\leqslant \int_{\mathbb{R}^{d+1}_+} \mathrm{e}^{ar\|x\|^2}\,{}^t\mathscr{R}_\beta(|g|^r)(x)({}^t\mathscr{R}_\beta(\mathrm{e}^{-ar'\|y\|^2})(x))^{r/r'}\,\mathrm{d}x,
\end{aligned}$$

where r' is the conjugate exponent of r. Since

$$ {}^t\mathscr{R}_\beta(\mathrm{e}^{-t\|y\|^2})(x) = C\mathrm{e}^{-t\|x\|^2} \tag{6.4}$$

for $t > 0$ (cf. [12]), it follows from (2.16) that

$$\begin{aligned}
\|\mathrm{e}^{a\|x\|^2}\,{}^t\mathscr{R}_\beta(\mathrm{e}^{-a\|y\|^2} g)\|^r_{L^r(\mathbb{R}^{d+1}_+)} &\leqslant C\int_{\mathbb{R}^{d+1}_+} {}^t\mathscr{R}_\beta(|g|^r)(x)\,\mathrm{d}x \\
&= C\int_{\mathbb{R}^{d+1}_+} |g(x)|^r\,\mathrm{d}\mu_\beta(x) < +\infty.
\end{aligned}$$

ii) If $r = \infty$, then it follows from (6.4) that

$$\begin{aligned}
\mathrm{e}^{a\|x\|^2}|{}^t\mathscr{R}_\beta(\mathrm{e}^{-a\|y\|^2} g)(x)| &\leqslant \mathrm{e}^{a\|x\|^2}\,{}^t\mathscr{R}_\beta(\mathrm{e}^{-a\|y\|^2})(x)\|g\|_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} \\
&= C\|g\|_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} < \infty.
\end{aligned}$$

This completes the proof. □

Lemma 9. *Let p, q be in $[1, +\infty]$ and f a measurable function on $\mathbb{R}^{d+1}_+$ such that*

$$\mathrm{e}^{a\|x\|^2} f \in L^p_\beta(\mathbb{R}^{d+1}_+) + L^q_\beta(\mathbb{R}^{d+1}_+) \tag{6.5}$$

for some $a > 0$. Then for all $z \in \mathbb{C}^{d+1}$, the integral

$$\mathscr{F}_W(f)(z) = \int_{\mathbb{R}^{d+1}_+} f(x)\Lambda(-x, z)\,\mathrm{d}\mu_\beta(x)$$

is well-defined. $\mathscr{F}_W(f)(z)$ *is entire and there exists* $C > 0$ *such that for all* ξ, η *in* $\mathbb{R}_+^{d+1}$,

$$|\mathscr{F}_W(f)(\xi + \mathrm{i}\eta)| \leqslant C\mathrm{e}^{\|\eta\|^2/4a}. \tag{6.6}$$

P r o o f. The first assertion easily follows from (2.5) and Hölder's inequality.

We shall prove (6.6). (6.5) implies that f belongs to $L^1_\beta(\mathbb{R}_+^{d+1})$ and thus, ${}^t\mathscr{R}_\beta(f)$ to $L^1(\mathbb{R}_+^{d+1})$ by (2.16). Hence by (2.20), for all $\xi, \eta \in \mathbb{R}_+^{d+1}$,

$$\mathscr{F}_W(f)(\xi + \mathrm{i}\eta) = \int_{\mathbb{R}_+^{d+1}} {}^t\mathscr{R}_\beta(f)(x)\mathrm{e}^{-\mathrm{i}\langle x, \xi + \mathrm{i}\eta\rangle}\,\mathrm{d}x$$

and then

$$\begin{aligned}
&|\mathscr{F}_W(f)(\xi + \mathrm{i}\eta)| \\
&\quad\leqslant \mathrm{e}^{\|\eta\|^2/(4a)} \int_{\mathbb{R}_+^{d+1}} \mathrm{e}^{a\|x\|^2} |{}^t\mathscr{R}_\beta(f)(x)| \mathrm{e}^{-a\|x\|^2 + \langle x, \eta\rangle - \|\eta\|^2/(4a)}\,\mathrm{d}x \\
&\quad\leqslant \mathrm{e}^{\|\eta\|^2/(4a)} \int_{\mathbb{R}_+^{d+1}} \mathrm{e}^{a\|x\|^2} |{}^t\mathscr{R}_\beta(f)(x)| \mathrm{e}^{-a\|x - \eta/2a\|^2}\,\mathrm{d}x.
\end{aligned}$$

Since (6.5) implies that there exist $u \in L^p_\beta(\mathbb{R}_+^{d+1})$ and $v \in L^q_\beta(\mathbb{R}_+^{d+1})$ such that

$$f(x) = \mathrm{e}^{-a\|x\|^2}u(x) + \mathrm{e}^{-a\|x\|^2}v(x),$$

it follows from Lemma 8 that

$$\begin{aligned}
&\int_{\mathbb{R}_+^{d+1}} \mathrm{e}^{a\|x\|^2} |{}^t\mathscr{R}_\beta(f)(x)| \mathrm{e}^{-a\|x - \eta/2a\|^2}\,\mathrm{d}x \\
&\qquad\leqslant C(\|u\|_{L^p_\beta(\mathbb{R}_+^{d+1})} + \|v\|_{L^q_\beta(\mathbb{R}_+^{d+1})}) < \infty.
\end{aligned}$$

Therefore, the desired result follows. □

P r o o f of Theorem 5. We will divide the proof in each case.

i) $ab > \frac{1}{4}$. Let h be a function on $\mathbb{C}^{d+1}$ defined by

$$h(z) = \left(\prod_{j=1}^{d+1} \mathrm{e}^{z_j^2/4a}\right)\mathscr{F}_W(f)(z). \tag{6.7}$$

This function is entire on $\mathbb{C}^{d+1}$ and by (6.6) we see that

$$|h(\xi + \mathrm{i}\eta)| \leqslant C\mathrm{e}^{\|\xi\|^2/(4a)} \tag{6.8}$$

for all $\xi \in \mathbb{R}_+^{d+1}$ and $\eta \in \mathbb{R}_+^{d+1}$. On the other hand, we note that

$$\int_{\mathbb{R}_+^{d+1}} \log^+ |h(y)| \, \mathrm{d}y$$
$$= \int_{\mathbb{R}_+^{d+1}} \log^+ |\mathrm{e}^{\|y\|^2/4a} \mathscr{F}_W(f)(y)| \, \mathrm{d}y$$
$$= \int_{\mathbb{R}_+^{d+1}} \log^+ \Big(\frac{\mathrm{e}^{b\|y\|^2} |\mathscr{F}_W(f)(y)|}{\lambda} \lambda \mathrm{e}^{(\frac{1}{4}a^{-1}-b)\|y\|^2} \Big) \, \mathrm{d}y$$
$$\leqslant \int_{\mathbb{R}_+^{d+1}} \log^+ \frac{\mathrm{e}^{b\|y\|^2} |\mathscr{F}_W(f)(y)|}{\lambda} \, \mathrm{d}y + \int_{\mathbb{R}_+^{d+1}} \lambda \mathrm{e}^{(\frac{1}{4}a^{-1}-b)\|y\|^2} \, \mathrm{d}y,$$

because $\log^+(cd) \leqslant \log^+(c) + d$ for all $c, d > 0$. Since $ab > \frac{1}{4}$, (6.2) implies that

$$\int_{\mathbb{R}_+^{d+1}} \log^+ |h(y)| \, \mathrm{d}y < +\infty. \tag{6.9}$$

Then it follows from (6.8) and (6.9) that h satisfies the assumptions in Lemma 7 and thus, h is a constant and

$$\mathscr{F}_W(f)(y) = C \mathrm{e}^{-\frac{1}{4}a^{-1}\|y\|^2}.$$

Since $ab > \frac{1}{4}$, (6.2) holds whenever $C = 0$ and the injectivity of $\mathscr{F}_W$ implies that $f = 0$ almost everywhere.

ii) $ab = \frac{1}{4}$. As in the previous case, it follows that $\mathscr{F}_W(f)(\xi) = C\mathrm{e}^{-\|\xi\|^2/(4a)}$. Then (6.2) holds whenever $|C| \leqslant \lambda$. Hence $f = CN_\beta(b, \cdot)$ with $|C| \leqslant \lambda$.

iii) $ab < \frac{1}{4}$. If f is of the given form, then $\mathscr{F}_W(f)(y) = Q(y)\mathrm{e}^{-\delta\|y\|^2}$ for some $Q \in \mathscr{P}_*$. Then f and $\mathscr{F}_W(f)$ satisfy (6.1) and (6.2) for all $\delta \in]b, \frac{1}{4}a^{-1}[$. □

The following is an immediate consequence of Theorem 5.

Corollary 3. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that*

$$\mathrm{e}^{a\|x\|^2} f \in L^p_\beta(\mathbb{R}_+^{d+1}) + L^q_\beta(\mathbb{R}_+^{d+1}) \tag{6.10}$$

and

$$\int_{\mathbb{R}_+^{d+1}} |\mathscr{F}_W(f)(\xi)|^r \mathrm{e}^{br\|\xi\|^2} \, \mathrm{d}\xi < \infty, \tag{6.11}$$

for some constants $a, b > 0$, $1 \leqslant p, q \leqslant +\infty$ and $0 < r \leqslant \infty$.

i) *If $ab \geqslant \frac{1}{4}$, then $f = 0$ almost everywhere.*
ii) *If $ab < \frac{1}{4}$, then for all $\delta \in]b, \frac{1}{4}a^{-1}[$, all functions of the form $f(x) = P(x) \times N_\beta(\delta, x)$, $P \in \mathscr{P}_*$, satisfy (6.10) and (6.11).*

7. Beurling's theorem for the Weinstein transform

Beurling's theorem and Bonami's, Demange's, and Jaming's extension are generalized for the Weinstein transform as follows.

Theorem 6. *Let $N \in \mathbb{N}$, $\delta > 0$ and $f \in L^2_\beta(\mathbb{R}^{d+1}_+)$ satisfy*

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)||\mathscr{F}_W(f)(y)||R(y)|^\delta}{(1+\|x\|+\|y\|)^N}\, \mathrm{e}^{\|x\|\|y\|}\,\mathrm{d}\mu_\beta(x)\,\mathrm{d}y < +\infty, \tag{7.1}$$

where R is a polynomial of degree m. If $N \geqslant d + m\delta + 3$, then

$$f(y) = \sum_{|s|<\frac{1}{2}(N-d-1-m\delta)} a_s^\beta W_s^\beta(r,y) \quad \text{a.e.}, \tag{7.2}$$

where $r > 0$, $a_s^\beta \in \mathbb{C}$, $s \in \mathbb{N}^{d+1}$ and $W_s^\beta(r,\cdot)$ is given by (2.33). *Otherwise, $f(y) = 0$ almost everywhere.*

Proof. We start with the following lemma.

Lemma 10. *We suppose that $f \in L^2_\beta(\mathbb{R}^{d+1}_+)$ satisfies* (7.1). *Then $f \in L^1_\beta(\mathbb{R}^{d+1}_+)$.*

Proof. We may suppose that f is not negligible. (7.1) and Fubini's theorem imply that for almost every $y \in \mathbb{R}^{d+1}_+$,

$$\frac{|\mathscr{F}_W(f)(y)||R(y)|^\delta}{(1+\|y\|)^N}\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)|}{(1+\|x\|)^N}\,\mathrm{e}^{\|x\|\|y\|}\,\mathrm{d}\mu_\beta(x) < +\infty.$$

Since f and thus, $\mathscr{F}_W(f)$ are not negligible, there exist $y_0 \in \mathbb{R}^{d+1}_+$, $y_0 \neq 0$, such that $\mathscr{F}_W(f)(y_0)R(y_0) \neq 0$. Therefore,

$$\int_{\mathbb{R}^{d+1}_+} \frac{|f(x)|}{(1+\|x\|)^N}\,\mathrm{e}^{\|x\|\|y_0\|}\,\mathrm{d}\mu_\beta(x) < +\infty.$$

Since $\mathrm{e}^{\|x\|\|y_0\|}/(1+\|x\|)^N \geqslant 1$ for large $\|x\|$, it follows that $\int_{\mathbb{R}^{d+1}_+} |f(x)|\,\mathrm{d}\mu_\beta(x) < +\infty$. □

This lemma and Proposition 2 imply that ${}^t\mathscr{R}_\beta(f)$ is well-defined almost everywhere on $\mathbb{R}^{d+1}_+$. By the same techniques as used in [13], we can deduce that

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+} \frac{\mathrm{e}^{\|x\|\|y\|}|{}^t\mathscr{R}_\beta(f)(x)||\mathscr{F}({}^t\mathscr{R}_\beta)(f)(y)||R(y)|^\delta}{(1+\|x\|+\|y\|)^N}\,\mathrm{d}y\,\mathrm{d}x < +\infty.$$

According to Theorem 2.3 in [19], we conclude that for all $x \in \mathbb{R}_+^{d+1}$,

$$^t\mathscr{R}_\beta(f)(x) = P(x)\mathrm{e}^{-\|x\|^2/(4r)},$$

where $r > 0$ and P a polynomial of degree strictly lower than $\frac{1}{2}(N - d - 1 - m\delta)$. Then by (2.20),

$$\mathscr{F}_W(f)(y) = \mathscr{F} \circ {}^t\mathscr{R}_\beta(f)(y) = \mathscr{F}(P(x)\mathrm{e}^{-\|x\|^2/(4r)})(y) = Q(y)\mathrm{e}^{-r\|y\|^2},$$

where Q is a polynomial of degree $\deg P$. Then by using (2.34), we can find constants $a_{\alpha,j}^\beta$ such that

$$\mathscr{F}_W(f)(y) = \mathscr{F}_W\biggl(\sum_{|\alpha|+2j<(N-d-1-m\delta)/2} a_{\alpha,j}^\beta W_{\alpha,j}^\beta(r,\cdot)\biggr)(y).$$

By the injectivity of $\mathscr{F}_W$ the desired result follows. □

As an application of Theorem 6, by using the same techniques as in [13], we can deduce the following Gelfand-Shilov type theorem for the Weinstein transform.

Corollary 4. *Let $N, m \in \mathbb{N}$, $\delta > 0$, $a, b > 0$ with $ab \geqslant \frac{1}{4}$, and $1 < p, q < +\infty$ with $p^{-1} + q^{-1} = 1$. Let $f \in L_\beta^2(\mathbb{R}_+^{d+1})$ satisfy*

$$\int_{\mathbb{R}_+^{d+1}} \frac{|f(x)|\mathrm{e}^{(2a)^p\|x\|^p/p}}{(1+\|x\|)^N}\,\mathrm{d}\mu_\beta(x) < +\infty \tag{7.3}$$

and

$$\int_{\mathbb{R}_+^{d+1}} \frac{|\mathscr{F}_W(f)(y)|\mathrm{e}^{(2b)^q\|y\|^q/q}|R(y)|^\delta}{(1+\|y\|)^N}\,\mathrm{d}y < +\infty \tag{7.4}$$

for some $R \in \mathscr{P}_m$.

i) *If $ab > \frac{1}{4}$ or $(p,q) \neq (2,2)$, then $f(x) = 0$ almost everywhere.*
ii) *If $ab = \frac{1}{4}$ and $(p,q) = (2,2)$, then f is of the form (7.2) whenever $N \geqslant \frac{1}{2}(d + 3 + m\delta)$ and $r = 2b^2$. Otherwise, $f(x) = 0$ almost everywhere.*

Proof. Since

$$4ab\|x\|\|y\| \leqslant \frac{(2a)^p}{p}\|x\|^p + \frac{(2b)^q}{q}\|y\|^q,$$

it follows from (7.3) and (7.4) that

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \frac{|f(x)||\mathscr{F}_W(f)(y)||R(y)|^\delta}{(1+\|x\|+\|y\|)^{2N}}\,\mathrm{e}^{4ab\|x\|\|y\|}\,\mathrm{d}\mu_\beta(x)\,\mathrm{d}y < +\infty.$$

Then (7.1) is satisfied, because $4ab \geqslant 1$. Therefore, according to the proof of Theorem 6, we can deduce that

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \frac{e^{4ab\|x\|\|y\|}|{}^t\mathscr{R}_\beta(f)(x)||\mathscr{F}({}^t\mathscr{R}_\beta)(f)(y)||R(y)|^\delta}{(1+\|x\|+\|y\|)^{2N}}\,dy\,dx < +\infty,$$

and ${}^t\mathscr{R}_\beta(f)$ and f are of the forms

$${}^t\mathscr{R}_\beta(f)(x) = P(x)e^{-\|x\|^2/(4r)} \quad \text{and} \quad \mathscr{F}_W(f)(y) = Q(y)e^{-r\|y\|^2},$$

where $r > 0$ and P, Q are polynomials of the same degree strictly lower than $\frac{1}{2}(2N - d - 1 - m\delta)$. Therefore, substituting these forms, we can deduce that

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \frac{e^{-(\sqrt{r}\|y\|-\frac{1}{2}\|x\|/\sqrt{r})^2}e^{(4ab-1)\|x\|\|y\|}|P(x)||Q(x)||R(y)|^\delta}{(1+\|x\|+\|y\|)^{2N}}\,dy\,dx < +\infty.$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Moreover, it follows from (7.3) and (7.4) that

$$\int_{\mathbb{R}_+^{d+1}} \frac{|P(x)|e^{-\frac{1}{4}r^{-1}\|x\|^2}e^{(2a)^p\cdot p^{-1}\|x\|^p}}{(1+\|x\|)^N}\,d\mu_\beta(x) < +\infty$$

and

$$\int_{\mathbb{R}_+^{d+1}} \frac{|Q(y)|e^{-r\|y\|^2}e^{(2b)^q/q}\|y\|^q|R(y)|^\delta}{(1+\|y\|)^N}\,dy < +\infty.$$

One of these integrals is not finite unless $(p,q) = (2,2)$.

When $4ab = 1$ and $(p,q) = (2,2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 6. □

8. Donoho-Stark uncertainty principle for the Weinstein transform

We shall investigate the case where f and $\mathscr{F}_W(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows.

A function $f \in L^2_\beta(\mathbb{R}_+^{d+1})$ is ε-concentrated on a measurable set $E \subset \mathbb{R}_+^{d+1}$ if there is a measurable function g vanishing outside E such that $\|f-g\|_{L^2_\beta(\mathbb{R}_+^{d+1})} \leqslant \varepsilon\|f\|_{L^2_\beta(\mathbb{R}_+^{d+1})}$.

Therefore, if we introduce a projection operator P_E as

$$P_E f(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases}$$

then f is ε-concentrated on E if and only if $\|f - P_E f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} \leqslant \varepsilon \|f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)}$.

We define a projection operator Q_E as

$$Q_E f(x) = \mathscr{F}_W^{-1}(P_E(\mathscr{F}_W(f)))(x).$$

Then $\mathscr{F}_W(f)$ is ε-concentrated on F if and only if $\|f - Q_F f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} \leqslant \varepsilon \|f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)}$. We note that, for mesurable set $E, F \subset \mathbb{R}^{d+1}_+$,

$$Q_F P_E f(x) = \int_{\mathbb{R}^{d+1}_+} q(t,x) f(t) \, \mathrm{d}\mu_\beta(t),$$

where

$$q(t,x) = \begin{cases} \int_F \Lambda(-t,\xi)\Lambda(x,\xi) \, \mathrm{d}\mu_\beta(\xi) & \text{if } t \in E, \\ 0 & \text{if } t \notin E. \end{cases}$$

Indeed, by the Fubini theorem we see that

$$\begin{aligned} Q_F P_E f(x) &= \int_F \mathscr{F}_W(P_E f)(\xi)\Lambda(\xi,x) \, \mathrm{d}\mu_\beta(\xi) \\ &= \int_F \left(\int_E f(t)\Lambda(\xi,-t) \, \mathrm{d}\mu_\beta(t) \right) \Lambda(\xi,x) \, \mathrm{d}\mu_\beta(\xi) \\ &= \int_E f(t) \left(\int_F \Lambda(\xi,-t)\Lambda(\xi,x) \, \mathrm{d}\mu_\beta(\xi) \right) \mathrm{d}\mu_\beta(t). \end{aligned}$$

The Hilbert-Schmidt norm $\|Q_F P_E\|_{\mathrm{HS}}$ is given by

$$\|Q_F P_E\|_{\mathrm{HS}} = \left(\int_{\mathbb{R}^{d+1}_+} \int_{\mathbb{R}^{d+1}_+} |q(s,t)|^2 \, \mathrm{d}\mu_\beta(s) \, \mathrm{d}\mu_\beta(t) \right)^{1/2}.$$

We denote by $\|T\|_2$ the operator norm on $L^2_\beta(\mathbb{R}^{d+1}_+)$. Since P_E and Q_F are projections, it is clear that $\|P_E\|_2 = \|Q_F\|_2 = 1$. Moreover, it follows that

$$\|Q_F P_E\|_{\mathrm{HS}} \geqslant \|Q_F P_E\|_2. \tag{8.1}$$

If F is a set of finite measure of $\mathbb{R}^{d+1}_+$, we put $\mu_\beta(F) := \int_F \mathrm{d}\mu_\beta(x)$.

Lemma 11. *If E and F are sets of finite measure of $\mathbb{R}^{d+1}_+$, then*

$$\|Q_F P_E\|_{\mathrm{HS}} \leqslant \sqrt{C(\beta)\mu_\beta(E)\mu_\beta(F)},$$

where $C(\beta)$ the constant defined by the relation (2.23).

P r o o f. For $t \in E$, let $g_t(s) = q(s,t)$. From (2.22) we have $\mathscr{F}_W(g_t)(w) = P_F(\Lambda(t,-w))$. Then by Parseval's identity (2.25) and (2.5) it follows that

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}} |q(s,t)|^2 \,\mathrm{d}\mu_\beta(s) &= \int_{\mathbb{R}_+^{d+1}} |g_t(s)|^2 \,\mathrm{d}\mu_\beta(s) \\ &= C(\beta)\int_{\mathbb{R}_+^{d+1}} |\mathscr{F}_W(g_t)(w)|^2 \,\mathrm{d}\mu_\beta(w) \leqslant C(\beta)\mu_\beta(F).\end{aligned}$$

Hence, integrating over $t \in E$, we see that $\|Q_F P_E\|_{\mathrm{HS}}^2 \leqslant C(\beta)\mu_\beta(E)\mu_\beta(F)$. □

Proposition 6. *Let E and F be measurable sets and suppose that*

$$\|\mathscr{F}_W(f)\|_{L^2_\beta(\mathbb{R}_+^{d+1})} = 1.$$

Assume that $\varepsilon_E + \varepsilon_F < \sqrt{C(\beta)}$, f is ε_E-concentrated on E and $\mathscr{F}_W(f)$ is ε_F-concentrated on F. Then

$$\mu_\beta(E)\mu_\beta(F) \geqslant \frac{(1-\varepsilon_E-\varepsilon_F)^2}{C(\beta)}.$$

P r o o f. Since $\|\mathscr{F}_W(f)\|_{L^2_\beta(\mathbb{R}_+^{d+1})} = 1$ and $\varepsilon_E + \varepsilon_F < \sqrt{C(\beta)}$, the measures of E and F must both be non-zero. Indeed, if not, then the ε_E-concentration of f implies that $\|f - P_E f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} = \|f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} = \sqrt{C(\beta)} \leqslant \varepsilon_E$, which contradicts with $\varepsilon_E < \sqrt{C(\beta)}$, likewise for $\mathscr{F}_W(f)$. If at least one of $\mu_\beta(E)$ and $\mu_\beta(F)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and F have finite positive measures. Since $\|Q_F\|_2 = 1$, it follows that

$$\begin{aligned}\|f - Q_F P_E f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} &\leqslant \|f - Q_F f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} + \|Q_F f - Q_F P_E f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} \\ &\leqslant \varepsilon_F + \|Q_F\|_2 \|f - P_E f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} \leqslant \varepsilon_E + \varepsilon_F\end{aligned}$$

and thus,

$$\|Q_F P_E f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} \geqslant \|f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} - \|f - Q_F P_E f\|_{L^2_\beta(\mathbb{R}_+^{d+1})} \geqslant 1 - \varepsilon_E - \varepsilon_F.$$

Hence $\|Q_F P_E\|_2 \geqslant 1 - \varepsilon_E - \varepsilon_F$. (8.1) and Lemma 11 yield the desired inequality. □

In the following we shall consider the case of $f \in L^1_\beta(\mathbb{R}_+^{d+1})$. As in the L^2_β case, we say that $f \in L^1_\beta(\mathbb{R}_+^{d+1})$ is ε-concentrated to E if $\|f - P_E f\|_{L^1_\beta(\mathbb{R}_+^{d+1})} \leqslant \varepsilon \|f\|_{L^1_\beta(\mathbb{R}_+^{d+1})}$. Let $B^1_\beta(F)$ denote the subspace of $L^1_\beta(\mathbb{R}_+^{d+1})$ which consists of all $g \in L^1_\beta(\mathbb{R}_+^{d+1})$ such that $Q_F g = g$. We say that f is ε-bandlimited to F if there is a $g \in B^1_\beta(F)$ with $\|f - g\|_{L^1_\beta(\mathbb{R}_+^{d+1})} < \varepsilon \|f\|_{L^1_\beta(\mathbb{R}_+^{d+1})}$. Here we denote by $\|P_E\|_1$ the operator norm of P_E on $L^1_\beta(\mathbb{R}_+^{d+1})$ and by $\|P_E\|_{1,F}$ the operator norm of $P_E\colon B^1_\beta(F) \to L^1_\beta(\mathbb{R}_+^{d+1})$. Corresponding to (8.1) and Lemma 11 in the L^2_β case, we can obtain the following.

Lemma 12. $\|P_E\|_{1,F} \leqslant C(\beta)\mu_\beta(E)\mu_\beta(F)$.

Proof. For $f \in B^1_\beta(F)$ we see that

$$\begin{aligned} f(t) &= C(\beta)\int_F \Lambda(t,\xi)\mathscr{F}_W(f)(\xi)\,\mathrm{d}\mu_\beta(\xi) \\ &= C(\beta)\int_F \Lambda(t,\xi)\Big(\int_{\mathbb{R}^{d+1}_+} f(x)\Lambda(x,-\xi)\,\mathrm{d}\mu_\beta(x)\,\mathrm{d}\mu_\beta(\xi)\Big) \\ &= C(\beta)\int_{\mathbb{R}^{d+1}_+} f(x)\Big(\int_F \Lambda(t,\xi)\Lambda(x,-\xi)\,\mathrm{d}\mu_\beta(\xi)\Big)\,\mathrm{d}\mu_\beta(x). \end{aligned}$$

Therefore, $\|f\|_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} \leqslant C(\beta)\mu_\beta(F)\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}$ by (2.5) and then,

$$\begin{aligned} \|P_E f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} &= \int_E |f(x)|\,\mathrm{d}\mu_\beta(x) \\ &\leqslant \mu_\beta(E)\|f\|_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} \leqslant C(\beta)\mu_\beta(E)\mu_\beta(F)\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}. \end{aligned}$$

Then, it follows that for $f \in B^1_\beta(F)$,

$$\frac{\|P_E f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}}{\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}} \leqslant \frac{C(\beta)\mu_\beta(E)\mu_\beta(F)\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}}{\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}} = C(\beta)\mu_\beta(E)\mu_\beta(F),$$

which implies the desired inequality. □

Proposition 7. *Let $f \in L^1_\beta(\mathbb{R}^{d+1}_+)$. If f is ε_E-concentrated to E and ε_F-bandlimited to F, then*

$$C(\beta)\mu_\beta(E)\mu_\beta(F) \geqslant \frac{1-\varepsilon_E-\varepsilon_F}{1+\varepsilon_F}.$$

Proof. Without loss of generality, we may suppose that $\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} = 1$. Since f is ε_E-concentrated to E, it follows that

$$\|P_E f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} \geqslant \|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} - \|f - P_E\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} \geqslant 1-\varepsilon_E.$$

Moreover, since f is ε_F-bandlimited, there is a $g \in B^1_\beta(F)$ with $\|g-f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} \leqslant \varepsilon_F$. Therefore, it follows that

$$\begin{aligned} \|P_E g\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} &\geqslant \|P_E f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} - \|P_E(g-f)\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} \\ &\geqslant \|P_E f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} - \varepsilon_F \geqslant 1-\varepsilon_E-\varepsilon_F \end{aligned}$$

and

$$\|g\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} \leqslant \|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} + \varepsilon_F = 1 + \varepsilon_F.$$

Then, we see that

$$\frac{\|P_E g\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}}{\|g\|_{L^1_\beta(\mathbb{R}^{d+1}_+)}} \geqslant \frac{1-\varepsilon_E-\varepsilon_F}{1+\varepsilon_F}.$$

Hence $\|P_E\|_{1,F} \geqslant (1-\varepsilon_E-\varepsilon_F)/(1+\varepsilon_F)$ and Lemma 12 yields the desired inequality. □

Proposition 8. *Let $f \in L^2_\beta(\mathbb{R}^{d+1}_+) \cap L^1_\beta(\mathbb{R}^{d+1}_+)$ with $\|f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} = \sqrt{C(\beta)}$. If f is ε_E-concentrated to E in L^1_β-norm and $\mathscr{F}_W(f)$ is ε_F-concentrated to F in L^2_β-norm, then*

$$C(\beta)\mu_\beta(E) \geqslant (1-\varepsilon_E)^2\|f\|^2_{L^1_\beta(\mathbb{R}^{d+1}_+)} \quad \text{and} \quad \mu_\beta(F)\|f\|^2_{L^1_\beta(\mathbb{R}^{d+1}_+)} \geqslant (1-\varepsilon_F)^2.$$

In particular,

$$C(\beta)\mu_\beta(E)\mu_\beta(F) \geqslant (1-\varepsilon_E)^2(1-\varepsilon_F)^2.$$

Proof. Since $\|\mathscr{F}_W(f)\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} = 1$ and f is ε_F-concentrated to F in L^2_β-norm, it follows that

$$\begin{aligned}(8.2)\qquad \|P_F(\mathscr{F}_W(f))\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} &\\ &\geqslant \|\mathscr{F}_W(f)\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} - \|\mathscr{F}_W(f) - P_F(\mathscr{F}_W(f))\|_{L^2_\beta(\mathbb{R}^{d+1}_+)}\\ &\geqslant 1-\varepsilon_F,\end{aligned}$$

and thus,

$$\begin{aligned}(1-\varepsilon_F)^2 &\leqslant \int_F |\mathscr{F}_W(f)(\xi)|^2 \, \mathrm{d}\mu_\beta(\xi)\\ &\leqslant \mu_\beta(F)\|\mathscr{F}_W(f)\|^2_{L^\infty_\beta(\mathbb{R}^{d+1}_+)} \leqslant \mu_\beta(F)\|f\|^2_{L^1_\beta(\mathbb{R}^{d+1}_+)}.\end{aligned}$$

Similarly, since $\|f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} = \sqrt{C(\beta)}$ and f is ε_E-concentrated to E in L^1_β-norm,

$$(1-\varepsilon_E)\|f\|_{L^1_\beta(\mathbb{R}^{d+1}_+)} \leqslant \int_E |f(x)| \, \mathrm{d}\mu_\beta(x) \, \mathrm{d}x \leqslant \sqrt{C(\beta)\mu_\beta(E)}.$$

Here we used the Cauchy-Schwarz inequality and the fact that $\|f\|_{L^2_\beta(\mathbb{R}^{d+1}_+)} = \sqrt{C(\beta)}$. □

9. Applications

The last part of this paper is motivated by a different kind of uncertainty principles written via the generalized Schrödinger and heat semigroups. Indeed, we proceed as [15] to prove the following identity

$$(9.1)\qquad \begin{aligned} u(t,x) &= \mathrm{e}^{\mathrm{i}t\Delta_\beta}u_0(x) \\ &= \frac{2}{\pi^{d/2}\Gamma(\beta+1)(4t)^{\beta+1+d/2}}\,\mathrm{e}^{-\mathrm{i}(d+2\beta+2)\frac{1}{4}\pi\,\mathrm{sgn}\,t}\mathrm{e}^{\mathrm{i}\|\cdot\|^2/(4t)} \\ &\quad\times\big[\mathscr{F}_W(\mathrm{e}^{\mathrm{i}\|\cdot\|^2/(4t)}u_0)\big]\Big(\frac{x}{2t}\Big), \end{aligned}$$

which tells us that this kind of results for the free solution of the Weinstein-Schrödinger equation with data u_0

$$(9.2)\qquad \begin{cases} \mathrm{i}\partial_t u(t,x)+\Delta_\beta u(t,x)=0, & (t,x)\in\mathbb{R}\times\mathbb{R}_+^{d+1}, \\ u_{|t=0}=u_0 \end{cases}$$

is related to uncertainty principles. In this regards we use uncertainty principles for the Weinstein transform proved in previous sections to obtain the following.

Proposition 9.

i) *Let u_0 be a mesurable function on $\mathbb{R}_+^{d+1}$ and $a,b>0$ such that*

$$u_0(x)=O(\mathrm{e}^{-a\|x\|^2}),\quad \mathrm{e}^{\mathrm{i}t\Delta_\beta}u_0(x)=O(\mathrm{e}^{-b\|x\|^2}).$$

If $ab>\frac{1}{16}t^{-2}$, then $u_0\equiv 0$. Moreover, if $ab=\frac{1}{16}t^{-2}$, then u is solution with initial data $C\mathrm{e}^{-(a+\mathrm{i}/(4t))\|x\|^2}$.

ii) *Let u_0 a mesurable function on $\mathbb{R}_+^{d+1}$ and $a,b>0$ such that*

$$e^{a\|x\|^2}u_0(x)\in L^p_\beta(\mathbb{R}_+^{d+1}),\quad \mathrm{e}^{b\|x\|^2}\mathrm{e}^{\mathrm{i}t\Delta_\beta}u_0(x)\in L^q_\beta(\mathbb{R}_+^{d+1})$$

with $p,q\in[1,\infty]$, with at least one of them finite. If $ab\geqslant\frac{1}{16}t^{-2}$, then $u_0\equiv 0$.

iii) *If $u_0\in L^2_\beta(\mathbb{R}_+^{d+1})$, $p\in(1,2)$, $1/p+1/q=1$, and $a,b>0$ such that for some $t\neq 0$*

$$\int_{\mathbb{R}_+^{d+1}}|u_0(x)|\mathrm{e}^{(2a)^p/p\|x\|^p}\,\mathrm{d}\mu_\beta(x)+\int_{\mathbb{R}^d}|\mathrm{e}^{\mathrm{i}t\Delta_\beta}u_0(x)|\mathrm{e}^{(2b)^q/(2t)^q q\|x\|^q}\,\mathrm{d}\mu_\beta(x)<\infty.$$

If $ab\geqslant\frac{1}{4}$, then $u_0\equiv 0$.

iv) *If* $u_0 \in L^2_\beta(\mathbb{R}^{d+1}_+)$ *such that for some* $t \neq 0$

$$\int_{\mathbb{R}^{d+1}_+ \times \mathbb{R}^{d+1}_+} |u_0(x)||\mathrm{e}^{\mathrm{i}t\Delta_\beta} u_0(y)|\mathrm{e}^{\|x\|\|y\|/(2t)} \,\mathrm{d}\mu_\beta(x)\,\mathrm{d}y < \infty,$$

then $u_0 \equiv 0$.

v) *Let* u_0 *a mesurable function on* $\mathbb{R}^{d+1}_+$ *such that*

$$\mathrm{e}^{a\|x\|^2} u_0 \in L^1_\beta(\mathbb{R}^{d+1}_+) + L^\infty_\beta(\mathbb{R}^{d+1}_+)$$

and

$$\int_{\mathbb{R}^{d+1}_+} \log^+ \frac{|\mathrm{e}^{\mathrm{i}t\Delta_\beta} u_0(\xi)\mathrm{e}^{b\|\xi\|^2}|}{\lambda} \,\mathrm{d}\xi < \infty,$$

for some constants $a > 0$, $b > 0$, $\lambda > 0$.

If $ab > \frac{1}{16}t^{-2}$, *then* $u_0 = 0$ *almost everywhere.*
If $ab = \frac{1}{16}t^{-2}$, *then* u *is solution with initial data* $C\mathrm{e}^{-(a+\mathrm{i}/(4t))\|x\|^2}$.

P r o o f. We only prove the estimate (i), the proofs of (ii)–(v) being similar. Set $h(y) = \mathrm{e}^{\mathrm{i}(\|y\|^2/(4t))} u_0(y)$. Then from (9.1) we get

$$u(t,x) = \frac{2}{\pi^{d/2}\Gamma(\beta+1)(4t)^{\beta+1+d/2}} \,\mathrm{e}^{-\mathrm{i}(d+2\beta+2)\frac{\pi}{4}\operatorname{sgn} t} \mathrm{e}^{\mathrm{i}\|\cdot\|^2/(4t)} \big[\mathscr{F}_W(h)\big]\Big(\frac{x}{2t}\Big).$$

From the hypothesis on u_0, we have

$$|\mathscr{F}_W(h)|\Big(\frac{x}{2t}\Big) \leqslant C\mathrm{e}^{-b\|x\|^2}.$$

Thus

$$|\mathscr{F}_W(h)|(x) \leqslant C\mathrm{e}^{-4bt^2\|x\|^2}.$$

Clearly $|h(y)| \leqslant C\mathrm{e}^{-a\|y\|^2}$. Now we apply Hardy's uncertainty principle for the Weinstein transform (cf. [12]) for h to obtain the result. □

We conclude this section by the following results concerning application of uncertainty principles to the generalized heat equation. Consider the initial value problem

$$\begin{cases} \partial_t u(t,x) - \Delta_\beta u(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^{d+1}_+, \\ u_{|t=0} = u_0. \end{cases} \tag{9.3}$$

Proposition 10. *Let* $u_0 \in L^1_\beta(\mathbb{R}^{d+1}_+)$ *and let* $u(t,x) = (u_0 *_W N_\beta(t,\cdot))(x)$ *be the solution of the problem* (9.3). *If* $s < t$ *and the following estimate*

$$|u(t,x)| \leqslant C(1+\|x\|^2)^m N_\beta(s,x)$$

holds, then $u \equiv 0$.

P r o o f. We use the relations (2.28), (2.30) and (2.31) we obtain

$$|\mathscr{F}_W(u(t,\cdot))(\xi)| \leqslant C\mathrm{e}^{-t\|\xi\|^2}.$$

On the other hand the relations (2.29), (2.30) give

$$|u(t,x)| \leqslant C(t)\mathrm{e}^{\frac{1}{4}t^{-1}\|x\|^2}.$$

Now we apply Corollary 1 and we obtain $u(t,x) = C(t)P(x)\mathrm{e}^{-\frac{1}{4}t^{-1}\|x\|^2}$. But this is not possible in view of the estimate on $u(t,x)$ unless $t \leqslant s$. □

Proposition 11. *Let* $u_0 \in \mathscr{E}'_*(\mathbb{R}^{d+1})$ *and let* $u(t,x) = (u_0 *_W N_\beta(t,\cdot))(x)$ *be the solution of the problem* (9.3). *If* $s < t$ *and the estimate*

$$|u(t,x)| \leqslant C(1+\|x\|^2)^m N_\beta(s,x)$$

holds, then $u \equiv 0$.

P r o o f. We use the fact that the Weinstein transform of u_0 has polynomial growth and so

$$|\mathscr{F}_W(u(t,\cdot))(\xi)| \leqslant C(1+\|\xi\|)^n \mathrm{e}^{-t\|\xi\|^2}.$$

Therefore, in this case too the solution u cannot have the decay

$$|u(t,x)| \leqslant C(1+\|x\|^2)^m N_\beta(s,x)$$

for any $s < t$. □

Acknowledgment. The authors are deeply indebted to the referee for providing constructive comments and help in improving the contents of this paper.

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