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A NOTE ON THE MEAN VALUE OF THE GENERAL KLOOSTERMAN SUMS

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Abstract. The main purpose of this paper is to use the analytic method to study the calculating problem of the general Kloosterman sums, and give an exact calculating formula for it.

Keywords: the general Kloosterman sums, mean value, calculating formula

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1. Introduction

Let $q \geqslant 2$ be an integer, and let χ denote a Dirichlet character modulo q. For any integers m and n, the general Kloosterman sum $S(m,n,\chi;q)$ is defined by:

$$S(m,n,\chi;q)=\sum_{a=1}^{q}\chi(a)e\Big(\frac{ma+n\overline{a}}{q}\Big),$$

where $\overline{a}$ denotes the inverse of a modulo q and $e(y)=\mathrm{e}^{2\pi\mathrm{i}y}$. This summation is very important, because it is a generalization of the classical Kloosterman sums. Many authors studied the properties of $S(m,n,\chi;q)$. For instance, Chowla [2] proved that if χ is not the Legendre symbol, p is a prime and $(a,p)=1$, then

$$|S(1,a,\chi,p)| \leqslant 2p^{1/2}.$$

A. V. Malyshev [4] gave a sharper upper bounds for $|S(m,n,\chi;q)|$ only if χ is the Legendre symbol, but for arbitrary q. That is,

$$|S(m,n,\chi;p)| \ll (m,n,p)^{1/2}p^{1/2+\varepsilon},$$

where p is a prime, ε is any fixed positive number, and (m, n, p) denotes the greatest common divisor of m, n and p. H. M. Andruhaev [1] obtained a more general estimate for $S(m, n, \chi; q)$ provided χ satisfies some conditions. That is, suppose r is a positive integer such that $r^2 | \frac{q}{u}$, and χ is a character mod r. Then

$$|S(m, n, \chi; q)| \ll d(q)(uq)^{1/2},$$

where $d(q)$ is the divisor function.

However, for an arbitrary composite number q, we do not know how large $|S(m, n, \chi; q)|$ is. In fact, the value of $|S(m, n, \chi; q)|$ is quite irregular if q is not a prime. Fortunately, the mean value of it shows a good distribution properties. The mean value of $|S(m, n, \chi; q)|$ was investigated by many authors:

Zhang Wenpeng [6] studied the fourth power mean of $|S(m, n, \chi; q)|$, and proved

$$\sum_{\chi \bmod q} \sum_{m=1}^{q} |S(m, n, \chi; q)|^4 = \varphi^2(q) q^2 d(q) \prod_{p^\alpha \| q} \left(1 - \frac{2}{\alpha+1} \frac{p^{\alpha-1}-1}{p^\alpha(p-1)} + \frac{\alpha - 4p^{\alpha-1}}{(\alpha+1)p^\alpha}\right),$$

where $\varphi(q)$ is the Euler function, and $\prod\limits_{p^\alpha \| q}$ denotes the product over all p such that $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

Xu Zhefeng [5] obtained the identity for square-full number q. It reads

$$\sum_{m=1}^{q} \sum_{\chi \bmod q} |S(m, n, \chi; q)|^4 = \varphi^3(q) q d(q) \prod_{p^\alpha \| q} \left(1 - \frac{2(k, p-1) - 1}{(\alpha+1)(p-1)}\right).$$

It is natural to study the mean value of the general Kloosterman sums

$$\sum_{m=1}^{q}{}' |S(m, n, \chi, q)|^4 .$$

Here $\sum\limits_{m}{}'$ denotes that the sum is extended over those m that are relatively prime to q.

For any prime p, Zhang Wenpeng [7] studied the fourth power mean of $|S(m, n, \chi; p)|$ and gave a calculating formula

$$\sum_{m=1}^{p} |S(m, n, \chi; p)|^4 = \begin{cases} p(2p^2 - 3p - 3), & \text{if } \chi \text{ is a principal character modulo } p; \\ p^2(3p - 7), & \text{if } \chi \text{ is the Legendre symbol;} \\ 2p^2(p - 3), & \text{if } \chi \text{ is a complex character modulo } p. \end{cases}$$

For p^2, Liu Huaning [3] gave a formula for classic Kloosterman sums $S(m,1,p^2)$. Namely, for the principal character χ_0 of general Kloosterman sums, we obtained

$$\sum_{m=1}^{p^2}{}' \left|S(m,1,\chi_0,p^2)\right|^4 = 3\varphi(p^6).$$

This paper is a continuation of [7] and [3], but the method used in it is different. That is, we will use properties of primitive characters and analytic methods to study the general Kloosterman sums for a composite number q and for a non-principal character χ modulo q, and give an exact formula as follows:

Theorem. *Let q be a square-full number and $2 \nmid q$. Then for any non-primitive $\chi \neq \chi_0$ modulo q, we have the identity*

$$\sum_{m=1}^{q}{}' \left|S(m,1,\chi,q)\right|^4 = q^{\frac{3}{2}} \prod_{p|q} (p^3 + p^2 - 2p + 1).$$

Remark. Our method also works for the general integer, but the mathematical expression is rather complicated. So we do not give the general conclusion in this paper.

For a primitive character χ modulo q, whether there exists an exact formula for

$$\sum_{m=1}^{q}{}' \left|S(m,1,\chi,q)\right|^4$$

is an open problem.

2. PROOF OF THEOREM

Based on the arithmetical fundamental theorem and the multiplicative properties of the general Kloosterman sums, it is sufficient to treat every square-full number q as p^2.

We construct a new function

$$\sum_{\psi \bmod p^2}{}^{*} \left| \sum_{m=1}^{p^2}{}' \psi(m) \left|S(m,1,\chi,p^2)\right|^2 \right|^2, \tag{2.1}$$

where $\sum^{\star}$ denotes the summation over all primitive characters mod p^2, χ is a non-principal and also non-primitive character mod p^2. We study it from two ways, then compare the different results, and accordingly give the conclusion.

First we prove the following lemma.

Lemma 1. *Let $p \geqslant 3$ be a prime and χ a nonprincipal character modulo p^2. Then for any primitive character ψ modulo p^2, we have*

$$\sum_{m=1}^{p^2} \psi(m) \left|S(m,1,\chi,p^2)\right|^2 = \tau^2(\psi) \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)),$$

where $\tau(\psi)$ is the Gauss Sum.

P r o o f. From the definition of $S(m,1,\chi,p^2)$ and the properties of the reduced residue system mod p^2 we have

$$\begin{aligned}
\sum_{m=1}^{p^2} \psi(m) \left|S(m,1,\chi,p^2)\right|^2 &= \sum_{m=1}^{p^2} \psi(m) \Bigl| \sum_{a=1}^{p^2} \chi(a) e\Bigl(\frac{ma+\overline{a}}{p^2}\Bigr)\Bigr|^2 \\
&= \sum_{m=1}^{p^2} \psi(m) \sum_{a=1}^{p^2}\sum_{b=1}^{p^2} \chi(a\overline{b}) e\Bigl(\frac{mb(a\overline{b}-1)+\overline{b}(\overline{a}b-1}{p^2}\Bigr) \\
&= \sum_{m=1}^{p^2} \psi(m) \sum_{a=1}^{p^2} \chi(a) \sideset{}{'}\sum_{b=1}^{p^2} e\Bigl(\frac{mb(a-1)+\overline{b}(\overline{a}-1)}{p^2}\Bigr) \\
&= \sum_{a=1}^{p^2} \chi(a) \sum_{m=1}^{p^2} \psi(m) \sideset{}{'}\sum_{b=1}^{p^2} e\Bigl(\frac{mb(a-1)}{p^2}\Bigr) e\Bigl(\frac{\overline{b}(\overline{a}-1)}{p^2}\Bigr) \\
&= \sum_{a=1}^{p^2} \chi(a) \sum_{m=1}^{p^2} \sideset{}{'}\sum_{b=1}^{p^2} \psi(m) e\Bigl(\frac{mb(a-1)}{p^2}\Bigr) e\Bigl(\frac{\overline{b}(\overline{a}-1)}{p^2}\Bigr) \\
&= \sum_{a=1}^{p^2} \chi(a) \sum_{b=1}^{p^2} \overline{\psi}(b(a-1)) \tau(\psi) e\Bigl(\frac{\overline{b}(\overline{a}-1)}{p^2}\Bigr) \\
&= \tau(\psi) \sum_{a=1}^{p^2} \chi(a) \sum_{b=1}^{p^2} \overline{\psi}(b(a-1)) e\Bigl(\frac{\overline{b}(\overline{a}-1)}{p^2}\Bigr) \\
&= \tau^2(\psi) \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)),
\end{aligned}$$

where we have used the fact that

$$\sum_{a=1}^{p^2} \psi(a) e\Bigl(\frac{a}{p^2}\Bigr) = \tau(\psi).$$

□

This completes the proof of Lemma 1.

Remark. Using Lemma 1, let us consider the summation

$$\sum_{\psi \bmod p^2}^{*} \left| \sum_{m=1}^{p^2}{}' \psi(m) \left| S(m,1,\chi,p^2) \right|^2 \right|^2 = \sum_{\psi \bmod p^2}^{*} \left| \tau^2(\psi) \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2.$$

By the fact that $|\tau(\psi)| = p$ since ψ is a primitive character of modulo p^2, the above summation is

$$\begin{aligned}
&= p^4 \sum_{\psi \bmod p^2}^{*} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 \\
&= p^4 \Bigg(\sum_{\psi \bmod p^2} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 - \sum_{\psi \bmod p} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 \Bigg) \\
&:= p^4(E_1 - E_2).
\end{aligned}$$

Thus, the summation (2.1) reduces to E_1 and E_2, which is easier to compute. Now we shall calculate both E_1 and E_2.

Lemma 2. *Let $p \geqslant 3$ be a prime. Then for any non-primitive character $\chi \neq \chi_0$ modulo p^2 we have*

$$\sum_{\psi \bmod p^2} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 = \begin{cases} \varphi(p^2)(2\varphi(p^2)-1), & \text{if } \chi \text{ is the Legendre symbol mod } p; \\ \varphi(p^2)(\varphi(p^2)-1), & \text{if } \chi \text{ is any non-real character mod } p. \end{cases}$$

P r o o f. From the orthogonality relations for characters we have

$$\begin{aligned}
(2.2) \qquad E_1 &= \sum_{\psi \bmod p^2} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 \\
&= \sum_{\psi \bmod p^2} \sum_{a=1}^{p^2} \sum_{b=1}^{p^2} \chi(a\overline{b})\overline{\psi}((a-1)(\overline{a}-1))\psi((b-1)(\overline{b}-1))
\end{aligned}$$

$$= \varphi(p^2) \sum_{\substack{a=1 \\ \overline{b}-\overline{a}+b-a\equiv 0 \pmod{p^2} \\ a\not\equiv 1, b\not\equiv 1 \pmod{p^2}}}^{p^2} \sum_{b=1}^{p^2} \chi(a\overline{b})$$

$$= \varphi(p^2)\Bigg(\sum_{\substack{a=1 \\ (1-\overline{ab})(b-a)\equiv 0 \pmod{p^2}}}^{p^2} \sum_{b=1}^{p^2} \chi(a\overline{b}) - 2\sum_{\substack{a=1 \\ (1-\overline{a})(1-a)\equiv 0 \pmod{p^2}}}^{p^2} \chi(a) +1\Bigg),$$

where a, b run through the reduced residue system modulo p^2.

So we split it into three parts: $p \nmid b-a$, $p \parallel b-a$ and $p^2 \mid b-a$, then

$$\sum_{\substack{a=1 \\ (1-\overline{ab})(b-a)\equiv 0 \pmod{p^2}}}^{p^2} \sum_{b=1}^{p^2} \chi(a\overline{b}) = \sum_{a=1}^{p^2}{}' 1 + \sum_{a=1}^{p^2} \chi(a^2) - \sum_{\substack{a=1 \\ a\equiv b\equiv \overline{b} \pmod{p^2}}}^{p^2}{}' \sum_{b=1}^{p^2}{}' 1 + \sum_{\substack{a=1 \\ b\equiv a \pmod p \\ \overline{b}\equiv a \pmod p}}^{p^2} \sum_{b=1}^{p^2} \chi(a\overline{b}) \tag{2.3}$$

$$= \varphi(p^2) + \sum_{a=1}^{p^2} \chi(a^2) - 2 + 2p.$$

From the properties of congruences, we immediately get

$$\sum_{\substack{a=1 \\ (a-1)(\overline{a}-1)\equiv 0 \pmod{p^2}}}^{p^2} \chi(a) = \sum_{k=0}^{p-1} \chi(kp+1) = p. \tag{2.4}$$

Based on the above formulas (2.2), (2.3) and (2.4), we obtain

$$E_1 = \begin{cases} \varphi(p^2)(2\varphi(p^2)-1), & \text{if } \chi \text{ is the Legendre symbol mod } p; \\ \varphi(p^2)(\varphi(p^2)-1), & \text{if } \chi \text{ is any non-real character mod } p. \end{cases}$$

This proves Lemma 2. □

Lemma 3. *Let p be an odd prime. Then for any non-primitive character $\chi \neq \chi_0$ modulo p^2 we have*

$$\sum_{\psi \bmod p} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2$$

$$= \begin{cases} 2p\varphi(p^2)(\varphi(p)-1), & \text{if } \chi \text{ is the Legendre symbol mod } p; \\ p\varphi(p^2)(\varphi(p)-2), & \text{if } \chi \text{ is any non-real character mod } p. \end{cases}$$

Proof. From the properties of characters and noting that any non-primitive character modulo p^2 must be a character modulo p, and if a and l pass through a reduced residue system modulo p then $lp+a$ also passes through a reduced residue system modulo p^2, we obtain

$$\begin{aligned}
E_2 &= \sum_{\psi \bmod p} \left| \sum_{a=1}^{p^2} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 \\
&= \sum_{\psi \bmod p} \left| \sum_{l=0}^{p-1} {\sum_{a=1}^{p}}' \chi(lp+a)\overline{\psi}((lp+a-1)(\overline{lp+a}-1)) \right|^2 \\
&= p^2 \sum_{\psi \bmod p} \left| \sum_{a=1}^{p} \chi(a)\overline{\psi}((a-1)(\overline{a}-1)) \right|^2 \\
&= p^2 {\sum_{a=1}^{p}}' {\sum_{b=1}^{p}}' \sum_{\psi \bmod p} \chi(a\overline{b})\overline{\psi}((a-1)(\overline{a}-1))\psi((b-1)(\overline{b}-1)) \\
&= \varphi(p)p^2 \mathop{{\sum_{a=1}^{p}}' {\sum_{b=1}^{p}}'}_{\substack{(b-a)(1-\overline{ab})\equiv 0 \pmod p \\ a\not\equiv 1, b\nmid 1 \pmod{p^2}}} \chi(a\overline{b}).
\end{aligned}$$

Similarly to Lemma 2, we can get

$$E_2 = \begin{cases} 2p\varphi(p^2)(\varphi(p)-1), & \text{if } \chi \text{ is the Legendre symbol mod } p; \\ p\varphi(p^2)(\varphi(p)-2), & \text{if } \chi \text{ is any non-real character mod } p. \end{cases}$$

□

This completes the proof of Lemma 3.

Remark. From Lemma 2 and Lemma 3 we can deduce

$$\sum_{\psi \bmod p^2}^{*} \left| {\sum_{m=1}^{p^2}}' \psi(m) \left| S(m,1,\chi,p^2) \right|^2 \right|^2 = p^4\varphi(p^2)(2p-1). \tag{2.5}$$

On the other hand, applying the properties of a primitive character, we have

$$\begin{aligned}
&\sum_{\psi \bmod p^2}^{*} \left| {\sum_{m=1}^{p^2}}' \psi(m) \left| S(m,1,\chi,p^2) \right|^2 \right|^2 \qquad (2.6)\\
&= \sum_{\psi \bmod p^2}^{*} \left| {\sum_{m=1}^{p^2}}' \psi(h) \left| \sum_{a=1}^{p^2} \chi(a) e\Big(\frac{ha+\overline{a}}{p^2}\Big) \right|^2 \right|^2
\end{aligned}$$

$$= \sum_{\psi \bmod p^2} \Big| \sideset{}{'}\sum_{m=1}^{p^2} \psi(m) \Big| \sum_{a=1}^{p^2} \chi(a) e\Big(\frac{ha+\overline{a}}{p^2}\Big)\Big|^2\Big|^2$$

$$- \sum_{\psi \bmod p} \Big| \sideset{}{'}\sum_{h=1}^{p^2} \psi(m) \Big| \sum_{a=1}^{p^2} \chi(a) e\Big(\frac{ha+\overline{a}}{p^2}\Big)\Big|^2\Big|^2$$

$$= \varphi(p^2) \sideset{}{'}\sum_{m=1}^{p^2} \big|S(m,1,\chi,p^2)\big|^4 - \sum_{\psi \bmod p} \Big| \sideset{}{'}\sum_{h=1}^{p^2} \psi(m) \Big| \sum_{a=1}^{p^2} \chi(a) e\Big(\frac{ha+\overline{a}}{p^2}\Big)\Big|^2\Big|^2$$

$$:= \varphi(p^2) \sideset{}{'}\sum_{m=1}^{p^2} \big|S(m,1,\chi,p^2)\big|^4 - E_3.$$

Now we shall compute E_3.

Lemma 4. *Let p be an odd prime and $\alpha \geqslant 2$ an integer. Then for any nonprimitive character $\chi \neq \chi_0$ modulo p^α we have*

$$\sum_{\psi \bmod p^{\alpha-1}} \Big| \sideset{}{'}\sum_{m=1}^{p^\alpha} \psi(m) \Big| \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{ma+\overline{a}}{p^\alpha}\Big)\Big|^2\Big|^2 = \varphi^2(p^\alpha) p^{2\alpha-2}(p^2-1).$$

Proof. From the properties of characters modulo p^α and noting that

$$\overline{a} - 1 = \overline{a}(1-a) \bmod p^\alpha,$$

and if h and l pass through a reduced residue system modulo $p^{\alpha-1}$ and modulo p respectively, then $lp^{\alpha-1}+h$ passes through a reduced residue system modulo p^α, we have

$$\sum_{\psi \bmod p^{\alpha-1}} \Big| \sideset{}{'}\sum_{m=1}^{p^\alpha} \psi(m) \Big| \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{ma+\overline{a}}{p^\alpha}\Big)\Big|^2\Big|^2$$

$$= \sum_{\psi \bmod p^{\alpha-1}} \Big| \sideset{}{'}\sum_{h=1}^{p^{\alpha-1}} \sum_{l=0}^{p-1} \psi(lp^{\alpha-1}+h) \Big| \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{(lp^{\alpha-1}+h)a+\overline{a}}{p^\alpha}\Big)\Big|^2\Big|^2$$

$$= \sum_{\psi \bmod p^{\alpha-1}} \Big| \sideset{}{'}\sum_{h=1}^{p^{\alpha-1}} \psi(h) \sum_{l=0}^{p-1} \Big| \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{(lp^{\alpha-1}+h)a+\overline{a}}{p^\alpha}\Big)\Big|^2\Big|^2$$

$$= \varphi(p^{\alpha-1}) \sideset{}{'}\sum_{h=1}^{p^{\alpha-1}} \Big| \sum_{l=0}^{p-1} \Big| \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{la}{p} + \frac{ha+\overline{a}}{p^\alpha}\Big)\Big|^2\Big|^2$$

$$= \varphi(p^{\alpha-1}) \sideset{}{'}\sum_{h=1}^{p^{\alpha-1}} \Big| \sum_{l=0}^{p-1} \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \chi(a\overline{b}) e\Big(\frac{l(a-b)}{p} + \frac{h(a-b)+\overline{a}-\overline{b}}{p^\alpha}\Big)\Big|^2$$

$$= \varphi(p^{\alpha-1}) \sum_{h=1}^{p^{\alpha-1}}{}' \Big| \sum_{l=0}^{p-1} \sum_{a=1}^{p^\alpha} \chi(a) \sum_{b=1}^{p^\alpha}{}' e\Big(\frac{lb(a-1)}{p} + \frac{hb(a-1) - \overline{a}\overline{b}(a-1)}{p^\alpha} \Big) \Big|^2$$

$$= \varphi(p^{\alpha-1}) p^2 \sum_{h=1}^{p^{\alpha-1}}{}' \Big| \sum_{\substack{a=1 \\ p \mid a-1}}^{p^\alpha} \chi(a) \sum_{b=1}^{p^\alpha}{}' e\Big(\frac{hb(a-1) - \overline{a}\overline{b}(a-1)}{p^\alpha} \Big) \Big|^2.$$

Let $(a-1, p^\alpha) = p^\beta$, where $1 \leqslant \beta \leqslant \alpha - 1$. Then $a = up^\beta + 1$ with $(u,p) = 1$ and $1 \leqslant u < p^{\alpha-\beta}$, therefore we have

$$\sum_{\psi \bmod p^{\alpha-1}} \Big| \sum_{m=1}^{p^\alpha}{}' \psi(m) \Big| \sum_{a=1}^{p^\alpha} \chi(a) e\Big(\frac{ma + \overline{a}}{p^\alpha} \Big) \Big|^2 \Big|^2$$

$$= \varphi(p^{\alpha-1}) p^2 \sum_{h=1}^{p^{\alpha-1}}{}' \Big| \sum_{\beta=1}^{\alpha-1} \sum_{u=1}^{p^{\alpha-\beta}}{}' \chi(up^\beta + 1) \sum_{b=1}^{p^\alpha}{}' e\Big(\frac{hbup^\beta - \overline{b}(\overline{up^\beta + 1})up^\beta}{p^\alpha} \Big) \Big|^2$$

$$= \varphi(p^{\alpha-1}) p^2 \sum_{h=1}^{p^{\alpha-1}}{}' \Big| \sum_{\beta=1}^{\alpha-1} p^\beta \sum_{u=1}^{p^{\alpha-\beta}}{}' \chi(up^\beta + 1) \sum_{b=1}^{p^{\alpha-\beta}}{}' e\Big(\frac{hb - \overline{b}(\overline{up^\beta + 1})u^2}{p^{\alpha-\beta}} \Big) \Big|^2$$

$$= \varphi(p^{\alpha-1}) p^2 \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\alpha-1} p^{\beta+\alpha} \sum_{u=1}^{p^{\alpha-\beta}}{}' \sum_{v=1}^{p^{\alpha-\gamma}}{}' \chi(up^\beta + 1) \overline{\chi}(vp^\gamma + 1)$$

$$\times \sum_{b=1}^{p^{\alpha-\beta}}{}' \sum_{c=1}^{p^{\alpha-\gamma}}{}' \sum_{h=1}^{p^{\alpha-1}}{}' e\Big(\frac{h(bp^\beta - cp^\gamma) - \overline{b}(\overline{up^\beta + 1})u^2 p^\beta - \overline{c}(\overline{vp^\gamma + 1})v^2 p^\gamma}{p^\alpha} \Big)$$

$$= \varphi(p^{\alpha-1}) p^2 \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=1}^{\alpha-1} p^{\beta+\alpha} \sum_{u=1}^{p^{\alpha-\beta}}{}' \sum_{v=1}^{p^{\alpha-\gamma}}{}' \chi(up^\beta + 1) \overline{\chi}(vp^\gamma + 1)$$

$$\times \sum_{b=1}^{p^{\alpha-\beta}}{}' \sum_{c=1}^{p^{\alpha-\gamma}}{}' e\Big(\frac{-\overline{b}(\overline{up^\beta + 1})u^2 p^\beta - \overline{c}(\overline{vp^\gamma + 1})v^2 p^\gamma}{p^\alpha} \Big)$$

$$\times \left(\sum_{h=1}^{p^{\alpha-1}} e\left(\frac{h(bp^\beta - cp^\gamma)}{p^\alpha} \right) - \sum_{h=1}^{p^{\alpha-2}} e\left(\frac{hp(bp^\beta - cp^\gamma)}{p^\alpha} \right) \right)$$

$$= \varphi^2(p^{\alpha-1}) p^2 \sum_{\beta=1}^{\alpha-1} p^{2\beta} \sum_{u=1}^{p^{\alpha-\beta}}{}' \sum_{v=1}^{p^{\alpha-\beta}}{}' \chi(up^\beta + 1) \overline{\chi}(vp^\beta + 1)$$

$$\times \sum_{b=1}^{p^{\alpha-\beta}}{}' e\Big(\frac{\overline{b}(\overline{vp^\beta + 1})v^2 - (\overline{up^\beta + 1})u^2}{p^{\alpha-\beta}} \Big)$$

$$= \varphi^2(p^\alpha)p^{2\alpha-2}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\chi((up^{\alpha-1}+1)(\overline{vp^{\alpha-1}+1}))\sum_{b=1}^{p-1}e\Big(\frac{b(v^2-u^2)}{p}\Big)$$

$$+ \varphi^2(p^\alpha)\sum_{\beta=1}^{\alpha-2}p^{2\beta}\sideset{}{'}\sum_{u=1}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{v=1}^{p^{\alpha-\beta}}\chi((up^\beta+1)(\overline{vp^\beta+1}))$$

$$\times \sideset{}{'}\sum_{b=1}^{p^{\alpha-\beta}}e\Big(\frac{b((\overline{vp^\beta+1})u^2-(\overline{up^\beta+1})v^2)}{p^{\alpha-\beta}}\Big)$$

$$:= \varphi^2(p^\alpha)(B_1+B_2),$$

where we have used the fact that $p^\alpha \mid bp^\beta - cp^\gamma$ if and only if $\beta=\gamma$ and $c\equiv b \pmod{p^{\alpha-\beta}}$. Now we compute E_1 and E_2. From the trigonometric identity

$$\sum_{a=1}^{q}e\Big(\frac{sa}{q}\Big)=\begin{cases} q, & \text{if } q \mid s;\\ 0, & \text{otherwise,}\end{cases}$$

we get

$$\begin{aligned}B_1 &= p^{2\alpha-2}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\chi((up^{\alpha-1}+1)(\overline{vp^{\alpha-1}+1}))\sum_{b=1}^{p-1}e\Big(\frac{b(v^2-u^2)}{p}\Big)\\ &= p^{2\alpha-2}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\left(\sum_{b=1}^{p}e\Big(\frac{b(v^2-u^2)}{p}\Big)-1\right)\\ &= p^{2\alpha-2}\left(p\sum_{\substack{u=1\\ p\mid v^2-u^2}}^{p-1}\sum_{v=1}^{p-1}1-\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}1\right)\\ &= p^{2\alpha-2}(p\varphi(p)+p\varphi(p)-(p-1)^2)=p^{2\alpha-2}(p^2-1).\end{aligned}$$

Note that if $1\leqslant\beta\leqslant\alpha-2$ and $p^{\alpha-\beta}\mid(u-v)(p^\beta uv+u+v)$, then $(u-v,p^\beta uv+u+v,p)=1$; therefore we have

$$\begin{aligned}B_2 &= \sum_{\beta=1}^{\alpha-2}p^{2\beta}\sideset{}{'}\sum_{u=1}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{v=1}^{p^{\alpha-\beta}}\chi((up^\beta+1)(\overline{vp^\beta+1}))\sideset{}{'}\sum_{b=1}^{p^{\alpha-\beta}}e\Big(\frac{b((\overline{vp^\beta+1})u^2-(\overline{up^\beta+1})v^2)}{p^{\alpha-\beta}}\Big)\\ &= \sum_{\beta=1}^{\alpha-2}p^{\alpha+\beta}\sideset{}{'}\sum_{\substack{u=1\\ p^{\alpha-\beta}\mid(u-v)(p^\beta uv+u+v)}}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{v=1}^{p^{\alpha-\beta}}\chi((up^\beta+1)(\overline{vp^\beta+1}))\\ &\quad -\sum_{\beta=1}^{\alpha-2}p^{\alpha+\beta-1}\sideset{}{'}\sum_{\substack{u=1\\ p^{\alpha-\beta-1}\mid(u-v)(p^\beta uv+u+v)}}^{p^{\alpha-\beta}}\sideset{}{'}\sum_{v=1}^{p^{\alpha-\beta}}\chi((up^\beta+1)(\overline{vp^\beta+1}))\end{aligned}$$

$$= \sum_{\beta=1}^{\alpha-2} \left(p^{\alpha+\beta} \varphi(p^{\alpha-\beta}) + p^{\alpha+\beta} \sum_{v=1}^{p^{\alpha-\beta}}{}' \chi(\overline{vp^{\beta}+1}^{2}) \right)$$

$$- \sum_{\beta=1}^{\alpha-2} \left(p^{\alpha+\beta} \varphi(p^{\alpha-\beta}) + p^{\alpha+\beta} \sum_{v=1}^{p^{\alpha-\beta}}{}' \chi(\overline{vp^{\beta}+1}^{2}) \right) = 0.$$

Hence, it is easy to compute that

$$\sum_{\psi \bmod p^{\alpha-1}} \left| \sum_{m=1}^{p^{\alpha}}{}' \psi(m) \left| \sum_{a=1}^{p^{\alpha}} \chi(a) e\Big(\frac{ha+\overline{a}}{p^{\alpha}} \Big) \right|^2 \right|^2 = \varphi^2(p^{\alpha}) p^{2\alpha-2}(p^2-1).$$

This completes the proof of Lemma 4. □

From Lemma 4, we have

$$E_3 = \varphi^2(p^2)p^2(p^2-1).$$

Combining (5) with (6), we get

$$\sum_{m=1}^{p^2}{}' |S(m,1,\chi,p^2)|^4 = p^3(p^3+p^2-2p+1).$$

Applying the multiplicative property of Kloosterman sums, we can complete the proof of the theorem. In fact, let q have the prime power decomposition

$$q = \prod_{i=1}^{r} p_i^2 \qquad \text{and} \qquad m = \sum_{i=1}^{r} \frac{m_i q}{p_i^2}.$$

It is clear that if m_i $(i=1,2,\ldots,r)$ pass through a reduced residue system modulo p_i^2, then m passes through a reduced residue system modulo q, so we have

$$\begin{aligned}
\sum_{m=1}^{q}{}' |S(m,1,\chi,q)|^4 &= \prod_{i=1}^{r} \sum_{m_i=1}^{p_i^2} \left| \sum_{a=1}^{p_i^2} \chi_i\Big(\frac{qa}{p_i}\Big) e\Bigg(\frac{\frac{m_i q}{p_i^2}\frac{aq}{p_i^2} + \frac{\overline{aq}}{p_i^2}}{q} \Bigg) \right|^4 \\
&= \prod_{i=1}^{r} \sum_{m_i=1}^{p_i^2} \left| \sum_{a=1}^{p_i^2} \chi_i(a) e\Big(\frac{m_i a + \overline{a}}{p_i^2} \Big) \right|^4 \\
&= \prod_{i=1}^{r} p_i^3(p_i^3+p_i^2-2p_i+1) \\
&= q^{\frac{3}{2}} \prod_{p|q} (p^3+p^2-2p+1).
\end{aligned}$$

This completes the proof of the theorem. □

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