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SOLVABILITY OF A CLASS OF ELASTIC BEAM EQUATIONS WITH STRONG CARATHÉODORY NONLINEARITY

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Abstract. We study the existence of a solution to the nonlinear fourth-order elastic beam equation with nonhomogeneous boundary conditions

$$\begin{cases} u^{(4)}(t) = f\big(t, u(t), u'(t), u''(t), u'''(t)\big), & \text{a.e. } t \in [0,1], \\ u(0) = a,\ u'(0) = b,\ u(1) = c,\ u''(1) = d, \end{cases}$$

where the nonlinear term $f(t, u_0, u_1, u_2, u_3)$ is a strong Carathéodory function. By constructing suitable height functions of the nonlinear term $f(t, u_0, u_1, u_2, u_3)$ on bounded sets and applying the Leray-Schauder fixed point theorem, we prove that the equation has a solution provided that the integration of some height function has an appropriate value.

Keywords: nonlinear ordinary differential equation, boundary value problem, existence, fixed point theorem

MSC 2010: 34B15, 34B16

1. INTRODUCTION

In this paper we consider the nonlinear elastic beam equation with nonhomogeneous Dirichlet boundary conditions on the left and nonhomogeneous Navier boundary conditions on the right

$$\text{(P)} \qquad \begin{cases} u^{(4)}(t) = f\big(t, u(t), u'(t), u''(t), u'''(t)\big), & \text{a.e. } t \in [0,1], \\ u(0) = a,\ u'(0) = b,\ u(1) = c,\ u''(1) = d. \end{cases}$$

In the homogeneous case, (P) corresponds to the elastic beam equation rigidly fixed at the left and simply supported at the right.

Throughout this paper, $\|u\| = \max_{0\leqslant t\leqslant 1} |u(t)|$ for $u \in C[0,1]$ and

$$p(t) = \frac{1}{4}(2a + 2b - 2c + d)t^3 + \frac{1}{4}(6c - 6a - 6b - d)t^2 + bt + a.$$

Thus, $p(0) = a$, $p'(0) = b$, $p(1) = c$, $p''(1) = d$, $p'''(t) \equiv 3a + 3b - 3c + \frac{3}{2}d$.

Denote $\gamma_i = \max_{0\leqslant t\leqslant 1} |p^{(i)}(t)|$, $i = 0,1,2,3$, and

$$\mu_0 = \max_{0\leqslant t\leqslant 1} p(t), \quad \nu_0 = \min_{0\leqslant t\leqslant 1} p(t), \quad \mu_1 = \max_{0\leqslant t\leqslant 1} p'(t), \quad \nu_1 = \min_{0\leqslant t\leqslant 1} p'(t),$$
$$\mu_2 = \max_{0\leqslant t\leqslant 1} p''(t), \quad \nu_2 = \min_{0\leqslant t\leqslant 1} p''(t), \quad \mu_3 = \nu_3 = 3a + 3b - 3c + \frac{3}{2}d.$$

Let us introduce constants

$$K = \frac{65536}{39 + 55\sqrt{33}} \approx 184.6340, \quad M = \frac{3}{17 - 12\sqrt{2}} \approx 101.9117,$$
$$k_0 = 1, \quad k_1 = \frac{65536}{45(39 + 55\sqrt{33})} \approx 4.1030, \quad k_2 = \frac{8192}{39 + 55\sqrt{33}} \approx 23.0793,$$
$$k_3 = \frac{40960}{39 + 55\sqrt{33}} \approx 115.3963,$$
$$m_0 = 1, \quad m_1 = \frac{1}{9[17 - 12\sqrt{2}]} \approx 3.7745, \quad m_2 = \frac{\sqrt{3}}{3[17 - 12\sqrt{2}]} \approx 19.6129,$$
$$m_3 = \frac{3}{17 - 12\sqrt{2}} \approx 101.9117.$$

In addition, let $G(t,s)$ be the Green function of the linear homogeneous boundary value problem $u^{(4)}(t) = 0$, $0 \leqslant t \leqslant 1$, $u(0) = u'(0) = u(1) = u''(1) = 0$, that is,

$$G(t,s) = \begin{cases} \frac{1}{12}(1-t)s^2[3(1-s) - (1-t)^2(3-s)], & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{1}{12}t^2(1-s)[3(1-t) - (1-s)^2(3-t)], & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Clearly, $G(t,s) > 0$, $0 < t, s < 1$.

The nonlinear beam equation (P) has been studied by some authors when $f\colon [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous, for example, see [1], [4], [5], [7], [9], [10], [11], [14]. Particularly, Agarwal [1] introduced the maximum height

$$\Phi(r_{0-3}) = \max\{|f(t,u_0,u_1,u_2,u_3)|\colon\ 0 \leqslant t \leqslant 1,\ |u_i| \leqslant 2r_i,\ i = 0,1,2,3\}$$

of the nonlinear term $f(u_0,u_1,u_2,u_3)$ on the bounded set $[0,1] \times [-2r_0, 2r_0] \times [-2r_1, 2r_1] \times [-2r_2, 2r_2] \times [-2r_3, 2r_3]$ and proved the following local existence theorem.

Theorem 1.1. *Suppose that* $f\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ *is continuous and there exist four positive numbers* $r_i>0$, $i=0,1,2,3$, *such that* $r_i\geqslant\gamma_i$ *and*

$$\Phi(r_{0-3})\leqslant K\min\{k_i^{-1}r_i,\ i=0,1,2,3\}.$$

Then equation (P) *has at least one solution* $u^*\in C^3[0,1]$ *and* $\|(u^*)^{(i)}\|\leqslant 2r_i$, $i=0,1,2,3$.

Theorem 1.1 shows that the equation (P) has a solution if there exist four positive numbers $r_i>0$, $i=0,1,2,3$, such that the maximum height $\Phi(r_{0-3})$ has an appropriate value.

The aim of this paper is to study the existence of a solution to the equation (P) when $f\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ is a strong Carathéodory function. Here, $f\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ is referred to as a strong Carathéodory function if

(C1) for a.e. $t\in[0,1]$, $f(t,\cdot,\cdot,\cdot,\cdot)\colon \mathbb{R}^4\to\mathbb{R}$ is continuous;
(C2) for all $(u_0,u_1,u_2,u_3)\in\mathbb{R}^4$, $f(\cdot,u_0,u_1,u_2,u_3)\colon [0,1]\to\mathbb{R}$ is measurable;
(C3) for every $r>0$ there exists a nonnegative function $j_r\in L^1[0,1]$, such that

$$|f(t,u_0,u_1,u_2,u_3)|\leqslant j_r(t),\quad (t,u_0,u_1,u_2,u_3)\in[0,1]\times[-r,r]^4.$$

In the ordinary way, $f(t,u_0,u_1,u_2,u_3)$ is called a Carathéodory function if $f\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ only satisfies the conditions (C1) and (C2).

If $f\colon (0,1)\times\mathbb{R}^4\to\mathbb{R}$ is continuous and satisfies the assumption (C3), then it is a strong Carathéodory function. In this case, the function $f(t,u_0,u_1,u_2,u_3)$ may be singular at $t=0$, $t=1$.

Therefore, we allow that there exists a zero measure set $E\subset[0,1]$ such that the nonlinear term $f(t,u_0,u_1,u_2,u_3)$ is discontinuous or singular at $t\in E$.

It is impossible to describe the growth change of a singular function on a bounded set by the maximum height $\Phi(r_{0-3})$. Therefore, we must introduce new tool in order to deal with the singular equation (P).

In this paper, we introduce a height function $\varphi(t,r_{0-3})$ to estimate the growth change of the nonlinear term $f(t,u_0,u_1,u_2,u_3)$ on bounded set. Here, the height function is

$$\varphi(t,r_{0-3})=\max\{|f(t,u_0,u_1,u_2,u_3)|\colon -r_i+\nu_i\leqslant u_i\leqslant r_i+\mu_i,\ i=0,1,2,3\}.$$

When $f\colon [0,1]\times\mathbb{R}^4\to\mathbb{R}$ is a strong Carathéodory function, the height function $\varphi(\cdot,r_{0-3})$ is integrable on $[0,1]$.

In geometry, the height function $\varphi(t,r_{0-3})$ is the maximum height of the nonlinear term $f(t,u_0,u_1,u_2,u_3)$ on the set $\{t\}\times[-r_0+\nu_0,r_0+\mu_0]\times[-r_1+\nu_1,r_1+\mu_1]\times[-r_2+\nu_2,r_2+\mu_2]\times[-r_3+\nu_3,r_3+\mu_3]$.

By applying the height function, we obtain a new result. More precisely, we prove the following local existence theorem.

Theorem 1.2. *Suppose that* $f\colon [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ *is a strong Carathéodory function and there exist four positive numbers* $r_i > 0$, $i = 0,1,2,3$, *such that one of the following conditions is satisfied:*

(1) $\max\limits_{0\leqslant t\leqslant 1} \int_0^1 |(\partial^i/\partial t^i)G(t,s)|\varphi(s,r_{0-3})\,\mathrm{d}s \leqslant r_i$, $i = 0,1,2,3$,

(2) $\int_0^1 \varphi(t,r_{0-3})\,\mathrm{d}t \leqslant M\min\{m_i^{-1}r_i,\ i=0,1,2,3\}$.

Then the equation (P) *has at least one solution* $u^* \in C^3[0,1]$ *and* $\|(u^*)^{(i)} - p^{(i)}\| \leqslant r_i$, $i = 0,1,2,3$. *In addition, the solution* u^* *is nontrivial if* $a^2+b^2+c^2+d^2 > 0$ *or* $\int_0^1 |f(t,0,0,0,0)|\,\mathrm{d}t > 0$.

Theorem 1.2 shows that the equation (P) has a solution if there exist four positive numbers $r_i > 0$, $i = 0,1,2,3$, such that the integration of the height function $\varphi(t,r_{0-3})$ has an appropriate value.

Obviously, Theorem 1.2 can deal with more complex cases than Theorem 1.1. We will verify that Theorem 1.2 extends Theorem 1.1 and illustrate that the extension is true by an example. The localization idea of this work comes from the papers [2], [8], [12], [13], [14], [15].

2. Preliminaries

After direct computations, we get

$$\max_{0\leqslant t,s\leqslant 1} G(t,s) = G(2-\sqrt{2}, 2-\sqrt{2}) = \frac{1}{3}[17-12\sqrt{2}].$$

Computing the partial derivatives of $G(t,s)$ in t, we have

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \frac14 t(1-s)[2-3t-(2-t)(1-s)^2], & 0\leqslant t\leqslant s\leqslant 1,\\ \frac14 s^2[(1-t)^2(3-s)-(1-s)], & 0\leqslant s\leqslant t\leqslant 1;\end{cases}$$

$$\frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} \frac12 (1-s)[1-3t-(1-t)(1-s)^2], & 0\leqslant t\leqslant s\leqslant 1,\\ \frac12 s^2(1-t)(s-3), & 0\leqslant s\leqslant t\leqslant 1;\end{cases}$$

$$\frac{\partial^3}{\partial t^3}G(t,s) = \begin{cases} \frac12 (1-s)(s^2-2s-2), & 0\leqslant t< s\leqslant 1,\\ \frac12 s^2(3-s), & 0\leqslant s< t\leqslant 1.\end{cases}$$

This yields

$$\max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial}{\partial t}G(t,s)\Big| = \Big|\frac{\partial}{\partial t}G\Big(1,\frac{2}{3}\Big)\Big| = \frac{1}{27},$$
$$\max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial^2}{\partial t^2}G(t,s)\Big| = \Big|\frac{\partial^2}{\partial t^2}G\Big(0,\frac{3-\sqrt{3}}{3}\Big)\Big| = \frac{\sqrt{3}}{9},$$
$$\sup_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial^3}{\partial t^3}G(t,s)\Big| = \lim_{s\to 1}\Big|\frac{\partial^3}{\partial t^3}G(1,s)\Big| = 1.$$

Therefore,

$$\begin{aligned}
M &= \frac{3}{17-12\sqrt{2}} = \Big[\max_{0\leqslant t,s\leqslant 1} G(t,s)\Big]^{-1},\\
m_1 &= \frac{1}{9[17-12\sqrt{2}]} = M\max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial}{\partial t}G(t,s)\Big|,\\
m_2 &= \frac{\sqrt{3}}{3[17-12\sqrt{2}]} = M\max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial^2}{\partial t^2}G(t,s)\Big|,\\
m_3 &= \frac{3}{17-12\sqrt{2}} = M\max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial^3}{\partial t^3}G(t,s)\Big|.
\end{aligned}$$

In this paper, $C^3[0,1]$ is the Banach space with the norm

$$|||u||| = \max\{\|u\|, \|u'\|, \|u''\|, \|u'''\|\}.$$

If $u \in C^3[0,1]$ and the assumption (C3) holds, then there exists a nonnegative function $j_{|||u|||+1} \in L^1[0,1]$ such that

$$|f(t,u(t),u'(t),u''(t),u'''(t))| \leqslant j_{|||u|||+1}(t), \quad 0\leqslant t\leqslant 1.$$

Define an operator T as follows, for $u \in C^3[0,1]$ and $0\leqslant t\leqslant 1$:

$$\begin{aligned}(Tu)(t) = \int_0^1 G(t,s) f\big(s,u(s)+p(s),u'(s)+p'(s),u''(s)+p''(s),\\
u'''(s)+p'''(s)\big)\,\mathrm{d}s.\end{aligned}$$

For convenience, we use the abbreviation

$$F(t,u(t)+p(t)) = f\big(t,u(t)+p(t),u'(t)+p'(t),u''(t)+p''(t),u'''(t)+p'''(t)\big).$$

Thus

$$(Tu)(t) = \int_0^1 G(t,s)F(s,u(s)+p(s))\,\mathrm{d}s, \quad 0\leqslant t\leqslant 1.$$

For $u \in C^3[0,1]$ and $0 \leqslant t \leqslant 1$, let

$$\begin{aligned}
(Au)(t) &= f\bigl(t, u(t)+p(t), u'(t)+p'(t), u''(t)+p''(t), u'''(t)+p'''(t)\bigr),\\
(Bu)(t) &= \int_0^1 G(t,s)u(s)\,\mathrm{d}s,\\
(Cu)(t) &= \int_0^1 \frac{\partial}{\partial t}G(t,s)u(s)\,\mathrm{d}s,\\
(Du)(t) &= \int_0^1 \frac{\partial^2}{\partial t^2}G(t,s)u(s)\,\mathrm{d}s,\\
(Eu)(t) &= \int_0^1 \frac{\partial^3}{\partial t^3}G(t,s)u(s) = \,\mathrm{d}s.
\end{aligned}$$

Proposition 2.1. *The operator $A\colon C^3[0,1] \to L^1[0,1]$ is continuous.*

P r o o f. Let $u \in C^3[0,1]$. Since $u+p \in C^3[0,1]$, there exist four sequences $\{\xi_n^i\}_{n=1}^{\infty}$, $i=0,1,2,3$, of simple functions such that

$$\lim_{n\to\infty} \xi_n^i(t) = u^{(i)}(t) + p^{(i)}(t), \quad \text{a.e. } t \in [0,1],\ i=0,1,2,3$$

(see (11.35), [6]). By the assumption (C2), $f\bigl(t, \xi_n^0(t), \xi_n^1(t), \xi_n^2(t), \xi_n^3(t)\bigr)$ is a measurable function on $[0,1]$ for each $n=1,2,\ldots$. By the assumption (C1),

$$\lim_{n\to\infty} f\bigl(t, \xi_n^0(t), \xi_n^1(t), \xi_n^2(t), \xi_n^3(t)\bigr) = (Au)(t), \quad \text{a.e. } t \in [0,1].$$

Since the limit of measurable functions is measurable (see (11.18), [6]), we assert that $(Au)(t)$ is a measurable function on $[0,1]$.

By making use of the above abbreviation, we have

$$(Au)(t) = F(t, u(t)+p(t)), \quad 0 \leqslant t \leqslant 1.$$

Let $r_u = |||u||| + |||p||| + 1$. Since

$$\max\{\|u+p\|, \|u'+p'\|, \|u''+p''\|, \|u'''+p'''\|\} \leqslant |||u||| + |||p||| + 1$$

and (C3), there exists $j_{r_u} \in L^1[0,1]$ such that

$$|F(t, u(t)+p(t))| \leqslant j_{r_u}(t), \quad 0 \leqslant t \leqslant 1.$$

Hence $Au \in L^1[0,1]$. It follows that $A\colon C^3[0,1] \to L^1[0,1]$.

Let $u_n \in C^3[0,1]$ and $|||u_n - u||| \to 0$. Then $\lim\limits_{n\to\infty} u_n^{(i)}(t) = u^{(i)}(t)$, $0 \leqslant t \leqslant 1$, $i = 0,1,2,3$. By (C1),

$$\lim_{n\to\infty} F(t, u_n(t) + p(t)) = F(t, u(t) + p(t)), \quad \text{a.e. } t \in [0,1].$$

Since $|||u_n - u||| \to 0$, there exists a positive integer $N(u)$ such that, for any $n \geqslant N(u)$, $|||u_n + p||| \leqslant |||u||| + |||p||| + 1$. So, for any $n \geqslant N(u)$,

$$|F(t, u_n(t) + p(t))| \leqslant j_{r_u}(t), \quad 0 \leqslant t \leqslant 1.$$

By the Lebesgue dominated convergence theorem (see (12.24), [6]), we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^1 |(Au_n)(t) - (Au)(t)| \,\mathrm{d}t \\
&\quad = \lim_{n\to\infty} \int_0^1 |F(t, u_n(t) + p(t)) - F(t, u(t) + p(t))| \,\mathrm{d}s \\
&\quad = \int_0^1 \lim_{n\to\infty} |F(t, u_n(t) + p(t)) - F(t, u(t) + p(t))| \,\mathrm{d}s = 0.
\end{aligned}$$

It follows that $A\colon C^3[0,1] \to L^1[0,1]$ is continuous. □

Proposition 2.2. *$B, C, D, E\colon L^1[0,1] \to C[0,1]$ are completely continuous operators.*

P r o o f. Obviously, $B, C, D, E\colon L^1[0,1] \to C[0,1]$ are bounded linear operators. Applying the Arzela-Ascoli theorem, we prove easily that B, C, D, E are completely continuous operators. □

Proposition 2.3. *For any $u \in L^1[0,1]$ and a.e. $t \in [0,1]$,*

$$(Bu)'(t) = (Cu)(t), \quad (Cu)'(t) = (Du)(t), \quad (Du)'(t) = (Eu)(t).$$

P r o o f. By the generalized mean value theorem in the nonsmooth analysis (see Theorem 2.3.7, [3]), we have, for every $i = 0,1,2$,

$$\Bigl|\frac{\partial^i}{\partial t^i} G(t + \Delta t, s) - \frac{\partial^i}{\partial t^i} G(t,s)\Bigr| \leqslant \max_{0 \leqslant t,s \leqslant 1} \Bigl|\frac{\partial^{i+1}}{\partial t^{i+1}} G(t,s)\Bigr| |\Delta t| \leqslant |\Delta t|.$$

From this inequality we obtain the estimate, for $i = 0,1,2$,

$$\Bigl|\frac{1}{\Delta t}\Bigl[\frac{\partial^i}{\partial t^i} G(t + \Delta t, s) - \frac{\partial^i}{\partial t^i} G(t,s)\Bigr] u(s)\Bigr| \leqslant |u(s)|, \quad \text{a.e. } t \in [0,1].$$

Since $u \in L^1[0,1]$, the Lebesgue dominated convergence theorem yields

$$\begin{aligned}
(Bu)'(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[(Bu)(t+\Delta t) - (Bu)(t)] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_0^1 [G(t+\Delta t, s) - G(t,s)]u(s)\,\mathrm{d}s \\
&= \int_0^1 \lim_{\Delta t \to 0} \frac{1}{\Delta t}[G(t+\Delta t, s) - G(t,s)]u(s)\,\mathrm{d}s \\
&= \int_0^1 \frac{\partial}{\partial t} G(t,s)u(s)\,\mathrm{d}s = (Cu)(t).
\end{aligned}$$

Similarly, $(Cu)'(t) = (Du)(t)$, $(Du)'(t) = (Eu)(t)$. □

Lemma 2.4. *$T\colon C^3[0,1] \to C^3[0,1]$ is a completely continuous operator.*

P r o o f. By Proposition 2.3, we have the compositions

$$T = B \circ A, \quad (T(\cdot))' = C \circ A, \quad (T(\cdot))'' = D \circ A, \quad (T(\cdot))''' = E \circ A.$$

By Proposition 2.1 and 2.2, we assert that $T, (T(\cdot))', (T(\cdot))'', (T(\cdot))'''\colon C^3[0,1] \to C[0,1]$ are completely continuous operators. Therefore, $T\colon C^3[0,1] \to C^3[0,1]$ is a completely continuous operator. □

3. The proof of Theorem 1.2

Let $V_{r_{0-3}} = \{u \in C^3[0,1]\colon \|u^{(i)}\| \leqslant r_i,\ i = 0,1,2,3\}$. We need to prove $T\colon V_{r_{0-3}} \to V_{r_{0-3}}$.

If $u \in V_{r_{0-3}}$ then $\|u^{(i)}\| \leqslant r_i$, $i = 0,1,2,3$. This implies that, for $0 \leqslant t \leqslant 1$,

$$\begin{aligned}
-r_0 + \nu_0 &\leqslant u(t) + p(t) \leqslant r_0 + \mu_0, \\
-r_1 + \nu_1 &\leqslant u'(t) + p'(t) \leqslant r_1 + \mu_1, \\
-r_2 + \nu_2 &\leqslant u''(t) + p''(t) \leqslant r_2 + \mu_2, \\
-r_3 + \nu_3 &\leqslant u'''(t) + p'''(t) \leqslant r_3 + \mu_3.
\end{aligned}$$

By the definition of $\varphi(t, r_{0-3})$, we have

$$|F(t, u(t) + p(t))| \leqslant \varphi(t, r_{0-3}), \quad t \in [0,1].$$

Applying this fact and the assumption (1), we get

$$
\begin{aligned}
\|Tu\| &\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)|F(s,u(s)+p(s))|\,\mathrm{d}s \\
&\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)\varphi(s,r_{0-3})\,\mathrm{d}s \leqslant r_0, \\
\|(Tu)'\| &\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial}{\partial t}G(t,s)\Big||F(s,u(s)+p(s))|\,\mathrm{d}s \\
&\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial}{\partial t}G(t,s)\Big|\varphi(s,r_{0-3})\,\mathrm{d}s \leqslant r_1, \\
\|(Tu)''\| &\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial^2}{\partial t^2}G(t,s)\Big||F(s,u(s)+p(s))|\,\mathrm{d}s \\
&\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial^2}{\partial t^2}G(t,s)\Big|\varphi(s,r_{0-3})\,\mathrm{d}s \leqslant r_2, \\
\|(Tu)'''\| &\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial^3}{\partial t^3}G(t,s)\Big||F(s,u(s)+p(s))|\,\mathrm{d}s \\
&\leqslant \max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial^3}{\partial t^3}G(t,s)\Big|\varphi(s,r_{0-3})\,\mathrm{d}s \leqslant r_3.
\end{aligned}
$$

Consequently, $T\colon\ V_{r_{0-3}} \to V_{r_{0-3}}$.

If the assumption (2) holds, then

$$
\begin{aligned}
\max_{0\leqslant t\leqslant 1}\int_0^1 G(t,s)\varphi(s,r_{0-3})\,\mathrm{d}s &\leqslant \max_{0\leqslant t,s\leqslant 1} G(t,s)\int_0^1 \varphi(s,r_{0-3})\,\mathrm{d}s \\
&\leqslant m_0M^{-1}\cdot Mm_0^{-1}r_0 = r_0, \\
\max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial}{\partial t}G(t,s)\Big|\varphi(s,r_{0-3})\,\mathrm{d}s &\leqslant \max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial}{\partial t}G(t,s)\Big|\int_0^1 \varphi(s,r_{0-3})\,\mathrm{d}s \\
&\leqslant m_1M^{-1}\cdot Mm_1^{-1}r_1 = r_1, \\
\max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial^2}{\partial t^2}G(t,s)\Big|\varphi(s,r_{0-3})\,\mathrm{d}s &\leqslant \max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial^2}{\partial t^2}G(t,s)\Big|\int_0^1 \varphi(s,r_{0-3})\,\mathrm{d}s \\
&\leqslant m_2M^{-1}\cdot Mm_2^{-1}r_2 = r_2, \\
\max_{0\leqslant t\leqslant 1}\int_0^1 \Big|\frac{\partial^3}{\partial t^3}G(t,s)\Big|\varphi(s,r_{0-3})\,\mathrm{d}s &\leqslant \max_{0\leqslant t,s\leqslant 1}\Big|\frac{\partial^3}{\partial t^3}G(t,s)\Big|\int_0^1 \varphi(s,r_{0-3})\,\mathrm{d}s \\
&\leqslant m_3M^{-1}\cdot Mm_3^{-1}r_3 = r_3.
\end{aligned}
$$

Therefore, the assumption (1) holds and $T\colon\ V_{r_{0-3}} \to V_{r_{0-3}}$.

Now, we prove that the equation (P) has a solution.

Since $V_{r_{0-3}}$ is a bounded, closed and convex set, by Lemma 2.4 and the Leray-Schauder fixed point theorem there exists $\bar{u} \in V_{r_{0-3}}$ such that $T\bar{u} = \bar{u}$. Let $u^* = \bar{u}+p$. Then $u^* \in C^3[0,1]$, $\|(u^*)^{(i)} - p^{(i)}\| \leqslant r_i$, $i = 0,1,2,3$, and

$$\begin{aligned} u^*(t) - p(t) &= \bar{u}(t) = (T\bar{u})(t) \\ &= \int_0^1 G(t,s) f\big(s, \bar{u}(s)+p(s), \bar{u}'(s)+p'(s), \bar{u}''(s)+p''(s), \bar{u}'''(s)+p'''(s)\big)\,\mathrm{d}s \\ &= \int_0^1 G(t,s) f\big(s, u^*(s), (u^*)'(s), (u^*)''(s), (u^*)'''(s)\big)\,\mathrm{d}s. \end{aligned}$$

Let $e = p'''(t) \equiv 3a + 3b - 3c + \frac{3}{2}d$. Then, for $0 \leqslant t \leqslant 1$,

$$\begin{aligned} (u^*)''' &= e + \int_0^1 \frac{\partial^3}{\partial t^3} G(t,s) f\big(s, u^*(s), (u^*)'(s), (u^*)''(s), (u^*)'''(s)\big)\,\mathrm{d}s \\ &= e + \frac{1}{2}\int_0^t s^2(3-s) F(s, u^*(s))\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_t^1 (1-s)(s^2-2s-2) F(s, u^*(s))\,\mathrm{d}s. \end{aligned}$$

Applying the properties of the indefinite integral (see (18.3), [6]), we get

$$(u^*)^{(4)}(t) = f\big(t, u^*(t), (u^*)'(t), (u^*)''(t), (u^*)'''(t)\big), \quad \text{a.e. } t \in [0,1].$$

Noticing that $G(0,s) = G(1,s) = (\partial/\partial t)G(0,s) = (\partial^2/\partial t^2)G(1,s) = 0$, $0 \leqslant s \leqslant 1$, we have $u^*(0) = p(0) = a$, $(u^*)'(0) = p'(0) = b$, $u^*(1) = p(1) = c$, $(u^*)''(1) = p''(1) = d$. Therefore, u^* is a solution of the equation (P).

If $a^2+b^2+c^2+d^2 > 0$ or $\int_0^1 |f(t,0,0,0,0)|\,\mathrm{d}t > 0$, then $u^*(t) \not\equiv 0$. In other words, the solution u^* is nontrivial.

The proof is completed. $\square$

4. Remark and example

Remark 4.1. Theorem 1.1 is a special case of Theorem 1.2 (1).

Assume that the conditions of Theorem 1.1 are satisfied. Then $f\colon [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ is continuous and there exist four positive numbers $r_i \geqslant \gamma_i$, $i = 0,1,2,3$, such that $\Phi(r_{0-3}) \leqslant K \min\{k_i^{-1} r_i,\ i = 0,1,2,3\}$.

By the definitions, $|\mu_i| \leqslant \gamma_i$ and $|\nu_i| \leqslant \gamma_i = 0,1,2,3$. So,

$$\begin{aligned} &[-r_0+\nu_0, r_0+\mu_0] \times [-r_1+\nu_1, r_1+\mu_1] \times [-r_2+\nu_2, r_2+\mu_2] \times [-r_3+\nu_3, r_3+\mu_3] \\ &\qquad \subset [-2r_0, 2r_0] \times [-2r_1, 2r_1] \times [-2r_2, 2r_2] \times [-2r_3, 2r_3]. \end{aligned}$$

Using this fact, we get

$$\varphi(t, r_{0-3}) \leqslant \Phi(r_{0-3}) \leqslant K \min\{k_i^{-1} r_i,\ i = 0, 1, 2, 3\}, \quad 0 \leqslant t \leqslant 1.$$

From [1] we have $\max\limits_{0 \leqslant t \leqslant 1} \int_0^1 |(\partial^i/\partial t^i) G(t, s)| \, \mathrm{d}s \leqslant K^{-1} k_i$, $i = 0, 1, 2, 3$. It follows that, for $i = 0, 1, 2, 3$,

$$\begin{aligned}\max_{0 \leqslant t \leqslant 1} \int_0^1 |(\partial^i/\partial t^i) G(t, s)| \varphi(s, r_{0-3}) \, \mathrm{d}s &\leqslant K k_i^{-1} r_i \max_{0 \leqslant t \leqslant 1} \int_0^1 |(\partial^i/\partial t^i) G(t, s)| \, \mathrm{d}s \\ &\leqslant K k_i^{-1} r_i \cdot K^{-1} k_i = r_i.\end{aligned}$$

By Theorem 1.2 (1), the equation (P) has one solution $u^* \in C^3[0, 1]$ and $\|(u^*)^{(i)} - p^{(i)}\| \leqslant r_i$, $i = 0, 1, 2, 3$. So,

$$\|(u^*)^{(i)}\| \leqslant \|(u^*)^{(i)} - p^{(i)}\| + \|p^{(i)}\| \leqslant r_i + \gamma_i \leqslant 2r_i.$$

Theorem 1.1 is proved.

Example 4.2. Consider the nonlinear beam equation with homogeneous boundary condition

$$\begin{cases} u^{(4)}(t) = \dfrac{(u^2(t) + 2) \mathrm{e}^{\frac{3}{2}((u''(t) + u'''(t))} \cos(u'(t) + \pi t)}{180 \sqrt{|t - \frac{1}{2}|}}, & t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1], \\ u(0) = u'(0) = u(1) = u''(1) = 0. \end{cases}$$

In this equation, $p = 0$ and the nonlinear term is

$$f(t, u_0, u_1, u_2, u_3) = \frac{(u_0^2 + 2) \mathrm{e}^{\frac{3}{2}(u_2 + u_3)} \cos(u_1 + \pi t)}{180 \sqrt{|t - \frac{1}{2}|}}.$$

Thus, $f(t, u_0, u_1, u_2, u_3)$ is a strong Carathéodory function and it is discontinuous on the set $\{\frac{1}{2}\} \times \mathbb{R}$.

Let $r_0 = r_1 = r_2 = r_3 = 1$. Since $\mu_i = \nu_i = 0$, $i = 0, 1, 2, 3$, we have

$$\begin{aligned}\varphi(t, r_{0-3}) &\leqslant \sup \left\{ \left| \frac{(u_0^2 + 2) \mathrm{e}^{\frac{3}{2}(u_2 + u_3)} \cos(u_1 + \pi t)}{180 \sqrt{|t - \frac{1}{2}|}} \right| : \begin{array}{l} -1 \leqslant u_0 \leqslant 1, \ -1 \leqslant u_1 \leqslant 1, \\ -1 \leqslant u_2 \leqslant 1, \ -1 \leqslant u_3 \leqslant 1 \end{array} \right\} \\ &\leqslant \frac{\mathrm{e}^3}{60 \sqrt{|t - \frac{1}{2}|}}.\end{aligned}$$

Further,

$$\int_0^1 \varphi(t, r_{0-3})\,\mathrm{d}t \leqslant \frac{\mathrm{e}^3}{60}\int_0^1 \frac{\mathrm{d}t}{\sqrt{|t-\frac{1}{2}|}} = \frac{\mathrm{e}^3}{30}\int_0^{1/2}\frac{\mathrm{d}t}{\sqrt{t}} = \frac{\sqrt{2}\mathrm{e}^3}{30}$$
$$\approx 0.9468 < 1 = Mm_3^{-1} = M\min\{m_i^{-1}r_i,\ i = 0,1,2,3\}.$$

By Theorem 1.2 (2), the equation has one solution $u^* \in C^3[0,1]$ such that $\|(u^*)^{(i)}\| \leqslant 1$, $i = 0,1,2,3$. Since $f(t,0,0,0,0) = \frac{2}{180}|t-\frac{1}{2}|^{-1/2}\cos\pi t \neq 0$, $t \in [0,\frac{1}{2}) \cup (\frac{1}{2},1]$, the solution u^* is nontrivial.

The conclusion cannot be derived from Theorem 1.1 and the existing literature because of the singularity of the nonlinear term.

The example illustrates that Theorem 1.2 is an effective and practical tool for the discontinuous equation (P).

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