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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 4, 923–940

Persistent URL: http://dml.cz/dmlcz/141798

Czechoslovak Mathematical Journal, 61 (136) (2011), 923–940

COMPACT EMBEDDINGS OF BESOV SPACES INVOLVING ONLY SLOWLY VARYING SMOOTHNESS

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(Received June 18, 2010)

Abstract. We characterize compact embeddings of Besov spaces $B^{0,b}_{p,r}(\mathbb{R}^n)$ involving the zero classical smoothness and a slowly varying smoothness b into Lorentz-Karamata spaces $L_{p,q;\bar{b}}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ and $\bar{b}$ is another slowly varying function.

Keywords: Besov spaces with generalized smoothness, Lorentz-Karamata spaces, compact embeddings

MSC 2010: 46E35, 46E30

1. Introduction

Classical Besov spaces play a significant role in numerous parts of mathematics. These spaces are particular cases of Besov spaces of generalized smoothness. The latter spaces have been studied especially by the Soviet mathematical school (cf. [21, Sect. 8]). A lot of attention has been paid to optimal embeddings and to growth and continuity envelopes of such spaces (see, e.g., [15], [17], [22], [6], [7], [14], [2], [20], [5], [18], [19], [24, Chapt. 1], [16], [3], [4], etc.). This paper is a direct continuation of [4], where local embeddings of Besov spaces $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$ into classical Lorentz spaces were characterized. These results have been applied to establish sharp local embeddings of Besov spaces in question into Lorentz-Karamata spaces and to determine growth envelopes of spaces $B^{0,b}_{p,r}$. Besov spaces $B^{0,b}_{p,r}$ are defined by means of the modulus of continuity and they involve the zero classical smoothness and a slowly varying smoothness b.[1] In particular, the following two theorems are proved there.

[1] We refer to Section 2 for precise definitions.

Theorem 1.1 ([4, Theorem 3.3]). *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$, $0 < q \leqslant \infty$ and let b be a slowly varying function on the interval $(0,1)$ (notation $b \in SV(0,1)$) satisfying*

$$\|t^{-1/r} b(t)\|_{r,(0,1)} = \infty. \tag{1.1}$$

Put $b(t) = 1$ if $t \in [1,2)$. Define, for all $t \in (0,1)$,

$$b_r(t) := \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)} \tag{1.2}$$

and

$$\tilde{b}(t) := \begin{cases} b_r(t)^{1-r/q+r/\max\{p,q\}} b(t^{1/n})^{r/q-r/\max\{p,q\}} & \text{if } r \neq \infty \\ b_\infty(t) & \text{if } r = \infty. \end{cases} \tag{1.3}$$

Then the inequality

$$\|t^{1/p-1/q} \tilde{b}(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{1.4}$$

holds for all $f \in B^{0,b}_{p,r}$ if and only if $q \geqslant r$.

Theorem 1.2 ([4, Theorem 3.4(i)]). *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant q \leqslant \infty$ and let $b \in SV(0,1)$ satisfy (1.1). Put $b(t) = 1$ if $t \in [1,2)$, define b_r and $\tilde{b}$ by (1.2) and (1.3). Let κ be a non-negative and non-increasing function on $(0,1)$. Then the inequality*

$$\|t^{1/p-1/q} \tilde{b}(t) \kappa(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{1.5}$$

holds for all $f \in B^{0,b}_{p,r}$ if and only if κ is bounded.

In the whole paper we assume that any slowly varying function on $(0,1)$ is extended by 1 to the interval $(0,\infty)$.

Theorems 1.1 and 1.2 describe the *optimal* continuous embeddings of the Besov space $B^{0,b}_{p,r}(\mathbb{R}^n)$ into the Lorentz-Karamata space $L_{p,q;\tilde{b}}(\Omega)$, where Ω is a domain in $\mathbb{R}^n$ of finite Lebesgue measure. Namely, these theorems imply that[2]

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow L_{p,q;\tilde{b}}(\Omega) \tag{1.6}$$

and that this embedding is optimal within the scale of Lorentz-Karamata spaces.

The aim of this paper is to characterize *compact* embeddings of the Besov space $B^{0,b}_{p,r}(\mathbb{R}^n)$ into Lorentz-Karamata spaces. Our main result reads as follows.

[2] Note that (1.6) means that the mapping $u \mapsto u|_\Omega$ from $B^{0,b}_{p,r}(\mathbb{R}^n)$ into $L_{p,q;\tilde{b}}(\Omega)$ is continuous.

Theorem 1.3. *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant q \leqslant \infty$ and let $b \in SV(0,1)$ satisfy (1.1). Define functions b_r and $\tilde{b}$ by (1.2) and (1.3). Let Ω be a bounded domain in $\mathbb{R}^n$ and let $0 < P \leqslant p$. Assume that $\bar{b} \in SV(0,1)$ and, if $P = p > q$, that $\bar{b}/\tilde{b}$ is a non-negative and non-decreasing function on the interval $(0,\delta)$ for some $\delta \in (0,1)$. Then*[3]

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{P,q;\bar{b}}(\Omega) \tag{1.7}$$

if and only if

$$\lim_{t\to 0+} \frac{t^{1/P}\bar{b}(t)}{t^{1/p}\tilde{b}(t)} = 0. \tag{1.8}$$

In fact, Theorem 1.3 is a corollary of more general Theorems 3.3, 4.4 and Remark 3.4 below. The sufficiency part of Theorem 1.3 follows from Theorem 3.3 and Remark 3.4, while the necessity part from Theorem 4.4 and Remark 3.4.

In particular, Theorem 1.3 shows that the optimal embedding (1.6) is not compact. Such assertions about optimal embeddings of Sobolev-type spaces into Banach function spaces are known. It seems that the same is true for optimal embeddings of Besov-type spaces but it is almost impossible to find the corresponding references to a proof of this property in the existing literature. This is even the case of optimal embeddings of classical Besov spaces into Lebesgue spaces L_q with $q \in [1,\infty)$. (For example, in such a case the result can be proved by contradiction using [24, Proposition 4.6, p. 197], combined with the relationship between Besov and Triebel-Lizorkin spaces [23, (22), p. 96] and the fact that $F^0_{q2} = L_q$ if $1 < q < \infty$ [23, Remark 2, p. 25]). Note also that target spaces of our embeddings need not be Banach function spaces.

The paper is organized as follows. Section 2 contains notation and preliminaries. In Section 3 we prove the sufficiency part of Theorem 1.3, while Section 4 is devoted to the proof of the necessity part of this theorem.

2. Notation and preliminaries

Whenever convenient, we use the abbreviation LHS($*$) (RHS($*$)) for the left-(right-)hand side of the relation ($*$).

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[3] Note that (1.7) means that the mapping $u \mapsto u|_\Omega$ from $B^{0,b}_{p,r}(\mathbb{R}^n)$ into $L_{P,q;\bar{b}}(\Omega)$ is compact.

For two non-negative expressions $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \leqslant c\mathcal{B}$ (or $c\mathcal{A} \geqslant \mathcal{B}$), where c is a positive constant independent of the appropriate quantities involved in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

Given a set A, its characteristic function is denoted by χ_A. If $a \in \mathbb{R}^n$ and $r \geqslant 0$, the symbol $B(a,r)$ stands for the closed ball in $\mathbb{R}^n$ centred at a with the radius r. The volume of $B(0,1)$ in $\mathbb{R}^n$ is denoted by V_n though, in general, we use the notation $|\cdot|_n$ for the Lebesgue measure in $\mathbb{R}^n$.

Let Ω be a measurable subset of $\mathbb{R}^n$. The symbol $\mathcal{M}(\Omega)$ is used to denote the family of all complex-valued or extended real-valued (Lebesgue-)measurable functions defined a.e. on Ω. By $\mathcal{M}^+(\Omega)$ we mean the subset of $\mathcal{M}(\Omega)$ consisting of those functions which are non-negative a.e. on Ω. If $\Omega = (a,b)$, we write simply $\mathcal{M}(a,b)$ and $\mathcal{M}^+(a,b)$ instead of $\mathcal{M}(\Omega)$ and $\mathcal{M}^+(\Omega)$, respectively. By $\mathcal{M}^+(a,b;\downarrow)$ or $\mathcal{M}^+(a,b;\uparrow)$ we mean the collection of all $f \in \mathcal{M}^+(a,b)$ which are non-increasing or non-decreasing on (a,b), respectively. Finally, by $\mathcal{W}(\Omega)$ or by $\mathcal{W}(a,b)$ we denote the class of *weight functions* on Ω or on (a,b), consisting of all measurable functions which are positive a.e. on Ω or on (a,b), respectively. A subscript 0 is added to the previous notation (as in $\mathcal{M}_0(\Omega)$, for example) if one restricts to functions in the considered class which are finite a.e.

Given two quasi-Banach spaces X and Y, we write $X = Y$ (and say that X and Y coincide) if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ or $X \hookrightarrow\hookrightarrow Y$ means that $X \subset Y$ and the natural embedding of X in Y is continuous or compact, respectively.

Let either $a = 1$ or $a = \infty$. A function $b \in \mathcal{M}_0^+(0,a)$, $b \not\equiv 0$, is said to be *slowly varying* on $(0,a)$, notation $b \in SV(0,a)$, if, for each $\varepsilon > 0$, there are functions $g_\varepsilon \in \mathcal{M}_0^+(0,a;\uparrow)$ and $g_{-\varepsilon} \in \mathcal{M}_0^+(0,a;\downarrow)$ such that

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \quad \text{for all } t \in (0,a).$$

Let $p, q \in (0,\infty]$, let Ω be a domain in $\mathbb{R}^n$ and let $w \in \mathcal{W}(0, |\Omega|_n)$ be such that

$$W_{p,q;w}(t) := \|\tau^{1/p-1/q} w(\tau)\|_{q;(0,t)} < \infty \quad \text{for all } t \in (0, |\Omega|_n], \tag{2.1}$$

where $\|\cdot\|_{q;E}$ is the usual L_q-(quasi-)norm on the measurable set E. The *Lorentz-type space* $L_{p,q;w}(\Omega)$ consists of all (classes of) functions $f \in \mathcal{M}(\Omega)$ for which the quantity

$$\|f\|_{p,q;w;\Omega} := \|t^{1/p-1/q} w(t) f^*(t)\|_{q;(0,|\Omega|_n)} \tag{2.2}$$

is finite; here f^* denotes the *non-increasing rearrangement* of f given by

$$f^*(t) = \inf\{\lambda > 0 \colon |\{x \in \Omega \colon |f(x)| > \lambda\}|_n \leqslant t\}, \quad t \geqslant 0. \tag{2.3}$$

We shall also need the maximal function f^{**} of f^* defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad t > 0.$$

It is known (see, e.g., [9, Cor. 2] for the case $q \in (0,\infty)$) that the functional (2.2) is a *quasi-norm* on $L_{p,q;w}(\Omega)$ *if and only if* the function $W_{p,q;w}$ given by (2.1) satisfies

$$W_{p,q;w} \in \Delta_2, \tag{2.4}$$

that is, $W_{p,q;w}(2t) \lesssim W_{p,q;w}(t)$ for all $t \in (0, |\Omega|_n/2)$. One can easily verify that this is satisfied provided that

$$w(2t) \lesssim w(t) \quad \text{for a.e. } t \in (0, |\Omega|_n/2).$$

Moreover, since the relation $w \in \mathcal{W}(0,|\Omega|_n)$ yields $W_{p,q;w}(t) > 0$ for all $t \in (0, |\Omega|_n)$, one can prove that the space $L_{p,q;w}(\Omega)$ is *complete* (cf. the proof of [8, Prop. 2.2.9]).

If $q \in [1,\infty)$, the spaces $L_{p,q;w}(\Omega)$ coincide with the *classical Lorentz spaces* $\Lambda^q(\omega)$. On the other hand, if w is a slowly varying function, then $L_{p,q;w}(\Omega)$ is the so-called *Lorentz-Karamata* space introduced in [13]. The scale of Lorentz-Karamata spaces involves as particular cases a lot of well-known spaces (cf., e.g., [13], [11]).

If $\Omega = \mathbb{R}^n$, we sometimes omit this symbol in the notation and, for example, simply write $\|\cdot\|_{p,q;w}$ or $L_{p,q;w}$ instead of $\|\cdot\|_{p,q;w;\mathbb{R}^n}$ or $L_{p,q;w}(\mathbb{R}^n)$, respectively.

Definition 2.1. A subset K of a Lorentz-type space $Y = Y(\Omega)$, with $|\Omega|_n < \infty$, is said to have *a uniformly absolutely continuous quasi-norm* in the space Y, written $K \subset UAC(Y)$, if

$$\forall \varepsilon > 0, \exists \delta > 0\colon\ f \in K,\ |E|_n < \delta \Rightarrow \|f\chi_E\|_Y < \varepsilon.$$

Lemma 2.2 ([12, Lemma 2.2]). *Let $K \subset UAC(Y)$, where $Y = L_{p,q;w}(\Omega)$ is a Lorentz-type space such that $W_{p,q;w} \in \Delta_2$ and $\|\chi_\Omega\|_Y \equiv W_{p,q;w}(|\Omega|_n) < \infty$. Then every sequence $\{u_i\} \subset K$ which converges in measure on Ω converges also in the space Y.*

Given $f \in L_p$, $1 \leqslant p < \infty$, the *first difference operator* Δ_h of step $h \in \mathbb{R}^n$ transforms f to $\Delta_h f$ defined by

$$(\Delta_h f)(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n,$$

whereas the *modulus of continuity* of f is given by

$$\omega_1(f,t)_p := \sup_{\substack{h \in \mathbb{R}^n \\ |h| \leqslant t}} \|\Delta_h f\|_p, \quad t > 0.$$

Definition 2.3. Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$ and let $b \in SV(0,1)$ satisfy (1.1). The Besov space $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$ consists of those functions $f \in L_p$ for which the norm

$$\|f\|_{B^{0,b}_{p,r}} := \|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,1)} \tag{2.5}$$

is finite.

3. Proof of the sufficiency part of Theorem 1.3

We shall start with some auxiliary statements. Our first assertion is an analogue of the well-known result which states that the classical Besov space $B^s_{p,r}(\mathbb{R}^n)$ is compactly embedded into the Lebesgue space $L_p(\Omega)$ when Ω is a bounded domain in $\mathbb{R}^n$, $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$ and $s > 0$. While such a statement can be easily proved from the corresponding one for Sobolev spaces by interpolation of compactness, such an argument does not work in the limiting case when the classical Besov space is replaced by the Besov space $B^{0,b}_{p,r}(\mathbb{R}^n)$ involving only slowly varying smoothness. Nevertheless, the result continues to hold.

Lemma 3.1. *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$ and let $b \in SV(0,1)$ satisfy* (1.1). *If Ω is a bounded domain in $\mathbb{R}^n$, then*

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_p(\Omega).$$

Proof. Put $X := B^{0,b}_{p,r}(\mathbb{R}^n)$ and $B(R) := \{x \in \mathbb{R}^n \colon |x| < R\}$ for $R \in (0,\infty)$. Since Ω is bounded, there is $R_0 \in (0,\infty)$ such that $\overline{\Omega} \subset B(R_0)$. Take a Lipschitz continuous function φ in $\mathbb{R}^n$ satisfying

$$0 \leqslant \varphi \leqslant 1, \quad \varphi = 1 \quad \text{on } \Omega \qquad \text{and} \qquad \varphi = 0 \quad \text{on } \mathbb{R}^n \setminus B(R_0+1). \tag{3.1}$$

As

$$\|u\|_{p,\Omega} \leqslant \|\varphi u\|_{p,\mathbb{R}^n},$$

it is sufficient to prove that the set

$$K := \{\varphi u \colon \|u\|_X \leqslant 1\}$$

is precompact in $L_p(\mathbb{R}^n)$.

By [10, Thm. IV.8.21], it is enough to verify that

(i) K is bounded in $L_p(\mathbb{R}^n)$;

(ii) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\Delta_h(\varphi u)\|_{p,\mathbb{R}^n} < \varepsilon$$

for every $u \in X$, $\|u\|_X \leqslant 1$, and all $h \in \mathbb{R}^n$ with $|h| < \delta$;

(iii) given $\varepsilon > 0$, there exists $R_1 \in (0, \infty)$ such that

$$\|\varphi u\|_{p,\mathbb{R}^n \setminus B(R)} < \varepsilon$$

for every $u \in X$, $\|u\|_X \leqslant 1$, and all $R \in (R_1, \infty)$.

Condition (i) is satisfied, since

$$\|\varphi u\|_{p,\mathbb{R}^n} \lesssim \|u\|_{p,\mathbb{R}^n} \leqslant \|u\|_X \qquad \text{for all } u \in X.$$

Condition (iii) holds as well. Indeed, taking $R_1 := R_0 + 1$ and using (3.1), we obtain for all $u \in X$ and $R \in (R_1, \infty)$ that

$$\|\varphi u\|_{p,\mathbb{R}^n \setminus B(R)} \leqslant \|\varphi u\|_{p,\mathbb{R}^n \setminus B(R_0+1)} = 0$$

and condition (iii) follows.

To verify condition (ii), first note that, for all $u \in X$ and $x, h \in \mathbb{R}^n$,

$$\begin{aligned} |\Delta_h(\varphi u)(x)| &\leqslant \|\varphi\|_{\infty,\mathbb{R}^n} |\Delta_h u(x)| + \|\Delta_h \varphi\|_{\infty,\mathbb{R}^n} |u(x)| \\ &\lesssim |\Delta_h u(x)| + |h||u(x)|, \end{aligned}$$

which implies that

$$\|\Delta_h(\varphi u)\|_{p,\mathbb{R}^n} \lesssim \|\Delta_h u\|_{p,\mathbb{R}^n} + |h|\|u\|_{p,\mathbb{R}^n}. \tag{3.2}$$

Second, if $u \in X$ and $\|u\|_X \leqslant 1$, then, for any $T \in (0, 1)$,

$$1 \geqslant \|u\|_X \geqslant \|t^{-1/r} b(t) \omega_1(u, t)_p\|_{r,(T,1)} \geqslant \omega_1(u, T)_p \|t^{-1/r} b(t)\|_{r,(T,1)}.$$

Hence, for any $T \in (0, 1)$ and all $u \in X$ with $\|u\|_X \leqslant 1$,

$$\omega_1(u, T)_p \leqslant \|t^{-1/r} b(t)\|_{r,(T,1)}^{-1}.$$

Together with (1.1), this shows that given $\varepsilon_1 > 0$ there is $\delta_1 > 0$ such that

$$\|\Delta_h u\|_{p,\mathbb{R}^n} < \varepsilon_1 \tag{3.3}$$

for all $h \in \mathbb{R}^n$, $|h| < \delta_1$, and every $u \in X$ with $\|u\|_X \leqslant 1$. Now, making use of (3.2) and (3.3), we can easily verify condition (ii). $\square$

Lemma 3.2. *Let* $0 < q \leqslant \infty$, $I = (\alpha, \beta) \subset \mathbb{R}$, $w, v \in \mathcal{M}^+(I)$ *and let* $C \in (0, \infty)$. *Then*

$$\|fw\|_{q,I} \leqslant C\|fv\|_{q,I} \quad \text{for all } f \in \mathcal{M}^+(I;\downarrow) \tag{3.4}$$

if and only if

$$\|w\|_{q,(\alpha,t)} \leqslant C\|v\|_{q,(\alpha,t)} \quad \text{for all } t \in I. \tag{3.5}$$

P r o o f. To prove the necessity part, test inequality (3.4) with $f := \chi_{(\alpha,t)}$, where $t \in I$.

To prove the sufficiency part, we distinguish two cases:

(i) Let $0 < q < \infty$. Then the proof is analogous to that of [1, Chapt. 2, Prop. 3.6]. Start with

$$f^q = \sum_{j=1}^{k} c_j \chi_{(\alpha,t_j)},$$

where the coefficients c_j are positive and $\alpha < t_1 < \ldots < t_k < \beta$, and verify the result. Then apply the monotone convergence theorem to prove the general case.

(ii) Let $q = \infty$. Put $W(t) := \|w\|_{\infty,(\alpha,t)}$, $t \in I$. Since $f \in \mathcal{M}^+(I;\downarrow)$, exchanging the essential suprema, we obtain that

$$\|fW\|_{\infty,I} = \|fw\|_{\infty,I}. \tag{3.6}$$

Moreover, by (3.5),

$$f(t)W(t) \leqslant Cf(t)\|v\|_{\infty,(\alpha,t)} \leqslant C\|fv\|_{\infty,(\alpha,t)} \leqslant C\|fv\|_{\infty,I}$$

for a.e. $t \in I$. Consequently,

$$\|fW\|_{\infty,I} \leqslant C\|fv\|_{\infty,I} \quad \text{for all } f \in \mathcal{M}^+(I;\downarrow).$$

Together with (3.6), this yields the result. $\square$

Theorem 3.3. *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant q \leqslant \infty$ and let $b \in SV(0,1)$ satisfy* (1.1). *Define b_r and $\tilde{b}$ by* (1.2) *and* (1.3). *Let Ω be a bounded domain in $\mathbb{R}^n$, $0 < P \leqslant p$ and let $w \in \mathcal{W}(0, |\Omega|_n)$ be such that the function*

$$W_{P,q,w}(t) := \|\tau^{1/P-1/q} w(\tau)\|_{q,(0,t)}, \qquad t \in (0, |\Omega|_n],$$

satisfies

$$W_{P,q,w} \in \Delta_2 \qquad \text{and} \qquad W_{P,q,w}(|\Omega|_n) < \infty.$$

If

$$\lim_{t\to 0+} \frac{W_{P,q,w}(t)}{W_{p,q,\tilde{b}}(t)} = \lim_{t\to 0+} \frac{\|\tau^{1/P-1/q} w(\tau)\|_{q,(0,t)}}{\|\tau^{1/p-1/q} \tilde{b}(\tau)\|_{q,(0,t)}} = 0, \tag{3.7}$$

then

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{P,q,w}(\Omega).$$

Proof. By Theorem 1.2,

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow L_{p,q,\tilde{b}}(\Omega). \tag{3.8}$$

Put

$$K := \{u \in B^{0,b}_{p,r}(\mathbb{R}^n) \colon \|u\|_{B^{0,b}_{p,r}(\mathbb{R}^n)} \leqslant 1\}.$$

If $\{u'_i\}_{i\in\mathbb{N}} \subset K$, then Lemma 3.1 implies that there is a subsequence $\{u_i\}_{i\in\mathbb{N}} \subset \{u'_i\}_{i\in\mathbb{N}}$ such that $u_i \to u$ in $L_p(\Omega)$. Thus, by [1, Chapt. 1, Thm. 1.4], $u_i \overset{\text{meas}}{\longrightarrow} u$ on Ω. In view of Lemma 2.2, it is sufficient to show that

$$K \subset UAC(L_{P,q,w}(\Omega)). \tag{3.9}$$

Let $\varepsilon > 0$. By (3.7), there is $\delta \in (0, |\Omega|_n)$ such that

$$\|\tau^{1/P-1/q} w(\tau)\|_{q,(0,t)} \leqslant \varepsilon \|\tau^{1/p-1/q} \tilde{b}(\tau)\|_{q,(0,t)} \qquad \text{for all } t \in (0, \delta]. \tag{3.10}$$

Assume that $u \in K$ and $M \subset \Omega$ with $|M|_n < \delta$. Since $(u\chi_M)^* \leqslant u^* \chi_{[0,\delta)}$, we obtain

$$\|u\chi_M\|_{P,q,w;\Omega} \leqslant \|t^{1/P-1/q} w(t) u^*(t)\|_{q,(0,\delta)}. \tag{3.11}$$

Moreover, using (3.10) and Lemma 3.2, we arrive at

$$\|t^{1/P-1/q} w(t) u^*(t)\|_{q,(0,\delta)} \leqslant \varepsilon \|t^{1/p-1/q} \tilde{b}(t) u^*(t)\|_{q,(0,\delta)} \tag{3.12}$$

for all $u \in L_{p,q,\tilde{b}}(\Omega)$. Estimates (3.11), (3.12) and embedding (3.8) imply that

$$\|u\chi_M\|_{P,q,w;\Omega} \lesssim \varepsilon \|u\|_{B^{0,b}_{p,r}(\mathbb{R}^n)} \leqslant \varepsilon \qquad \text{for all } u \in K$$

and (3.9) follows. □

Remark 3.4. (i) Let $P = p \in (0,\infty)$ and $w = \bar{b} \in SV(0,|\Omega|_n)$. Then

$$\frac{W_{P,q,w}(t)}{W_{p,q,\tilde{b}}(t)} \approx \frac{t^{1/P}\bar{b}(t)}{t^{1/p}\tilde{b}(t)} = \frac{\bar{b}(t)}{\tilde{b}(t)} \qquad \text{for all } t \in (0,|\Omega|_n)$$

and (3.7) is equivalent to

$$\lim_{t\to 0+} \frac{\bar{b}(t)}{\tilde{b}(t)} = 0. \tag{3.13}$$

(ii) Let $0 < P < p < \infty$ and $w = \bar{b} \in SV(0,|\Omega|_n)$. Then

$$\frac{W_{P,q,w}(t)}{W_{p,q,\tilde{b}}(t)} \approx \frac{t^{1/P}\bar{b}(t)}{t^{1/p}\tilde{b}(t)} \qquad \text{for all } t \in (0,|\Omega|_n)$$

and condition (3.7) holds since

$$\lim_{t\to 0+} \frac{W_{P,q,w}(t)}{W_{p,q,\tilde{b}}(t)} = \lim_{t\to 0+} \frac{t^{1/P}\bar{b}(t)}{t^{1/p}\tilde{b}(t)} = 0.$$

Proof of the sufficiency part of Theorem 1.3. The sufficiency part of Theorem 1.3 follows from Theorem 3.3 and Remark 3.4. □

4. Proof of the necessity part of Theorem 1.3

We start with some auxiliary results.

Lemma 4.1 ([3, Proposition 3.5]). (i) *Let $f \in L_1(\mathbb{R}^n)$ and let $F(x) := f^*(V_n|x|^n)$, $x \in \mathbb{R}^n$. Then*

$$\begin{aligned}\omega_1(F,t)_1 &\lesssim n\int_0^{t^n} f^*(s)\,\mathrm{d}s + (n-1)t\int_{t^n}^\infty f^*(s)s^{-1/n}\,\mathrm{d}s\\ &= t\biggl(\int_{t^n}^\infty s^{-1/n}\int_0^s (f^*(u)-f^*(s))\,\mathrm{d}u\,\frac{\mathrm{d}s}{s}\biggr)\end{aligned}$$

for all $t > 0$ and $f \in L_1(\mathbb{R}^n)$.

(ii) *Let $1 < p < \infty$, $f \in L_p(\mathbb{R}^n)$ and let $F(x) = f^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$. Then*

$$\omega_1(F,t)_p \lesssim t\biggl(\int_{t^n}^\infty s^{-p/n}\int_0^s (f^*(u)-f^*(s))^p\,\mathrm{d}u\,\frac{\mathrm{d}s}{s}\biggr)^{1/p}$$

for all $t > 0$ and $f \in L_p(\mathbb{R}^n)$.

Making use of Lemma 4.1, one can prove the next statement.

Lemma 4.2. *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$ and let $b \in SV(0,1)$ satisfy (1.1). If $f \in L_p(0,1)$ and the function F is defined on $\mathbb{R}^n$ by*

$$F(x) = f^*(V_n|x|^n) \quad \text{when } p = 1 \quad \text{and} \quad F(x) = f^{**}(V_n|x|^n) \quad \text{when } 1 < p < \infty,$$

then

$$\|F\|_{B^{0,b}_{p,r}} \lesssim \left\| t^{1-1/r} b(t) \left(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\frac{\mathrm{d}s}{s} \right)^{1/p} \right\|_{r,(0,1)}.$$

Proof. Let $f \in L_p(0,1)$. Then $f^*(t) = 0$ for $t \geqslant 1$. Therefore, $\int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u = \int_0^s f^*(u)^p \,\mathrm{d}u = \int_0^1 f^*(u)^p \,\mathrm{d}u$ when $s \in (1,\infty)$. Hence,

$$\begin{aligned} \|f\|_p &= \left(\int_0^1 f^*(u)^p \,\mathrm{d}u \right)^{1/p} \\ &\approx \left(\int_1^2 s^{-p/n-1} \,\mathrm{d}s \int_0^1 f^*(u)^p \,\mathrm{d}u \right)^{1/p} \\ &\approx \|t^{1-1/r} b(t)\|_{r,(0,1)} \left(\int_1^2 s^{-p/n-1} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\mathrm{d}s \right)^{1/p} \\ &\leqslant \left\| t^{1-1/r} b(t) \left(\int_{t^n}^2 s^{-p/n-1} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\mathrm{d}s \right)^{1/p} \right\|_{r,(0,1)}. \end{aligned} \tag{4.1}$$

Moreover, using (4.1), we obtain

$$\begin{aligned} &\left\| t^{1-1/r} b(t) \left(\int_{t^n}^\infty s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\frac{\mathrm{d}s}{s} \right)^{1/p} \right\|_{r,(0,1)} \\ &\qquad \leqslant \left\| t^{1-1/r} b(t) \left(\int_{t^n}^1 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\frac{\mathrm{d}s}{s} \right)^{1/p} \right\|_{r,(0,1)} \\ &\qquad\quad + \left\| t^{1-1/r} b(t) \left(\int_1^\infty s^{-p/n} \int_0^1 f^*(u)^p \,\mathrm{d}u \,\frac{\mathrm{d}s}{s} \right)^{1/p} \right\|_{r,(0,1)} \\ &\qquad \lesssim \left\| t^{1-1/r} b(t) \left(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\frac{\mathrm{d}s}{s} \right)^{1/p} \right\|_{r,(0,1)}. \end{aligned} \tag{4.2}$$

Now, since $\|F\|_p \lesssim \|f\|_p$, the desired result follows from (2.5), Lemma 4.1, [1, Chapt. 2, Corollary 7.8] and estimates (4.1) and (4.2). $\square$

We shall also need the following assertion.

Lemma 4.3. *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant \infty$, and let $b \in SV(0,1)$. Then*

$$\begin{aligned}\Big\| t^{1-1/r} b(t) \Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \,\mathrm{d}u \,\frac{\mathrm{d}s}{s}\Big)^{1/p} \Big\|_{r,(0,1)} \\ \approx \|f\|_p + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p \,\mathrm{d}u \Big)^{1/p} \Big\|_{r,(0,1)} \\ \approx \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t f^*(u)^p \,\mathrm{d}u \Big)^{1/p} \Big\|_{r,(0,2)}\end{aligned}$$

for all $f \in L_p(0,1)$.

P r o o f. Lemma is a consequence of [4, Lemmas 4.4 and 4.6] and [1, Chapt. 2, Corollary 7.8]. $\square$

Theorem 4.4. *Let $1 \leqslant p < \infty$, $1 \leqslant r \leqslant q \leqslant \infty$ and let $b \in SV(0,1)$ satisfy* (1.1). *Define b_r and $\tilde{b}$ by* (1.2) *and* (1.3). *Let Ω be a bounded domain in $\mathbb{R}^n$, $0 < P \leqslant p$ and let $w \in \mathcal{W}(0, |\Omega|_n)$ be such that the function*

$$W_{P,q,w}(t) := \|\tau^{1/P-1/q} w(\tau)\|_{q,(0,t)}, \qquad t \in (0, |\Omega|_n],$$

satisfies

$$W_{P,q,w} \in \Delta_2 \qquad \text{and} \qquad W_{P,q,w}(|\Omega|_n) < \infty.$$

If

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{P,q,w}(\Omega), \tag{4.3}$$

then (3.7) *holds provided that one of the following conditions is satisfied:*

(A) $p \leqslant q$;

(B) $q < p$, $P < p$,
there exists $\bar{b} \in SV(0,\delta)$ with $\delta \in (0, |\Omega|_n)$ such that $w = \bar{b}$ on $(0,\delta)$;

(C) $q < p$, $P = p$,
there exists $\bar{b} \in SV(0,\delta)$ with $\delta \in (0, |\Omega|_n)$ such that $w = \bar{b}$ on $(0,\delta)$ and $\bar{b}/\tilde{b} \in \mathcal{M}^+_0(0,\delta;\uparrow)$.

P r o o f. Without loss of generality, we can suppose that $|\Omega|_n = 1$.

Assume that (4.3) holds but (3.7) does not and seek for a contradiction. It is enough to find a sequence $\{F_k\}_{k\in\mathbb{N}} \subset B^{0,b}_{p,r}(\mathbb{R}^n)$ with

$$\sup_{k\in\mathbb{N}} \|F_k\|_{B^{0,b}_{p,r}(\mathbb{R}^n)} \lesssim 1 \tag{4.4}$$

such that $\{F_k\}_{k\in\mathbb{N}}$ has no convergent subsequence in $L_{P,q,w}(\Omega)$.

To this end, it is sufficient to construct a sequence $\{F_k\}_{k\in\mathbb{N}} \subset B^{0,b}_{p,r}(\mathbb{R}^n)$ satisfying (4.4) and

$$\|F_k\|_{P,q,w,\Omega} \gtrsim 1 \quad \text{for all sufficiently large } k \in \mathbb{N}, \tag{4.5}$$

$$F_k \xrightarrow{\text{meas}} 0 \quad \text{on } \Omega. \tag{4.6}$$

Indeed, suppose that $F \in L_{P,q,w}(\Omega)$ is the limit of a convergent subsequence $\{F_{\sigma(k)}\}_{k\in\mathbb{N}}$ in the space $L_{P,q,w}(\Omega)$, that is,

$$\|F_{\sigma(k)}(x) - F(x)\|_{P,q;w;\Omega} \to 0 \quad \text{as } k \to \infty. \tag{4.7}$$

Then, by (2.1) and (2.2),

$$\begin{aligned} W_{P,q;w}(|\{x \in \Omega\colon\ |F_{\sigma(k)}(x) - F(x)| > \alpha\}|_n) \\ &= \|\tau^{1/P-1/q} w(\tau)\|_{q;\ (0,|\{x\in\Omega\colon\ |F_{\sigma(k)}(x)-F(x)|>\alpha\}|_n)} \\ &= \|\tau^{1/P-1/q} w(\tau)\chi^*_{\{x\in\Omega\colon\ |F_{\sigma(k)}(x)-F(x)|>\alpha\}}(\tau)\|_{q;(0,|\Omega|_n)} \\ &= \|\chi_{\{x\in\Omega\colon\ |F_{\sigma(k)}(x)-F(x)|>\alpha\}}\|_{P,q;w;\Omega} \\ &\leqslant \alpha^{-1}\|F_{\sigma(k)}(x) - F(x)\|_{P,q;w;\Omega}. \end{aligned} \tag{4.8}$$

Since the function $W_{P,q,w}$ satisfies $W_{P,q,w}(t) > 0$ if $t \in (0, |\Omega|_n]$, (4.8) and (4.7) imply that $F_{\sigma(k)} \xrightarrow{\text{meas}} F$ on Ω (otherwise (4.8) and (4.7) lead to a contradiction). Together with (4.6), this means that $F = 0$ a.e. on Ω, which contradicts (4.5).

So, to prove our theorem, we will construct a sequence $\{F_k\}_{k\in\mathbb{N}} \subset B^{0,b}_{p,r}(\mathbb{R}^n)$ satisfying (4.4), (4.5) and (4.6).

As (3.7) does not hold, there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset (0,1)$, $t_{k+1} < t_k$, $k \in \mathbb{N}$, $\lim\limits_{k\to\infty} t_k = 0$, satisfying

$$W_{P,q,w}(t_k) \gtrsim W_{p,q,\tilde{b}}(t_k) \quad \text{for all } k \in \mathbb{N}. \tag{4.9}$$

Let $(f_k)_{k\in\mathbb{N}} \subset L_p(0,1)$. If $p = 1$, put $F_k := f_k^*(V_n|x|^n)$, $x \in \mathbb{R}^n$, and, if $1 < p < \infty$, put $F_k := f_k^{**}(V_n|x|^n)$, $x \in \mathbb{R}^n$. Then, by Lemmas 4.2 and 4.3,

$$\begin{aligned} \|F_k\|_{B^{0,b}_{p,r}(\mathbb{R}^n)} &\lesssim \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(0,2)} \\ &\lesssim \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(0,t_k)} \\ &\quad + \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(t_k,2)} \\ &=: N_1 + N_2. \end{aligned} \tag{4.10}$$

(i) Let $p \leqslant q$. Put $f_k(t) := t_k^{-1/p} b_r(t_k)^{-1} \chi_{(0,t_k]}(t)$, $t \in (0,1)$. Then

$$
\begin{aligned}
(4.11)\qquad N_1 &= t_k^{-1/p} b_r(t_k)^{-1} \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t (\chi_{(0,t_k]}(u))^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(0,t_k)} \\
&= t_k^{-1/p} b_r(t_k)^{-1} \| t^{1/p-1/r} b(t^{1/n}) \|_{r,(0,t_k)} \\
&\approx b_r(t_k)^{-1} b(t_k^{1/n}) \lesssim 1 \quad \text{for all } k \in \mathbb{N}.
\end{aligned}
$$

(The last estimate in (4.11) follows from the properties of slowly varying functions—cf. [4, Lemma 2.2, part 7].) Moreover, for all $k \in \mathbb{N}$,

$$
(4.12)\qquad N_2 = t_k^{-1/p} b_r(t_k)^{-1} \left(\int_0^{t_k} \mathrm{d}u \right)^{1/p} \| t^{-1/r} b(t^{1/n}) \|_{r,(t_k,2)} = 1.
$$

Thus, by (4.10)–(4.12), condition (4.4) is satisfied.

On the other hand, for all $k \in \mathbb{N}$,

$$
\begin{aligned}
(4.13)\qquad \|F_k\|_{P,q,w,\Omega} &= \| t^{1/P-1/q} w(t) F_k^*(t) \|_{q,(0,1)} \\
&\geqslant \| t^{1/P-1/q} w(t) f_k(t) \|_{q,(0,t_k)} \\
&= f_k(t_k) \| t^{1/P-1/q} w(t) \|_{q,(0,t_k)} \\
&= t_k^{-1/p} b_r(t_k)^{-1} W_{P,q,w}(t_k).
\end{aligned}
$$

Using estimate (4.9), the facts that $\tilde{b} = b_r$ if $p \leqslant q$ and $W_{p,q,\tilde{b}}(t_k) \approx t_k^{1/p} b_r(t_k)$ for all $k \in \mathbb{N}$, we obtain from (4.13) that (4.5) holds.

Given any $\alpha > 0$, we have

$$
\begin{aligned}
|\{x \in \Omega\colon\ |F_k(x)| > \alpha\}|_n &= |\{t \in (0,1)\colon\ F_k^*(t) > \alpha\}|_1 \\
&= |\{t \in (0,t_k)\colon\ t_k^{-1/p} b_r(t_k)^{-1} > \alpha\}|_1 \\
&\quad + \chi_{(1,\infty)}(p) |\{t \in (t_k,1)\colon\ t^{-1} t_k^{1-1/p} b_r(t_k)^{-1} > \alpha\}|_1 \\
&\leqslant t_k + \chi_{(1,\infty)}(p) \alpha^{-1} t_k^{1-1/p} b_r(t_k)^{-1}.
\end{aligned}
$$

Thus, using the properties of slowly varying functions, we see that (4.6) is satisfied.

(ii) Let (B) hold. Together with the assumption $r \leqslant q$, this shows that $r < \infty$. Take $\gamma > 0$ and put $f_k(t) := b_r(t_k)^\gamma \varphi(t) \chi_{(0,t_k)}$, $t \in (0,1)$, where $\varphi \in \mathcal{M}_0^+(0,1;\downarrow)$ and $\varphi(t) \approx t^{-1/p} b_r(t)^{-\gamma-1-r/p} b(t^{1/n})^{r/p}$ for all $t \in (0,1)$.

It is easy to verify that, given $\beta > 0$, then

$$
(4.14)\qquad I_\beta(t) := \int_0^t u^{-1} b_r(u)^{-\beta-r} b(u^{1/n})^r \,\mathrm{d}u \approx b_r(t)^{-\beta} \quad \text{for all } t \in (0,1).
$$

Using this estimate, we obtain

$$(4.15) \qquad \begin{aligned} N_1 &= \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(0,t_k)} \\ &\approx b_r(t_k)^\gamma \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t \varphi(u)^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(0,t_k)} \\ &= b_r(t_k)^\gamma \| t^{-1/r} b(t^{1/n}) (I_{(\gamma+1)p}(t))^{1/p} \|_{r,(0,t_k)} \\ &\approx b_r(t_k)^\gamma \| t^{-1/r} b(t^{1/n}) b_r(t)^{-(\gamma+1)} \|_{r,(0,t_k)} \\ &= b_r(t_k)^\gamma (I_{\gamma r}(t_k))^{1/r} \\ &\approx b_r(t_k)^\gamma b_r(t_k)^{-\gamma} \\ &= 1 \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

Moreover,

$$(4.16) \qquad \begin{aligned} N_2 &= \left\| t^{-1/r} b(t^{1/n}) \left(\int_0^t f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} \right\|_{r,(t_k,2)} \\ &= \left(\int_0^{t_k} f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} b_r(t_k), \quad k \in \mathbb{N}. \end{aligned}$$

Since

$$\begin{aligned} \left(\int_0^{t_k} f_k^*(u)^p \,\mathrm{d}u \right)^{1/p} &\approx b_r(t_k)^\gamma \left(\int_0^{t_k} \varphi(u)^p \,\mathrm{d}u \right)^{1/p} \\ &= b_r(t_k)^\gamma (I_{(\gamma+1)p}(t_k))^{1/p} \\ &\approx b_r(t_k)^\gamma b_r(t_k)^{-(\gamma+1)} \\ &= b_r(t_k)^{-1} \quad \text{for all } k \in \mathbb{N}, \end{aligned}$$

(4.16) implies that

$$(4.17) \qquad N_2 \approx 1 \quad \text{for all } k \in \mathbb{N}.$$

By (4.10), (4.15) and (4.17), condition (4.4) is satisfied.

Assumption (1.1) implies that given any $k \in \mathbb{N}$, there exists $s_k \in \{t_{k+j} \colon j \in \mathbb{N}\}$ such that

$$(4.18) \qquad \frac{b_r(t_k)}{b_r(s_k)} \leqslant 2^{-1/(\gamma q)}.$$

Let $k_0 \in \mathbb{N}$ be such that $t_{k_0} \leqslant \delta$, $\mathbb{K}_0 := \{k \in \mathbb{N}\colon\ k \geqslant k_0\}$ (recall that δ is the number from condition (B)). Putting

$$M_k := \inf_{t\in(s_k,t_k)} t^{1/P-1/p}\frac{\bar b(t)}{\tilde b(t)}, \quad k \in \mathbb{K}_0,$$

and using the fact that the function

$$t \longmapsto t^{1/P-1/p}\frac{\bar b(t)}{\tilde b(t)}, \quad t \in (0,\delta), \tag{4.19}$$

is equivalent to a non-decreasing function on $(0,\delta)$, we obtain

$$M_k \gtrsim s_k^{1/P-1/p}\frac{\bar b(s_k)}{\tilde b(s_k)} \approx \frac{W_{P,q,\bar b}(s_k)}{W_{p,q,\tilde b}(s_k)} \quad \text{for all } k \in \mathbb{K}_0. \tag{4.20}$$

Now, making use of the definition of F_k and condition (B), we obtain, for all $k \in \mathbb{K}_0$,

$$\begin{aligned} \|F_k\|_{P,q,w,\Omega} &= \|t^{1/P-1/q}w(t)F_k^*(t)\|_{q,(0,1)} \\ &\gtrsim b_r(t_k)^\gamma \|t^{1/P-1/q}w(t)\varphi(t)\|_{q,(0,t_k)} \\ &\geqslant b_r(t_k)^\gamma \|t^{1/P-1/q}w(t)\varphi(t)\|_{q,(s_k,t_k)} \\ &\geqslant b_r(t_k)^\gamma \|t^{1/p-1/q}\tilde b(t)\varphi(t)\|_{q,(s_k,t_k)}\cdot M_k. \end{aligned} \tag{4.21}$$

As $r \leqslant q < p$, $\tilde b(t) = b_r(t)^{1-r/q+r/p} b(t^{1/n})^{r/q-r/p}$ for all $t \in (0,1)$. Using also the definition of φ, we arrive at

$$\|t^{1/p-1/q}\tilde b(t)\varphi(t)\|_{q,(s_k,t_k)} = \|t^{-1/q}b_r(t)^{-\gamma-r/q}b(t^{1/n})^{r/q}\|_{q,(s_k,t_k)}, \quad k \in \mathbb{N}. \tag{4.22}$$

By a change of variables and (4.18),

$$\begin{aligned} \mathrm{RHS}(4.22) &= \Big\{\frac{r}{\gamma q} b_r(t_k)^{-\gamma q}\Big[1-\Big(\frac{b_r(t_k)}{b_r(s_k)}\Big)^{\gamma q}\Big]\Big\}^{1/q} \\ &\geqslant \frac{r}{\gamma q} b_r(t_k)^{-\gamma}\Big(\frac{1}{2}\Big)^{1/q} \\ &\approx b_r(t_k)^{-\gamma} \quad \text{for all } k \in \mathbb{N}. \end{aligned}$$

Thus,

$$\|t^{1/p-1/q}\tilde b(t)\varphi(t)\|_{q,(s_k,t_k)} \gtrsim b_r(t_k)^{-\gamma} \quad \text{for all } k \in \mathbb{K}_0.$$

Together with (4.21), (4.20) and (4.9) (and the hypothesis (B)), this implies that

$$\|F_k\|_{P,q,w,\Omega} \gtrsim 1 \quad \text{for all } k \in \mathbb{K}_0,$$

which means that (4.5) holds.

Let $\alpha > 0$. Applying Hölder's inequality and (4.14), we get (with a convenient positive constant c) that, for all $k \in \mathbb{N}$,

$$\begin{aligned}|\{x \in \Omega\colon\ |F_k(x)| > \alpha\}|_n &= |\{t \in (0,1)\colon\ F_k^*(t) > \alpha\}|_1 \\ &\leqslant t_k + \chi_{(1,\infty)}(p)|\{t \in (t_k,1)\colon\ ct^{-1}t_k^{1-1/p}b_r(t_k)^{-1} > \alpha\}|_1 \\ &\leqslant t_k + \chi_{(1,\infty)}(p)c\alpha^{-1}t_k^{1-1/p}b_r(t_k)^{-1}.\end{aligned}$$

Thus, using the properties of slowly varying functions, we see that (4.6) holds.

(iii) Let (C) hold. The proof is the same as that of part (ii). Note that now the function (4.19) is non-decreasing on $(0,\delta)$ by our assumption in (C). □

P r o o f of the necessity part of Theorem 1.3. The necessity part of Theorem 1.3 follows from Theorem 4.4 and Remark 3.4. □

Acknowledgement. This work was partially supported by grant no. 201/08/0383 of the Grant Agency of the Czech Republic, by the Institutional Research Plan no. AV0Z10190503 of the Academy of Sciences of the Czech Republic (AS CR), by a joint project between AS CR and *Fundação para a Ciência e a Tecnologia* (Portugal), and by *Centro de I&D em Matemática e Aplicações* (formerly *Unidade de Investigação Matemática e Aplicações*) of University of Aveiro.

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