Aydin Huseynov

Second order nonlinear differential equations with linear impulse and periodic boundary conditions

Applications of Mathematics, Vol. 56 (2011), No. 6, 591–606

Persistent URL: `http://dml.cz/dmlcz/141769`

SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH LINEAR IMPULSE AND PERIODIC BOUNDARY CONDITIONS*

Aydin Huseynov, Baku

(Received April 20, 2009)

Abstract. In this study, we establish existence and uniqueness theorems for solutions of second order nonlinear differential equations on a finite interval subject to linear impulse conditions and periodic boundary conditions. The results obtained yield periodic solutions of the corresponding periodic impulsive nonlinear differential equation on the whole real axis.

Keywords: impulse conditions, periodic boundary conditions, Green's function, fixed point theorems

MSC 2010: 34B15, 34B37

1. Introduction

Impulsive differential equations are a basic tool to study dynamics of processes that are subjected to abrupt changes in their states. Theory of impulsive differential equations has been motivated by a number of applied problems. We point out names of a few representative examples, control theory [9], [10], population dynamics [20], chemotherapeutic treatment in medicine [14], and some physics problems [17]. A significant development has been made in the mathematical theory of impulsive differential equations in the last two decades; see the monographs [2], [3], [15], [24]. Periodic boundary value problems for nonlinear differential equations with impulse were earlier studied [4]–[7], [11], [13], [16], [18], [19], [21]–[23], [25], [26]. However, due to the special form of our problem we have developed in this paper more detailed analysis and established more explicit results.

* This work was supported by a NATO Science Reintegration Grant.

In this paper, we deal with the following boundary value problem with impulse (BVPI):

$$-[p(x)y']' + q(x)y = f(x,y), \quad x \in [a,c) \cup (c,b], \tag{1}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{2}$$

$$y(a) = y(b), \quad y^{[1]}(a) = y^{[1]}(b), \tag{3}$$

where $a < c < b$; $y = y(x)$ is a desired solution; $y^{[1]}(x) = p(x)y'(x)$ denotes the quasi-derivative of $y(x)$; $y(c^-)$ is the left-hand limit of $y(x)$ at c and $y(c^+)$ is the right-hand limit of $y(x)$ at c; the coefficients $p(x)$, $q(x)$ of the equation (1) are complex-valued functions defined on $[a,c) \cup (c,b]$, $f(x,\xi)$ is a complex-valued function defined on $([a,c) \cup (c,b]) \times \mathbb{C}$; d_1 and d_2 in the conditions (2) are nonzero complex numbers. Note that everywhere $\mathbb{C}$ denotes the set of complex numbers.

The conditions in (2) express an *impulse effect* at the point c. The conditions in (3) are called the *periodic boundary conditions* and they form an important representative of nonseparated boundary conditions.

Note that a complex-valued function $y(x)$ defined on $[a,c) \cup (c,b]$ is called a *solution* of (1)–(3) if its first derivative $y'(x)$ exists for each $x \in [a,c) \cup (c,b]$, $p(x)y'(x)$ is differentiable on $[a,c) \cup (c,b]$, there exist finite values $y(c^{\pm})$ and $y^{[1]}(c^{\pm})$, the impulse conditions in (2) and the boundary conditions in (3) are satisfied, and the equation (1) is satisfied on $[a,c) \cup (c,b]$.

The paper is organized as follows. In Section 2, following our paper [12] we present basic properties of solutions of the second order linear homogeneous differential equation with impulse

$$-[p(x)y']' + q(x)y = 0, \quad x \in (-\infty,c) \cup (c,\infty),$$
$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+),$$

and give the Green's function of the linear BVPI

$$-[p(x)y']' + q(x)y = h(x), \quad x \in [a,c) \cup (c,b], \tag{4}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{5}$$

$$y(a) = y(b), \quad y^{[1]}(a) = y^{[1]}(b). \tag{6}$$

In Section 3, the Green's function of the linear problem (4)–(6) is used to reduce the nonlinear BVPI (1)–(3) to a fixed point problem. In Section 4, by using the Contraction Mapping Theorem (Banach Fixed Point Theorem) we show that there is a unique solution of the BVPI (1)–(3) if $f(x,\xi)$ satisfies a Lipschitz condition. In Section 5, a theorem (Theorem 7) based on the Schauder Fixed Point Theorem is proved which gives a result that yields existence of solutions without implication that the solutions must be unique. In Section 6, Theorem 7 is illustrated by several examples. Finally, in Section 7, we end with some concluding remarks.

2. Auxiliary linear problem and its Green's function

Let c be a real number and d_1, d_2 nonzero complex numbers. Consider the second order linear homogeneous differential equation with impulse

$$-[p(x)y']' + q(x)y = 0, \quad x \in (-\infty, c) \cup (c, \infty), \tag{7}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{8}$$

where $y = y(x)$ is a desired solution and

$$y^{[1]}(x) = p(x)y'(x) \tag{9}$$

denotes the quasi-derivative of $y(x)$. We will assume that the coefficients $p(x)$ and $q(x)$ of the equation (7) are complex-valued continuous functions on $(-\infty, c) \cup (c, \infty)$ and $p(x) \neq 0$. In addition, it is assumed that there exist finite left-sided and right-sided limits $p(c^{\pm})$ and $q(c^{\pm})$, and that $p(c^{\pm}) \neq 0$.

A function $y(x)$ defined on $(-\infty, c) \cup (c, \infty)$ is called a *solution* of (7)–(8) if its first derivative $y'(x)$ exists, $p(x)y'(x)$ is continuously differentiable on $(-\infty, c) \cup (c, \infty)$, there exist finite values $y(c^{\pm})$, $y^{[1]}(c^{\pm})$ that satisfy the impulse conditions (8), and the equation (7) is satisfied on $(-\infty, c) \cup (c, \infty)$.

For any fixed point x_0 in $(-\infty, c) \cup (c, \infty)$ and any complex numbers c_0, c_1 the problem (7)–(8) has a unique solution $y(x)$ such that

$$y(x_0) = c_0, \quad y^{[1]}(x_0) = c_1.$$

For two differentiable functions y and z on $(-\infty, c) \cup (c, \infty)$ we define their Wronskian by

$$\begin{aligned} W_x(y, z) &= y(x)z^{[1]}(x) - y^{[1]}(x)z(x) \\ &= p(x)[y(x)z'(x) - y'(x)z(x)], \quad x \in (-\infty, c) \cup (c, \infty). \end{aligned}$$

The Wronskian of any two solutions y and z of (7)–(8) is constant on each of the intervals $(-\infty, c)$ and (c, ∞):

$$W_x(y, z) = \begin{cases} \omega^-, & x \in (-\infty, c), \\ \omega^+, & x \in (c, \infty), \end{cases}$$

where ω^- and ω^+ are constants such that

$$\omega^- = d_1 d_2 \omega^+.$$

It follows that if y and z are two solutions of (7)–(8), then either $W_x(y,z) = 0$ for all $x \in (-\infty, c) \cup (c, \infty)$ or $W_x(y,z) \neq 0$ for all $x \in (-\infty, c) \cup (c, \infty)$.

Any two solutions of (7)–(8) are linearly independent if and only if their Wronskian is not zero.

The problem (7)–(8) has two linearly independent solutions and every solution of (7)–(8) is a linear combination of these solutions.

Let a, b, and c be fixed real numbers with $a < c < b$. Consider the following linear boundary value problem with impulse (BVPI):

$$-[p(x)y']' + q(x)y = h(x), \quad x \in [a, c) \cup (c, b], \tag{10}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{11}$$

$$y(a) = y(b), \quad y^{[1]}(a) = y^{[1]}(b), \tag{12}$$

where $h(x)$ is a complex-valued continuous function on $[a, c) \cup (c, b]$ such that there exist finite limit values $h(c^{\pm})$ the coefficients $p(x)$, $q(x)$, d_1, and d_2 are as above.

Denote by $u(x)$ and $v(x)$ the solutions of the homogeneous problem

$$-[p(x)y']' + q(x)y = 0, \quad x \in [a, c) \cup (c, b], \tag{13}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{14}$$

satisfying the initial conditions

$$u(a) = 1, \quad u^{[1]}(a) = 0 \tag{15}$$

and

$$v(a) = 0, \quad v^{[1]}(a) = 1, \tag{16}$$

respectively. We have

$$W_x(u, v) = \begin{cases} 1, & x \in [a, c), \\ d_1^{-1} d_2^{-1}, & x \in (c, b]. \end{cases} \tag{17}$$

Let us set

$$D = u(b) + v^{[1]}(b) - d_1^{-1} d_2^{-1} - 1. \tag{18}$$

It follows that $D \neq 0$ if and only if the homogeneous problem (13)–(14) has only the trivial solution $y(x) \equiv 0$ satisfying the periodic boundary conditions in (12).

Theorem 1. *If $D \neq 0$ then the nonhomogeneous BVPI* (10)–(12) *has a unique solution $y(x)$ for which the formula*

$$y(x) = \int_a^b G(x,s)h(s)\,\mathrm{d}s, \quad x \in [a,c) \cup (c,b], \tag{19}$$

holds, where the function $G(x,s)$ is called the Green's function of the BVPI (10)–(12) *and it is defined for $x, s \in [a,c) \cup (c,b]$ by the formula*

$$\begin{aligned} G(x,s) &\\ = \frac{1}{DW_s(u,v)}&[v(b)u(x)u(s) - u^{[1]}(b)v(x)v(s)] \\ + \frac{1}{DW_s(u,v)}&\begin{cases} [v^{[1]}(b)-1]u(x)v(s) - [u(b)-1]u(s)v(x), & s \leqslant x, \\ [v^{[1]}(b) - d_1^{-1}d_2^{-1}]u(s)v(x) - [u(b) - d_1^{-1}d_2^{-1}]u(x)v(s), & \\ & x \leqslant s, \end{cases} \end{aligned} \tag{20}$$

the number D being defined by (18).

Proofs of all the statements given above in this section can be found in the author's paper [12]. For the case when there is no impulse see [1].

Remark 2. In the case of $p(x) \equiv 1$, $q(x) \equiv \alpha^2$ ($\alpha > 0$), $a = 0$, $d_1 = d_2 = 1$, where α is a constant, the Green's function $G(x,s)$ of the BVPI (10)–(12) has the form

$$G(x,s) = \frac{1}{2\alpha(\mathrm{e}^{\alpha b} - 1)} \begin{cases} \mathrm{e}^{\alpha(x-s)} + \mathrm{e}^{\alpha(b+s-x)}, & 0 \leqslant s \leqslant x \leqslant b, \\ \mathrm{e}^{\alpha(s-x)} + \mathrm{e}^{\alpha(b+x-s)}, & 0 \leqslant x \leqslant s \leqslant b. \end{cases}$$

3. Nonlinear problem

In this section, we consider the nonlinear BVPI

$$-[p(x)y']' + q(x)y = f(x,y), \quad x \in [a,c) \cup (c,b], \tag{21}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{22}$$

$$y(a) = y(b), \quad y^{[1]}(a) = y^{[1]}(b). \tag{23}$$

We will assume that the following conditions are satisfied.

(H1) $p(x)$ and $q(x)$ are complex-valued continuous functions on $(-\infty, c) \cup (c, \infty)$ and $p(x) \neq 0$. In addition, it is assumed that there exist finite left-sided and right-sided limits $p(c^{\pm})$ and $q(c^{\pm})$, and that $p(c^{\pm}) \neq 0$.

(H2) d_1 and d_2 are given nonzero complex numbers.

(H3) $f(x, \xi)$ is a complex-valued continuous function defined on $([a, c) \cup (c, b]) \times \mathbb{C}$, and such that for each $\xi_0 \in \mathbb{C}$ there exist finite limits

$$\lim_{\substack{(x,\xi)\to(c,\xi_0)\\ x<c}} f(x, \xi) = f(c^-, \xi_0) \quad \text{and} \quad \lim_{\substack{(x,\xi)\to(c,\xi_0)\\ x>c}} f(x, \xi) = f(c^+, \xi_0).$$

(H4) The linear homogeneous BVPI

$$-[p(x)y']' + q(x)y = 0, \quad x \in [a, c) \cup (c, b], \tag{24}$$

$$y(c^-) = d_1 y(c^+), \quad y^{[1]}(c^-) = d_2 y^{[1]}(c^+), \tag{25}$$

$$y(a) = y(b), \quad y^{[1]}(a) = y^{[1]}(b), \tag{26}$$

has only the trivial solution $y(x) \equiv 0$.

R e m a r k 3. If $p(x) > 0$, $q(x) \geqslant 0$, $q(x)$ is not identically zero, $d_1 > 0$, $d_2 > 0$, and $d_1 + d_2 \geqslant 1 + d_1 d_2$, then the condition (H4) is satisfied, see [12].

Let $u(x)$ and $v(x)$ be solutions of (24)–(25) satisfying the initial conditions (15) and (16), respectively, and let D be defined by (18). Then the condition (H4) is equivalent to the condition that $D \neq 0$. Define $G(x, s)$ by (20) for $x, s \in [a, c) \cup (c, b]$. Then the nonlinear BVPI (21)–(23) is equivalent, by Theorem 1, to the integral equation

$$y(x) = \int_a^b G(x, s) f(s, y(s))\, \mathrm{d}s, \quad x \in [a, c) \cup (c, b]. \tag{27}$$

We will investigate the equation (27) in the Banach space $\mathcal{B}$ of all complex-valued continuous functions $y(x)$ on $[a, c) \cup (c, b]$ for which the finite values $y(c^-)$ and $y(c^+)$ exist, with the norm

$$\|y\| = \sup |y(x)|, \quad x \in [a, c) \cup (c, b].$$

If we define the operator $A\colon \mathcal{B} \to \mathcal{B}$ by

$$(Ay)(x) = \int_a^b G(x, s) f(s, y(s))\, \mathrm{d}s, \quad x \in [a, c) \cup (c, b], \tag{28}$$

then the equation (27) can be written as

$$y = Ay, \quad y \in \mathcal{B}.$$

Therefore, solving the equation (27) is equivalent to finding fixed points of the operator A.

4. Existence and uniqueness of solutions

We will use the following well-known Contraction Mapping Theorem also called the Banach Fixed Point Theorem: *Let $\mathcal{B}$ be a Banach space and S be a nonempty closed subset of $\mathcal{B}$. Assume $A\colon S \to S$ is a contraction, i.e., there exists a λ, $0 < \lambda < 1$, such that $\|Au - Av\| \leqslant \lambda\|u - v\|$ for all u, v in S. Then A has a unique fixed point in S, that is, there is a unique element u_0 in S such that $Au_0 = u_0$.*

If the function $f(x,\xi)$ satisfies the Lipschitz condition

$$|f(x,\xi_1) - f(x,\xi_2)| \leqslant K|\xi_1 - \xi_2|, \quad x \in [a,c) \cup (c,b], \;\; \xi_1, \xi_2 \in \mathbb{C},$$

then for the operator $A\colon \mathcal{B} \to \mathcal{B}$ defined by (28) we easily get

$$\|Ay - Az\| \leqslant \lambda\|y - z\|,$$

where

$$\lambda = K \sup_{x\in[a,c)\cup(c,b]} \int_a^b |G(x,s)| \,\mathrm{d}s.$$

Therefore, if $\lambda < 1$, then A is a contraction mapping and hence, the BVPI (21)–(23) has a unique solution.

In the next theorem the function $f(x,\xi)$ satisfies a Lipschitz condition not on the whole $\mathbb{C}$ but only on a subset.

Theorem 4. *Assume that the conditions* (H1), (H2), *and* (H4) *are satisfied. Let the function $f(x,\xi)$ satisfy the following Lipschitz condition: for a number $R > 0$,*

$$|f(x,\xi_1) - f(x,\xi_2)| \leqslant K|\xi_1 - \xi_2| \tag{29}$$

for all $x \in [a,c) \cup (c,b]$ and all ξ_1, ξ_2 in the disc $\{\xi \in \mathbb{C}\colon\ |\xi| \leqslant R\}$, where $K > 0$ is a constant which may depend on R. If

$$\sup_{x\in[a,c)\cup(c,b]} \int_a^b |G(x,s)| \,\mathrm{d}s \cdot \sup_{(s,\xi)\in\Omega_R} |f(s,\xi)| \leqslant R, \tag{30}$$

where $\Omega_R = \{(s,\xi)\colon\ s \in [a,c) \cup (c,b],\ \xi \in \mathbb{C},\ |\xi| \leqslant R\}$, *and if*

$$K \cdot \sup_{x \in [a,c) \cup (c,b]} \int_a^b |G(x,s)|\,\mathrm{d}s < 1, \tag{31}$$

then the BVPI (21)–(23) *has a unique solution* $y(x)$ *such that*

$$|y(x)| \leqslant R \quad \text{for } x \in [a,c) \cup (c,b].$$

P r o o f. Let us set $S = \{y \in \mathcal{B}\colon\ \|y\| \leqslant R\}$. Obviously, S is a closed subset of $\mathcal{B}$. Let $A\colon\ \mathcal{B} \to \mathcal{B}$ be the operator defined by (28). For y and z in S we have $|y(s)| \leqslant R$, $|z(s)| \leqslant R$ for all s in $[a,c) \cup (c,b]$. Therefore, taking into account (29) and (31), we can easily get $\|Ay - Az\| \leqslant \lambda\|y - z\|$ for all y and z in S, where $0 < \lambda < 1$. It remains to show that $A\colon\ S \to S$, that is, A transforms the set S into itself. For y in S, we have

$$\begin{aligned}
|(Ay)(x)| &\leqslant \int_a^b |G(x,s)||f(s,y(s))|\,\mathrm{d}s \\
&\leqslant \sup_{(s,\xi)\in\Omega_R} |f(s,\xi)| \cdot \sup_{x \in [a,c) \cup (c,b]} \int_a^b |G(x,s)|\,\mathrm{d}s \leqslant R,
\end{aligned}$$

by (30). Hence, $\|Ay\| \leqslant R$ and therefore, $Ay \in S$. Now the Contraction Mapping Theorem can be applied to obtain a unique fixed point of A in S, and so the proof is completed. □

5. Existence of solutions

An operator (nonlinear, in general) is called *completely continuous* if it is continuous and maps bounded sets into relatively compact sets.

In this section, to get an existence theorem for solutions without uniqueness, we will use the following Schauder Fixed Point Theorem: *Let* $\mathcal{B}$ *be a Banach space and* S *be a nonempty bounded, closed, and convex subset of* $\mathcal{B}$. *Assume* $A\colon\ \mathcal{B} \to \mathcal{B}$ *is a completely continuous operator. If the operator* A *leaves the set* S *invariant, i.e., if* $A(S) \subset S$ *then* A *has at least one fixed point in* S.

Consider the BVPI (21)–(23) and let $A\colon\ \mathcal{B} \to \mathcal{B}$ be the operator defined by (28).

Lemma 5. *A subset* S *of the space* $\mathcal{B}$ *is relatively compact if and only if the functions belonging to* S *are equi-bounded and equi-continuous on each of the intervals* $[a,c)$ *and* $(c,b]$.

P r o o f. Let S be relatively compact. Then provided that $y(c) = y(c^-)$ the set of restrictions of functions $y \in S$ to $[a, c)$ will be a relatively compact set in the Banach space $C[a, c]$ of continuous functions on $[a, c]$. Also, provided that $y(c) = y(c^+)$ the set of restrictions of functions $y \in S$ to $(c, b]$ will be a relatively compact set in $C[c, b]$. Consequently, it follows from the well-known Arzela-Ascoli Theorem that the functions belonging to S are equi-bounded and equi-continuous on each of the intervals $[a, c)$ and $(c, b]$.

Conversely, let the functions belonging to S be equi-bounded and equi-continuous on each of the intervals $[a, c)$ and $(c, b]$. Take any sequence $\{y_n(x)\}$ of functions $y_n \in S$. We have to show that this sequence contains a convergent (in the metric of $\mathcal{B}$) subsequence. The functions $y_n(x)$ for $x \in [a, c]$ provided that $y_n(c) = y_n(c^-)$ are equi-bounded and equi-continuous on $[a, c]$. Therefore, by the Arzela-Ascoli Theorem there is a subsequence $\{u_n(x)\}$ of $\{y_n(x)\}$ that converges uniformly to a continuous function $u(x)$ on $[a, c]$. Next, the functions $u_n(x)$ for $x \in [c, b]$ provided that $u_n(c) = u_n(c^+)$ are equi-bounded and equi-continuous on $[c, b]$. Therefore, again by the Arzela-Ascoli Theorem, the sequence $\{u_n(x)\}$ contains a subsequence $\{v_n(x)\}$ that converges uniformly to a continuous function $v(x)$ on $[c, b]$. Consequently, if we define the function

$$y(x) = \begin{cases} u(x), & x \in [a, c), \\ v(x), & x \in (c, b], \end{cases}$$

then $y \in \mathcal{B}$, and $\{v_n(x)\}$ (subsequence of $\{y_n(x)\}$) converges to $y(x)$ in the metric of $\mathcal{B}$. The lemma is proved. $\square$

Note that the statement of Lemma 5 can be obtained also as a corollary of Proposition 2.3 in [8].

Lemma 6. *Suppose that the function $f(x, \xi)$ satisfies the condition* (H3) *formulated above. Then the operator $A\colon \mathcal{B} \to \mathcal{B}$ defined by* (28) *is completely continuous in the space $\mathcal{B}$.*

P r o o f. Let us take any fixed element $y_0 \in \mathcal{B}$ and show that A is continuous at y_0. Denote

$$M_1 = \sup_{x \in [a,c) \cup (c,b]} \int_a^b |G(x, s)| \, \mathrm{d}s \quad \text{and} \quad M_2 = \sup_{x \in [a,c) \cup (c,b]} |y_0(x)| = \|y_0\|.$$

The function $f(x, \xi)$ is continuous (admits continuous extension) on each of the bounded and closed regions $\Omega_1 = [a, c] \times D$ and $\Omega_2 = [c, b] \times D$, where

$$D = \{\xi \in \mathbb{C}\colon\ |\xi| \leqslant M_2 + 1\}.$$

Therefore, it is uniformly continuous on each of Ω_1 and Ω_2. Hence, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\xi_1, \xi_2 \in D \text{ and } |\xi_1 - \xi_2| \leqslant \delta \quad \text{imply} \quad |f(s,\xi_1) - f(s,\xi_2)| \leqslant \frac{\varepsilon}{M_1}$$

for all $s \in [a,c) \cup (c,b]$. Put $\delta_1 = \min\{1,\delta\}$ and take any $y \in \mathcal{B}$ with $\|y - y_0\| \leqslant \delta_1$. Then we have

$$\|y\| \leqslant \|y_0\| + \delta_1 = M_2 + \delta_1 \leqslant M_2 + 1$$

so that

$$y(s), y_0(s) \in D \text{ and } |y(s) - y_0(s)| \leqslant \delta_1 \leqslant \delta \quad \text{for all } s \in [a,c) \cup (c,b].$$

Therefore,

$$|f(s,y(s)) - f(s,y_0(s))| \leqslant \frac{\varepsilon}{M_1} \quad \text{for all } s \in [a,c) \cup (c,b].$$

Then we have, for all $x \in [a,c) \cup (c,b]$,

$$\begin{aligned}
|(Ay)(x) - (Ay_0)(x)| &\leqslant \int_a^b |G(x,s)||f(s,y(s)) - f(s,y_0(s))| \,\mathrm{d}s \\
&\leqslant \frac{\varepsilon}{M_1} \int_a^b |G(x,s)| \,\mathrm{d}s \leqslant \frac{\varepsilon}{M_1} M_1 = \varepsilon,
\end{aligned}$$

that is, $\|Ay - Ay_0\| \leqslant \varepsilon$. This means that A is continuous at y_0.

Next, let $Y \subset \mathcal{B}$ be a bounded set:

$$\|y\| \leqslant M_3 \quad \text{for all } y \in Y.$$

We have to prove that the set $A(Y)$ is relatively compact in $\mathcal{B}$. For this it is sufficient, by Lemma 5, to show that the functions belonging to $A(S)$ are equi-bounded and equi-continuous on each of the intervals $[a,c)$ and $(c,b]$. Let us set $\Omega = ([a,c) \cup (c,b]) \times \{\xi \in \mathbb{C} \colon |\xi| \leqslant M_3\}$ and

$$M_4 = \sup_{(s,\xi) \in \Omega} |f(s,\xi)|.$$

Then for an arbitrary y in Y and x in $[a,c) \cup (c,b]$, we have

$$|(Ay)(x)| \leqslant \int_a^b |G(x,s)||f(s,y(s))| \,\mathrm{d}s \leqslant M_4 \int_a^b |G(x,s)| \,\mathrm{d}s \leqslant M_4 M_1.$$

Hence, $\|Ay\| \leqslant M_4 M_1$ for all $y \in Y$. Therefore, $A(Y)$ is a bounded set in $\mathcal{B}$.

Further, the function $G(x,s)$ is uniformly continuous on each of the rectangles $[a,c)\times[a,c)$, $[a,c)\times(c,b]$, $(c,b]\times[a,c)$, and $(c,b]\times(c,b]$. Therefore, for a given $\varepsilon>0$ we can find a $\delta>0$ such that if $x_1,x_2\in[a,c)$ or $x_1,x_2\in(c,b]$, and $|x_1-x_2|\leqslant\delta$, and s is arbitrary in $[a,c)\cup(c,b]$, then

$$|G(x_1,s)-G(x_2,s)|\leqslant\frac{\varepsilon}{M_4(b-a)}.$$

Consequently, for arbitrary y in Y and x_1, x_2 as above, we have

$$\begin{aligned}|(Ay)(x_1)-(Ay)(x_2)| &\leqslant \int_a^b |G(x_1,s)-G(x_2,s)||f(s,y(s))|\,\mathrm{d}s\\ &\leqslant \frac{\varepsilon}{M_4(b-a)}\int_a^b |f(s,y(s))|\,\mathrm{d}s\\ &\leqslant \frac{\varepsilon}{M_4(b-a)}\,M_4(b-a)=\varepsilon.\end{aligned}$$

This proves that the functions belonging to $A(Y)$ are equi-continuous. □

Theorem 7. *In addition to the hypotheses* (H1), (H2), (H3), *and* (H4), *assume that there exists a number $R>0$ such that*

$$\sup_{x\in[a,c)\cup(c,b]}\int_a^b |G(x,s)|\,\mathrm{d}s\cdot\sup_{(s,\xi)\in\Omega_R}|f(s,\xi)|\leqslant R, \tag{32}$$

where $\Omega_R=\{(s,\xi)\colon\ s\in[a,c)\cup(c,b],\ \xi\in\mathbb{C},\ |\xi|\leqslant R\}$. Then the BVPI (21)–(23) *has at least one solution y such that*

$$|y(x)|\leqslant R\quad\textit{for }\ x\in[a,c)\cup(c,b].$$

P r o o f. Let $A\colon\ \mathcal{B}\to\mathcal{B}$ be the operator defined by (28). It follows from Lemma 6 that A is completely continuous. Using (32), we can see, as in the proof of Theorem 4, that A maps the set $S=\{y\in\mathcal{B}\colon\ \|y\|\leqslant R\}$ into itself. On the other hand, it is obvious that the set S is bounded, closed, and convex. Therefore, the Schauder Fixed Point Theorem can be applied to obtain a fixed point of A in S. This completes the proof. □

6. Examples

In this section, we discuss the condition (32) of Theorem 7 in some examples which summarize the situation in three cases: sublinear, linear, and quadratic growth.

It follows from the formula (20) that the Green's function $G(x,s)$ is bounded for $x,s \in [a,c) \cup (c,b]$. Therefore, we can define a finite positive number g by

$$g^{-1} = \sup_{x \in [a,c)\cup(c,b]} \int_a^b |G(x,s)|\,\mathrm{d}s \tag{33}$$

and the condition (32) can be written as

$$|f(x,\xi)| \leqslant gR \quad \text{for } (x,\xi) \in \Omega_R, \tag{34}$$

that is, for $x \in [a,c) \cup (c,b]$ and $\xi \in \mathbb{C}$ with $|\xi| \leqslant R$.

1. If the function $f(x,\xi)$ satisfies

$$|f(x,\xi)| \leqslant c_1 + c_2|\xi|^r \quad \text{for all } x \in [a,c) \cup (c,b] \text{ and } \xi \in \mathbb{C},$$

where c_1, c_2, and r are some positive constants, then the condition (34) will be satisfied if

$$c_1 + c_2 R^r \leqslant gR. \tag{35}$$

Obviously, the last inequality will hold if $r < 1$ and R is sufficiently large. We can determine how large must R be. Indeed, rewriting the inequality (35) in the form

$$R\Big(g - \frac{c_2}{R^{1-r}}\Big) \geqslant c_1, \tag{36}$$

we require that the inequality

$$g - \frac{c_2}{R^{1-r}} \geqslant \frac{g}{2}, \quad \text{that is,} \quad R \geqslant \Big(\frac{2c_2}{g}\Big)^{1/(1-r)}$$

be also satisfied. Then (36) yields

$$R \geqslant \frac{2c_1}{g}.$$

Therefore, if

$$R \geqslant \max\Big\{\frac{2c_1}{g}, \Big(\frac{2c_2}{g}\Big)^{1/(1-r)}\Big\},$$

then the inequality (35) will be satisfied.

2. Let

$$|f(x,\xi)| \leqslant c_1 + c_2|\xi| \quad \text{for all } x \in [a,c) \cup (c,b] \text{ and } \xi \in \mathbb{C},$$

where c_1, c_2 are some positive constants. Then the condition (34) will be satisfied if the positive number R can be chosen so that

$$c_1 + c_2 R \leqslant gR.$$

This will hold if

$$c_2 < g \quad \text{and} \quad R \geqslant \frac{c_1}{g - c_2}.$$

3. Let

$$|f(x,\xi)| \leqslant c_1 + c_2|\xi|^2 \quad \text{for all } x \in [a,c) \cup (c,b] \text{ and } \xi \in \mathbb{C},$$

where c_1, c_2 are some positive constants. Then the condition (34) will be satisfied if the positive number R can be chosen so that

$$c_1 + c_2 R^2 \leqslant gR,$$

that is,

$$c_2 R^2 - gR + c_1 \leqslant 0. \tag{37}$$

The quadratic function $c_2\lambda^2 - g\lambda + c_1$ has two distinct real zeros

$$\lambda_1 = \frac{g - \sqrt{g^2 - 4c_1c_2}}{2c_2} \quad \text{and} \quad \lambda_2 = \frac{g + \sqrt{g^2 - 4c_1c_2}}{2c_2}$$

if $4c_1c_2 < g^2$, and for any R satisfying

$$\lambda_1 \leqslant R \leqslant \lambda_2$$

the inequality (37) will be satisfied.

7. Concluding remarks

1. The condition (34) involves the number g defined by (33). Therefore, in order to make this condition explicit we need to calculate the number g explicitly or to get at least an explicit positive lower bound for g.

In the case of $p(x) \equiv 1$, $q(x) \equiv \alpha^2$ ($\alpha > 0$), $a = 0$, $d_1 = d_2 = 1$, where α is constant, the problem (21)–(23) takes the form

$$\begin{gathered} -y'' + \alpha^2 y = f(x, y), \quad x \in [0, b], \\ y(0) = y(b), \quad y'(0) = y'(b). \end{gathered}$$

The associated Green's function $G(x, s)$ is given in Remark 2 made above in Section 2. A straightforward calculation shows that

$$\int_0^b |G(x, s)| \, \mathrm{d}s = \int_0^b G(x, s) \, \mathrm{d}s = \alpha^{-2}$$

for all $x \in [0, b]$, and hence $g = \alpha^2$.

In the case of $p(x) \equiv 1$, $q(x) \equiv 1$, $a = -1$, $c = 0$, $b = 1$, the problem (21)–(23) takes the form

$$\begin{gathered} -y'' + y = f(x, y), \quad x \in [-1, 0) \cup (0, 1], \\ y(0^-) = d_1 y(0^+), \quad y'(0^-) = d_2 y'(0^+), \\ y(-1) = y(1), \quad y'(-1) = y'(1). \end{gathered}$$

Suppose that

$$d_1 > 0, \quad d_2 > 0, \quad \text{and} \quad d_1 + d_2 \geqslant 1 + d_1 d_2. \tag{38}$$

In this case the following can be shown.

(a) If $d_1 = 1$ and $d_2 > 0$ is arbitrary (such d_1 and d_2 satisfy (38)), then $g = 1$.

(b) If $d_2 = \mathrm{e}^2$ and $0 < d_1 \leqslant 1$ (such d_1 and d_2 satisfy (38)), then

$$g^{-1} = 1 + \frac{\mathrm{e}^{-2}(d_1 - 1)(1 - \mathrm{e}^4)}{1 + (\mathrm{e}^2 - \mathrm{e}^{-2})d_1 - \mathrm{e}^4}.$$

(c) If $d_2 = \mathrm{e}^{-2}$ and $d_1 \geqslant 1$ (such d_1 and d_2 satisfy (38)), then

$$g^{-1} = 1 + \frac{(d_1 - 1)(1 - \mathrm{e}^{-4})}{1 - (\mathrm{e}^2 - \mathrm{e}^{-2})d_1 - \mathrm{e}^{-4}}.$$

2. Theorem 7 yields a periodic solution of the corresponding perodic impulsive nonlinear differential equation on the whole real axis as follows. Let $\omega > 0$ be a fixed real number and $0 < c < \omega$. Let us set $c_i = c + i\omega$ for $i \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. Consider the periodic impulsive problem

$$-[p(x)y']' + q(x)y = f(x, y), \quad x \in \mathbb{R} \setminus \{c_i \colon i \in \mathbb{Z}\}, \tag{39}$$

$$y(c_i^-) = d_1 y(c_i^+), \quad y^{[1]}(c_i^-) = d_2 y^{[1]}(c_i^+), \quad i \in \mathbb{Z}, \tag{40}$$

where we assume that the following periodicity conditions are satisfied:

(H5) $p(x+\omega) = p(x)$, $q(x+\omega) = q(x)$, $x \in \mathbb{R} \setminus \{c_i \colon i \in \mathbb{Z}\}$,

(H6) $f(x+\omega, \xi) = f(x, \xi)$, $x \in \mathbb{R} \setminus \{c_i \colon i \in \mathbb{Z}\}$, $\xi \in \mathbb{C}$.

We are interested in the existence of ω-periodic (i.e. periodic with period ω) solutions of the problem (39)–(40). Together with the problem (39)–(40) consider the BVPI (21)–(23) with $a = 0$ and $b = \omega$. If the conditions (H5) and (H6) are satisfied, then every solution of the BVPI (21)–(23) with $a = 0$ and $b = \omega$, extended from $[a, c) \cup (c, b]$ to $\mathbb{R} \setminus \{c_i \colon i \in \mathbb{Z}\}$ as an ω-periodic function, will be a solution of the problem (39)–(40). Therefore, Theorem 7 yields the following result: Assume that the conditions of Theorem 7 with $a = 0$ and $b = \omega$, and (H5), (H6) are satisfied. Then the problem (39)–(40) has at least one ω-periodic solution $y(x)$ such that

$$|y(x)| \leqslant R \quad \text{for } x \in \mathbb{R} \setminus \{c_i \colon i \in \mathbb{Z}\}.$$

References

[1] *F. M. Atici, G. Sh. Guseinov*: On the existence of positive solutions for nonlinear differential equations with periodic coefficients. J. Comput. Appl. Math. *132* (2001), 341–356.

[2] *D. D. Bainov, P. S. Simeonov*: Impulsive Differential Equations: Periodic Solutions and Applications. Longman Scientific & Technical/John Wiley & Sons, Harlow/New York, 1993.

[3] *M. Benchohra, J. Henderson, S. Ntouyas*: Impulsive Differential Equations and Inclusions. Hindawi Publishing Corporation, New York, 2006.

[4] *J. Chen, C. C. Tisdell, R. Yuan*: On the solvability of periodic boundary value problems with impulse. J. Math. Anal. Appl. *331* (2007), 902–912.

[5] *Y. Dong*: Periodic solutions for second order impulsive differential systems. Nonlinear Anal., Theory Methods Appl. *27* (1996), 811–820.

[6] *P. W. Eloe, J. Henderson*: A boundary value problem for a system of ordinary differential equations with impulse effects. Rocky Mt. J. Math. *27* (1997), 785–799.

[7] *D. Franco, J. J. Nieto*: Maximum principles for periodic impulsive first order problems. J. Comput. Appl. Math. *88* (1998), 149–159.

[8] *D. Fraňková*: Regulated functions. Math. Bohem. *116* (1991), 20–59.

[9] *R. K. George, A. K. Nandakumaran, A. Arapostathis*: A note on controllability of impulsive systems. J. Math. Anal. Appl. *241* (2000), 276–283.

[10] *Z. H. Guan, G. Chen, T. Ueta*: On impulsive control of a periodically forced chaotic pendulum system. IEEE Trans. Autom. Control *45* (2000), 1724–1727.

[11] *S. Hu, V. Lakshmikantham*: Periodic boundary value problems for second order impulsive differential systems. Nonlinear Anal., Theory Methods Appl. *13* (1989), 75–85.

[12] *A. Huseynov*: On the sign of Green's function for an impulsive differential equation with periodic boundary conditions. Appl. Math. Comput. *208* (2009), 197–205.

[13] *S. G. Khristova, D. D. Bainov*: Existence of periodic solutions of nonlinear systems of differential equations with impulse effect. J. Math. Anal. Appl. *125* (1987), 192–202.

[14] *A. Lakmeche, O. Arino*: Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment. Dyn. Contin. Discrete Impuls. Syst. *7* (2000), 265–287.

[15] *V. Lakshmikantham, D. D. Bainov, P. S. Simeonov*: Theory of Impulsive Differential Equations. World Scientific, Singapore, 1989.

[16] *V. Lakshmikantham, K. N. Murty, S. Sivasundaram*: Impulsive differential systems for two-point boundary value problems. Appl. Math. Comput. *50* (1992), 157–166.

[17] *S. Lenci, G. Rega*: Periodic solutions and bifurcations in an impact inverted pendulum under impulsive excitation. Chaos Solitons Fractals *11* (2000), 2453–2472.

[18] *J. Li, J. J. Nieto, J. Shen*: Impulsive periodic boundary value problems of first-order differential equations. J. Math. Anal. Appl. *325* (2007), 226–236.

[19] *E. Liz, J. J. Nieto*: The monotone iterative technique for periodic boundary value problems of second order impulsive differential equations. Commentat. Math. Univ. Carol. *34* (1993), 405–411.

[20] *S. Nenov*: Impulsive controllability and optimization problems in population dynamics. Nonlinear Anal., Theory Methods Appl. *36* (1999), 881–890.

[21] *J. J. Nieto*: Basic theory for nonresonance impulsive periodic problems of first order. J. Math. Anal. Appl. *205* (1997), 423–433.

[22] *J. J. Nieto*: Impulsive resonance periodic problems of first order. Appl. Math. Lett. *15* (2002), 489–493.

[23] *J. J. Nieto*: Periodic boundary value problems for first-order impulsive ordinary differential equations. Nonlinear. Anal., Theory Methods Appl. *51* (2002), 1223–1232.

[24] *A. M. Samoilenko, N. A. Perestyuk*: Impulsive Differential Equations. World Scientific, Singapore, 1995.

[25] *Y. Tian*: Solutions to boundary-value problems for second-order impulsive differential equations at resonance. Electron. J. Differ. Equ. 2008, No. 18 (2008), 1–9.

[26] *A. S. Vatsala, Y. Sun*: Periodic boundary value problems of impulsive differential equations. Appl. Anal. *44* (1992), 145–158.

Author's address: *A. Huseynov*, Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, AZ1141 Baku, Azerbaijan, e-mail: `huseynov@email.com`.