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Applications of Mathematics, Vol. 56 (2011), No. 6, 513–531

Persistent URL: `http://dml.cz/dmlcz/141763`

56 (2011) APPLICATIONS OF MATHEMATICS No. 6, 513–531

FINITE ELEMENT DERIVATIVE INTERPOLATION RECOVERY TECHNIQUE AND SUPERCONVERGENCE*

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(Received May 11, 2009)

Abstract. A new finite element derivative recovery technique is proposed by using the polynomial interpolation method. We show that the recovered derivatives possess superconvergence on the recovery domain and ultraconvergence at the interior mesh points for finite element approximations to elliptic boundary problems. Compared with the well-known Z-Z patch recovery technique, the advantage of our method is that it gives an explicit recovery formula and possesses the ultraconvergence for the odd-order finite elements. Finally, some numerical examples are presented to illustrate the theoretical analysis.

Keywords: finite element method, derivative recovery technique, superconvergence and ultraconvergence

MSC 2010: 65M60, 65N30

1. Introduction

It is well known that the superconvergence property of finite element methods has attracted considerable attention because of its practical importance in enhancing the accuracy of finite element approximations and in constructing the adaptive algorithm of finite element methods via a posteriori error estimators [1], [4], [5], [8], [12]. In this field, many derivative recovery techniques have been established in order to obtain superconvergence for finite element approximations in derivative. For example, the averaging techniques [3], [6], the L_2-projection techniques [7], [9], the well-known Zienkiewicz-Zhu superconvergence patch recovery technique (SPR) [11], [12], the polynomial preserving recovery technique (PPR) [12], are popular. The basic idea of SPR and PPR is to use the least squares polynomial fitting method

* This work was supported in part by the National Basic Research Program (2012CB955804) and the National Natural Science Foundation of China (11071033 and 11171251.

to fit the derivative of the finite element solution at the Gauss points (SPR) or the solution itself at the Lobatto points (PPR). Both SPR and PPR techniques possess the superconvergence on the patch recovery domain and the ultraconvergence (two orders higher than the optimal global convergence order) at the interior mesh points for the derivative approximations of finite element solutions for the elliptic boundary value problems [12], [13]. However, these two techniques are only valid for even-order finite elements when the ultraconvergence is concerned. Recently, in [17] Zhu et al. have proposed a recovery technique for the odd-order finite elements of order $k > 1$. Once again, this technique employs the least squares method to fit the derivative of the finite element solution at some special points.

In this paper, we consider the kth-order rectangular finite element approximation to the following elliptic boundary value problem on a rectangular domain:

$$\begin{aligned} Au &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$A = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial}{\partial x_i} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a_0 I. \tag{1.2}$$

A new derivative recovery technique is provided by using the polynomial interpolation method on the local recovery domain. By means of this recovery method, we establish the superconvergence for the recovery derivatives of the finite element solutions, and the ultraconvergence if the orders of finite elements are odd and $A = -\Delta + a_0 I$. Compared with the SPR and the PPR techniques or Zhu's method, the advantage of our method is as follows: First, it gives an explicit derivative recovery formula while all the above three methods are implicit such that they cost some additional computations; secondly, it has the ultraconvergence for the odd-order finite element approximations. Moreover, Zhu's method is only valid for the odd-order finite elements of order $k > 1$. However, our method is applicable for the finite elements of order $k \geqslant 1$. As a by-product of the superconvergence estimates, we also obtain an asymptotically exact a posteriori error estimator for the finite element approximation to the elliptic boundary value problem.

Throughout this paper, we use the notations $H_0^1(\Omega)$ and $W_p^m(\Omega)$ to represent the usual Sobolev spaces on a domain Ω, and $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ the norm and the seminorm of the space $W_p^m(\Omega)$, respectively, and the letter C a generic positive constant independent of the mesh size h.

This paper is organized in the following way. In Section 2, we introduce the interpolation operator of projection type and give its approximation properties. In

Section 3, the derivative interpolation recovery operator is defined, and its super-approximation and ultra-approximation properties are analyzed. In Section 4, we first prove the superconvergence and the ultraconvergence properties of the recovered derivatives of the finite element solutions, and then give an asymptotically exact a posteriori error estimator. Section 5 is devoted to some numerical experiments to illustrate the theoretical results.

2. Interpolation operator of projection type and its approximation properties

Let $e = e_1 \times e_2 = (x_e - h_e, x_e + h_e) \times (y_e - \hbar_e, y_e + \hbar_e)$ be an arbitrary element, and $\{l_j(x)\}_{j=0}^{\infty}$ and $\{\tilde{l}_j(y)\}_{j=0}^{\infty}$ the normalized orthogonal Legendre polynomial systems in $L_2(e_1)$ and $L_2(e_2)$, respectively. Set

$$\omega_0(x) = \tilde{\omega}_0(y) = 1, \quad \omega_{j+1}(x) = \int_{x_e - h_e}^{x} l_j(t)\,\mathrm{d}t,$$
$$\tilde{\omega}_{j+1}(y) = \int_{y_e - \hbar_e}^{y} \tilde{l}_j(t)\,\mathrm{d}t, \quad j \geqslant 0.$$

It is well known that the polynomials $l_k(x)$ and $\omega_{k+1}(x)$ ($k \geqslant 1$) have k and $k+1$ zero points in e_1 and in the closure $\overline{e}_1$, respectively, and these zero points are symmetrically distributed with respect to the middle point x_e. Moreover, we know that these polynomials also possess the following symmetry and antisymmetry:

$$\omega_{2j}(x_e + x) = \omega_{2j}(x_e - x), \quad \omega_{2j-1}(x_e + x) = -\omega_{2j-1}(x_e - x), \tag{2.1}$$
$$l_{2j}(x_e + x) = l_{2j}(x_e - x), \quad l_{2j-1}(x_e + x) = -l_{2j-1}(x_e - x). \tag{2.2}$$

The completely parallel conclusions hold for the polynomials $\tilde{\omega}_{k+1}(y)$ and $\tilde{l}_k(y)$ in the element $e_2 = (y_e - \hbar_e, y_e + \hbar_e)$.

Now, let $u \in H^2(e)$. Then, we have the Fourier expansion [5]:

$$u(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_{ij}\omega_i(x)\tilde{\omega}_j(y), \quad (x,y) \in e, \tag{2.3}$$

$$\beta_{00} = u(x_e - h_e, y_e - \hbar_e), \tag{2.4}$$
$$\beta_{ij} = \int_e u_{xy} l_{i-1}(x)\tilde{l}_{j-1}(y)\,\mathrm{d}x\,\mathrm{d}y,$$

$$\beta_{i0} = \int_{e_1} u_x(x, y_e - \hbar_e) l_{i-1}(x)\,\mathrm{d}x, \tag{2.5}$$
$$\beta_{0j} = \int_{e_2} u_y(x_e - h_e, y)\tilde{l}_{j-1}(y)\,\mathrm{d}y, \quad i,j \geqslant 1.$$

Introduce the kth-order and the bi-complete kth-order polynomial spaces P_k and Q_k, respectively, by

$$p(x,y)=\sum_{i=0}^{k}\sum_{j=0}^{k-i}a_{ij}x^iy^j \ \ \forall\, p\in P_k;\quad q(x,y)=\sum_{i=0}^{k}\sum_{j=0}^{k}a_{ij}x^iy^j \ \ \forall\, q\in Q_k.$$

Define the kth-order interpolation operator of projection type

$$\pi_k\colon\ H^2(e)\to Q_k(e)$$

by

$$\pi_k u(x,y)=\sum_{i=0}^{k}\sum_{j=0}^{k}\beta_{ij}\omega_i(x)\tilde{\omega}_j(y),\quad (x,y)\in e. \tag{2.6}$$

Then, π_k is uniquely solvable with respect to $Q_k(e)$ and possesses the following properties taken from [5]. For all $k\geqslant 1$,

$$\pi_k u(x_e\pm h_e, y_e\pm\hbar_e)=u(x_e\pm h_e, y_e\pm\hbar_e), \tag{2.7}$$

$$|u-\pi_k u|_{m,p,e}\leqslant Ch^{k+1-m}|u|_{k+1,p,e},\quad 1\leqslant p\leqslant\infty,\ \ 0\leqslant m\leqslant k+1, \tag{2.8}$$

where $h=\sqrt{h_e^2+\hbar_e^2}$. From (2.3) and (2.6) we derive that

$$u-\pi_k u=\Bigg(\sum_{i=0}^{k}\sum_{j=k+1}^{\infty}+\sum_{i=k+1}^{\infty}\sum_{j=0}^{k}+\sum_{i=k+1}^{\infty}\sum_{j=k+1}^{\infty}\Bigg)\beta_{ij}\omega_i(x)\tilde{\omega}_j(y). \tag{2.9}$$

Lemma 2.1. *Let $u\in Q_k(e)\cup\{x^{k+1},y^{k+1}\}$, and $D_1=(\partial/\partial x)$, $D_2=(\partial/\partial y)$. Then, we have*

$$u-\pi_k u=\beta_{k+1,0}\omega_{k+1}(x)+\beta_{0,k+1}\tilde{\omega}_{k+1}(y),\quad D_1D_2(u-\pi_k u)=0,$$
$$D_1(u-\pi_k u)=\beta_{k+1,0}\, l_k(x),\quad D_2(u-\pi_k u)=\beta_{0,k+1}\tilde{l}_k(y).$$

Proof. Let $u\in Q_k(e)\cup\{x^{k+1},y^{k+1}\}$ be such that $D_1D_2u\in Q_{k-1}(e)$, $D_1u\in P_k(e_1)$, and $D_2u\in P_k(e_2)$. Then, it follows from (2.4)–(2.5) and the orthogonality of the system of Legendre polynomials that

$$\beta_{ij}=0,\quad i\geqslant k+1,\ j\geqslant 1\ \text{ or }\ i\geqslant 1,\ j\geqslant k+1,$$
$$\beta_{i0}=\beta_{0j}=0,\quad i\geqslant k+2,\ j\geqslant k+2,$$

which, together with (2.9), leads to the expression for $u-\pi_k u$ in Lemma 2.1.

The other expressions follow by taking partial derivatives of the formula for $u-\pi_k u$, and the proof is complete. □

From Lemma 2.1 and the orthogonality of the system of Legendre polynomials, we derive that for $u \in Q_k(e) \cup \{x^{k+1}, y^{k+1}\}$,

$$\int_e \nabla(u - \pi_k u)\nabla v \,\mathrm{d}x\,\mathrm{d}y = 0 \quad \forall\, v \in Q_k(e).$$

This shows that the interpolation approximation $\pi_k u$ can be regarded as the finite element solution of the Laplace equation when the exact solution u belongs to $Q_k \cup \{x^{k+1}, y^{k+1}\}$.

Moreover, when $u = u(x)$, from (2.4)–(2.5) we have $\beta_{ij} = \beta_{0j} = 0$, $i, j \geqslant 1$. Then, the expressions (2.3) and (2.6) become, respectively,

$$u(x) = \sum_{i=0}^{\infty} \beta_{i0}\omega_i(x), \quad \pi_k u(x) = \sum_{i=0}^{k} \beta_{i0}\omega_i(x), \quad x \in e_1. \tag{2.10}$$

Thus, we see that when restricted to e_1, π_k is identical to the interpolation operator of projection type in one dimensional space (see [5]).

3. Derivative interpolation recovery technique

In this section, we introduce the derivative interpolation recovery operator, and discuss its super-approximation and ultra-approximation properties. Let $e^{(s)}$ ($s = 1, 2, 3, 4$) be four elements which share a common interior nodal point (x_0, y_0). Corresponding to the point (x_0, y_0), we introduce a patch domain D_0 (see Fig. 1) as follows:

$$D_0 = \bigcup_{s=1}^{4} e^{(s)} = (x_0 - h_i, x_0 + h_{i+1}) \times (y_0 - \hbar_j, y_0 + \hbar_{j+1}).$$

Denote the patch recovery intervals by

$$E_1 = (x_0 - h_i, x_0 + h_{i+1}) \quad \text{and} \quad E_2 = (y_0 - \hbar_j, y_0 + \hbar_{j+1}).$$

On E_1 and E_2, we will define the derivative recovery operator in x-direction and y-direction, respectively. The main idea of the interpolation recovery technique is to choose some special interpolation nodes in E_l (in what follows, such interpolation nodes are called the *sample points*), and then use the polynomial interpolation method to recover the derivatives of finite element solutions. Let $\{g_i\}$ (or $\{\tilde{g}_j\}$) ($i = \pm 1, \ldots \pm k$) be the $2k$ sample points in E_1 (or E_2), that is

$$x_0 - h_i < g_{-k} < \ldots < g_{-1} < x_0 < g_1 < \ldots < g_k < x_0 + h_{i+1},$$
$$y_0 - \hbar_j < \tilde{g}_{-k} < \ldots < \tilde{g}_{-1} < y_0 < \tilde{g}_1 < \ldots < \tilde{g}_k < y_0 + \hbar_{j+1}.$$

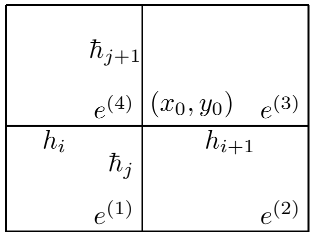

Figure 1. Patch domain.

Corresponding to the sample sets $\{g_i\}$ and $\{\tilde{g}_j\}$, we introduce the $(2k-1)$th-order Lagrange interpolation basis functions $\varphi_i(x) \in P_{2k-1}(E_1)$ and $\tilde{\varphi}_j(y) \in P_{2k-1}(E_2)$ defined by

$$\varphi_i(x) = \prod_{l=\pm 1, l\neq i}^{\pm k} \frac{(x - g_l)}{(g_i - g_l)},$$

$$\tilde{\varphi}_j(y) = \prod_{l=\pm 1, l\neq j}^{\pm k} \frac{(y - \tilde{g}_l)}{(\tilde{g}_j - \tilde{g}_l)}, \quad i, j = \pm 1, \ldots, \pm k.$$

Then, $\{\varphi_i(x)\}$ and $\{\tilde{\varphi}_j(y)\}$ form bases of the spaces $P_{2k-1}(E_1)$ and $P_{2k-1}(E_2)$, respectively.

Now, for any given piecewise smooth function $w \in W^1_\infty(D_0)$, we define the derivative interpolation recovery operator

$$R\colon W^1_\infty(E_l) \to P_{2k-1}(E_l)$$

according to the following conditions:

$$RD_1 w(x, y_0) = \sum_{i=\pm 1}^{\pm k} D_1 w(g_i, y_0)\varphi_i(x), \qquad x \in E_1, \tag{3.1}$$

$$RD_2 w(x_0, y) = \sum_{j=\pm 1}^{\pm k} D_2 w(x_0, \tilde{g}_j)\tilde{\varphi}_j(y), \qquad y \in E_2. \tag{3.2}$$

Note that $D_l u_h$ may be discontinuous across the node (x_0, y_0) when u_h is a continuous piecewise polynomial on D_0, and the recovered derivative $RD_l u_h$ is a $(2k-1)$th-order polynomial on E_l.

Lemma 3.1. *The derivative recovery operator R possesses the following properties:*

$$RD_l u_{k+s} = D_l u_{k+s} \ \ \forall\, u_{k+s} \in P_{k+s}(E_l), \quad 1 \leqslant s \leqslant k, \ \ l = 1, 2, \tag{3.3}$$

$$\|RD_l u\|_{0,\infty,E_l} \leqslant C\|D_l u\|_{0,\infty,E_l} \ \ \forall\, u \in W^1_\infty(E_l), \quad l = 1, 2. \tag{3.4}$$

Furthermore, if the sample points $\{g_i\}$ (or $\{\tilde{g}_j\}$) are chosen as the $2k$ Gauss points in the elements $(x_0 - h_i, x_0)$ and $(x_0, x_0 + h_{i+1})$ (or $(y_0 - \hbar_j, y_0)$ and $(y_0, y_0 + \hbar_{j+1})$), then

$$D_l u_{k+1} = RD_l \pi_k u_{k+1} \quad \forall\, u_{k+1} \in P_{k+1}(E_l), \ \ l = 1, 2. \tag{3.5}$$

P r o o f. Let $u_{k+s} \in P_{k+s}(E_l)$. Then, we have that $D_l u_{k+s} \in P_{k+s-1}(E_l)$. Because $k + s - 1 \leqslant 2k - 1$ when $s \leqslant k$, the equality (3.3) follows from the uniqueness of the interpolation polynomial (see (3.1)–(3.2)). The estimate (3.4) can be verified directly by mapping E_l onto the standard element $\hat{E} = (-1, 1)$. When $\{g_i\}$ are the Gauss points, from Lemma 2.1 we see that

$$D_1 u_{k+1}(g_i) = D_1 \pi_k u_{k+1}(g_i), \quad i = \pm 1, \ldots, \pm k.$$

Then, from (3.1) we obtain

$$RD_1 u_{k+1}(x, y_0) = RD_1 \pi_k u_{k+1}(x, y_0).$$

Thus, (3.5) follows from (3.3). □

Theorem 3.1. *Assume that $u \in W^{k+2}_\infty(D_0)$, and the recovery operator R is defined by* (3.1)–(3.2) *with the Gauss points as the sample points. Then, R possesses the following super-approximation property:*

$$\|D_l u - RD_l \pi_k u\|_{0,\infty,E_l} \leqslant Ch^{k+1}|u|_{k+2,\infty,D_0}, \quad k \geqslant 1, \ \ l = 1, 2. \tag{3.6}$$

P r o o f. The estimate (3.6) can be obtained by using Lemma 3.1 and the Bramble-Hilbert Lemma (see [2], Theorem 4.1.3). □

In [14], we have chosen the Gauss points as the interpolation sample points and obtained the ultraconvergence at (x_0, y_0) for the even-order finite elements. However, the method and the result there are not valid for the odd-order finite elements. We need to choose again the new sample points in order to obtain the ultraconvergence for the odd-order finite elements.

Lemma 3.2. *Let $\{l_k(x)\}$ be a system of the Legendre orthogonal polynomials on an interval $E = (a,b)$, and β_k and β_{k+1} any two constants with $\beta_k\beta_{k+1} \neq 0$. Then, the polynomial equation,*

$$F(x) \equiv \beta_k l_k(x) + \beta_{k+1} l_{k+1}(x) = 0, \quad x \in E, \ k \geqslant 1, \tag{3.7}$$

has k separated roots in E. Moreover, when setting

$$\begin{aligned} \beta_k^{\pm} &= (k+2)\int_{E^{\pm}} (x-x_0)^{k+1} l_k(x)\,\mathrm{d}x, \\ \beta_{k+1}^{\pm} &= (k+2)\int_{E^{\pm}} (x-x_0)^{k+1} l_{k+1}(x)\,\mathrm{d}x, \end{aligned} \tag{3.8}$$

where $E^- = (x_0 - h, x_0)$ and $E^+ = (x_0, x_0 + h)$ are two adjacent intervals, we further have that $\beta_k^+ = -\beta_k^-$, $\beta_{k+1}^+ = \beta_{k+1}^-$, and the $2k$ roots of $F(x) = 0$ in $E^- \cup E^+$ are symmetrically distributed with respect to the point x_0.

P r o o f. It is well known that the kth-order Legendre polynomial $l_k(x)$ has k single zeros in E for $k \geqslant 1$, and the k zeros $\{\xi_j\}$ of $l_k(x)$ and the $k+1$ zeros $\{\eta_j\}$ of $l_{k+1}(x)$ are alternately distributed in E (see, for example, [10]), that is,

$$a < \eta_1 < \xi_1 < \eta_2 < \xi_2 < \ldots < \eta_j < \xi_j < \eta_{j+1} < \ldots < \eta_k < \xi_k < \eta_{k+1} < b.$$

Since $l_k(x)$ only has the k single zeros $\xi_1, \ldots, \xi_k$, the symbol of $l_k(x)$ changes alternately on the intervals:

$$(a, \xi_1), \ (\xi_1, \xi_2), \ \ldots, \ (\xi_{k-1}, \xi_k), \ (\xi_k, b).$$

Thus, we have

$$F(\eta_j)F(\eta_{j+1}) = \beta_k^2 l_k(\eta_j)\, l_k(\eta_{j+1}) < 0, \quad j = 1, \ldots, k.$$

In addition, from Intermediate Value Theorem we know that $F(x)$ has k separated zeros in E.

In the following, we shall prove the symmetry. Let

$$F(x) = \begin{cases} \beta_k^+ l_k(x) + \beta_{k+1}^+ l_{k+1}(x), & x \in E^+, \\ \beta_k^- l_k(x) + \beta_{k+1}^- l_{k+1}(x), & x \in E^-. \end{cases}$$

Here and afterwards, when we use $l_k(x)$ restricted to the interval E^+ (or to the interval E^-), we imply that $l_k(x)$ is the Legendre polynomial on the interval E^+

(or the interval E^-), noting that $l_k(x)|_{E^+}$ and $l_k(x)|_{E^-}$ are two distinct Legendre polynomials. Since (see (3.8))

$$\beta_i^+ = (k+2)\int_{x_0}^{x_0+h}(x-x_0)^{k+1}l_i(x)\,\mathrm{d}x, \quad i=k,k+1,$$
$$\beta_i^- = (k+2)\int_{x_0-h}^{x_0}(x-x_0)^{k+1}l_i(x)\,\mathrm{d}x, \quad i=k,k+1,$$

it follows from the variable transformation

$$x=\frac{h}{2}t+x_0+\frac{h}{2} \quad (\text{or } x=\frac{h}{2}t+x_0-\frac{h}{2})$$

that

$$\beta_i^+ = (k+2)\Big(\frac{h}{2}\Big)^{k+2}\int_{-1}^{1}(t+1)^{k+1}\hat{l}_i(t)\,\mathrm{d}t, \quad i=k,k+1,$$
$$\beta_i^- = (k+2)\Big(\frac{h}{2}\Big)^{k+2}\int_{-1}^{1}(t-1)^{k+1}\hat{l}_i(t)\,\mathrm{d}t, \quad i=k,k+1,$$

where $\hat{l}_i(t)$ is the standard Legendre polynomial on $[-1,1]$. Then, by means of the variable transformation $t=-\tau$, and the symmetry and antisymmetry properties (2.2), it is easy to see that $\beta_k^+=-\beta_k^-$, and $\beta_{k+1}^+=\beta_{k+1}^-$. Hence, by using (2.2) we obtain that for $0<\tau<h$,

$$\begin{aligned} F(x_0+\tau) &= \beta_k^+ l_k(x_0+\tau)+\beta_{k+1}^+ l_{k+1}(x_0+\tau)\\ &= -\beta_k^- l_k(x_0+\tau)+\beta_{k+1}^- l_{k+1}(x_0+\tau)\\ &= \begin{cases} -\beta_k^- l_k(x_0-\tau)-\beta_{k+1}^- l_{k+1}(x_0-\tau)=-F(x_0-\tau), & k \text{ even},\\ \beta_k^- l_k(x_0-\tau)+\beta_{k+1}^- l_{k+1}(x_0-\tau)=F(x_0-\tau), & k \text{ odd}, \end{cases} \end{aligned}$$

which implies

$$F(x_0+\tau)=0 \iff F(x_0-\tau)=0, \quad 0<\tau<h.$$

This shows that the $2k$ roots of $F(x)=0$ in the intervals (x_0-h,x_0) and (x_0,x_0+h) are symmetrically distributed with respect to the point x_0.

Similar conclusions hold for $\tilde{F}(y)\equiv\tilde{\beta}_k\tilde{l}_k(y)+\tilde{\beta}_{k+1}\tilde{l}_{k+1}(y)=0$. □

Now let us investigate the super-approximation and the ultra-approximation properties of the derivative recovery operator at an interior nodal point (x_0,y_0). Let D_0 consist of four rectangular elements, which are local uniformly in x- and y-directions, respectively, that is, $D_0=(x_0-h_i,x_0+h_i)\times(y_0-\hbar_j,y_0+\hbar_j)$. When the sample points $\{g_i, \tilde{g}_j\}$ are symmetrically distributed with respect to the point (x_0, y_0), we have that

$$(3.9)\qquad \begin{aligned} &x_0 - g_{-i} = g_i - x_0,\ \ g_i - g_l = g_{-l} - g_{-i},\ \ i, l = 1, 2, \ldots, k,\\ &y_0 - \tilde{g}_{-j} = \tilde{g}_j - y_0,\ \ \tilde{g}_j - \tilde{g}_l = \tilde{g}_{-l} - \tilde{g}_{-j},\ \ j, l = 1, 2, \ldots, k. \end{aligned}$$

This yields that

$$\varphi_i(x_0) = \varphi_{-i}(x_0), \quad \tilde{\varphi}_j(y_0) = \tilde{\varphi}_{-j}(y_0), \quad i, j = 1, 2, \ldots, k.$$

In this case, at the point (x_0, y_0), the recovery operator R can be simplified as (see (3.1)–(3.2))

$$(3.10)\qquad RD_1 w(x_0, y_0) = \sum_{i=1}^{k} [D_1 w(g_i, y_0) + D_1 w(g_{-i}, y_0)] \varphi_i(x_0),$$

$$(3.11)\qquad RD_2 w(x_0, y_0) = \sum_{j=1}^{k} [D_2 w(x_0, \tilde{g}_j) + D_2 w(x_0, \tilde{g}_{-j})] \tilde{\varphi}_j(y_0).$$

Theorem 3.2. *Let $u \in W_\infty^{k+s}(D_0)$ with $k \geqslant 1$ being an odd number, $s = 1, 2$, (x_0, y_0) an interior nodal point, and $D_0 = (x_0 - h, x_0 + h) \times (y_0 - \hbar, y_0 + \hbar)$. Suppose further that the sample points are the $2k$ roots of (3.7)–(3.8) in E_l. Then, the derivative recovery operator R possesses the following super-approximation and ultra-approximation properties at the point (x_0, y_0):*

$$(3.12)\qquad |D_l u(x_0, y_0) - RD_l \pi_k u(x_0, y_0)| \leqslant C h^{k+s} |u|_{k+1+s,\infty,D_0},$$

where $s = 1, 2$, $k \geqslant 1$, $l = 1, 2$.

P r o o f. If, under the conditions of Theorem 3.2, we can prove that

$$(3.13)\qquad D_l u(x_0, y_0) - RD_l \pi_k u(x_0, y_0) = 0 \quad \forall\, u \in P_{k+s}(E_l),\ \ l = 1, 2,\ \ s = 1, 2,$$

then the conclusion of Theorem 3.2 follows from the Bramble-Hilbert Lemma directly. Below we only prove (3.13) for $l = 1$, since for $l = 2$ the argument is completely similar.

First, let $u \in P_{k+1}(E_1)$. Without loss of generality, we assume that $u = p_k(x) + a(x - x_0)^{k+1}$. Then, it follows from (2.10) and the orthogonality of the Legendre polynomial system that $\beta_{i0} = 0$ for $i \geqslant k+2$, which implies that $u - \pi_k u = \beta_{k+1,0} \omega_{k+1}(x)$. Thus, we obtain from (3.10) that

$$(3.14)\qquad RD_1(u - \pi_k u)(x_0, y_0) = \sum_{i=1}^{k} [\beta_{k+1,0}^{+} l_k(g_i) + \beta_{k+1,0}^{-} l_k(g_{-i})] \varphi_i(x_0),$$

where we have by utilizing the orthogonality of the Legendre polynomial system again that

$$\beta^+_{k+1,0} = a(k+1)\int_{x_0}^{x_0+h}(x-x_0)^k l_k(x)\,\mathrm{d}x,$$
$$\beta^-_{k+1,0} = a(k+1)\int_{x_0-h}^{x_0}(x-x_0)^k l_k(x)\,\mathrm{d}x.$$

Using a similar argument to that in Lemma 3.2, we find that $\beta^+_{k+1,0} = \beta^-_{k+1,0}$. Therefore, (3.14) becomes

$$RD_1(u-\pi_k u)(x_0,y_0) = \beta^+_{k+1,0}\sum_{i=1}^{k}[l_k(g_i)+l_k(g_{-i})]\varphi_i(x_0). \tag{3.15}$$

Noticing that k is odd, we have by means of (2.2) and (3.9) that

$$\begin{aligned} l_k(g_{-i}) &= l_k(x_0-(x_0-g_{-i})) = -l_k(x_0+(x_0-g_{-i})) \\ &= -\,l_k(x_0+(g_i-x_0)) = -l_k(g_i), \end{aligned}$$

which, together with (3.15) and Lemma 3.1, yields

$$RD_1(u-\pi_k u)(x_0,y_0) = D_1u(x_0,y_0) - RD_1\pi_k u(x_0,y_0) = 0 \quad \forall\, u\in P_{k+1}(E_1). \tag{3.16}$$

Thus, we obtain (3.13) for $s=1$.

For $s=2$, according to (3.16) we only need to verify (3.13) for $u=(x-x_0)^{k+2}$. In fact, it follows from (2.10) and the orthogonality of the Legendre polynomial system that

$$u-\pi_k u = \beta_{k+1,0}\omega_{k+1}(x) + \beta_{k+2,0}\omega_{k+2}(x) \quad \forall\, u\in P_{k+2}(E_1).$$

Then, when $u=(x-x_0)^{k+2}$, we have that

$$D_1(u-\pi_k u) = \beta_k l_k(x) + \beta_{k+1}l_{k+1}(x) = F(x), \quad x\in E_1,$$

where $\beta_k = \beta_{k+1,0}$ and $\beta_{k+1} = \beta_{k+2,0}$ are given in (3.8). Thus, from our special choice of the sample points (see Lemma 3.2), we derive that

$$D_1(u-\pi_k u)(g_i,y_0) = 0, \;\text{ or }\; D_1u(g_i,y_0) = D_1\pi_k u(g_i,y_0), \quad i=\pm1,\dots,\pm k,$$

which implies (see (3.1)) that $RD_1u(x,y_0) = RD_1\pi_k u(x,y_0)$. Using Lemma 3.1 again, we gain that $D_1u(x,y_0) = RD_1\pi_k u(x,y_0)$, which completes the proof of Theorem 3.2. □

R e m a r k 3.1. In fact, the recovery formula (3.1)–(3.2) is a simplified form of our original patch interpolation recovery formula with the Gauss sample points (see [14]):

$$RD_l w(x,y) = \sum_{i=\pm 1}^{\pm k} \sum_{j=\pm 1}^{\pm k} D_l w(g_i, \tilde{g}_j)\varphi_i(x)\tilde{\varphi}(y), \quad l = 1,2, \ (x,y) \in D_0. \tag{3.17}$$

The recovery formula (3.17) also possesses the super-approximation and the ultra-approximation (for even-order case) properties, but it requires more computational cost. The advantage of the formula (3.17) is in that it can recover the derivative on the whole patch domain D_0 and can be extended to quadrilateral meshes, see [15].

4. Superconvergence and ultraconvergence

Let $\Omega \subset R^2$ be a rectangular domain with sides parallel to the coordinate axes, and $J_h = \bigcup\{e = [x_e - h_e, x_e + h_e] \times [y_e - \hbar_e, y_e + \hbar_e]\}$ a regular family of finite element partitions of Ω parameterized by the mesh size $h = \max_{e \in J_h}\{h(e)\}$ with $h(e) = 2\sqrt{h_e^2 + \hbar_e^2}$, such that $\overline{\Omega} = \bigcup_{e \in J_h} \overline{e}$. Here we say that a partition J_h is regular, if $\{\hbar_e/h_e \colon e \in J_h\}$ has uniformly positive lower and upper bounds. Introduce the kth-order finite element space $S_h \subset H_0^1(\Omega)$ by

$$S_h = \{v \in C(\overline{\Omega}) \colon v|_e \in Q_k(e), \ v|_{\partial\Omega} = 0 \ \forall\, e \in J_h\}.$$

On each element $e \in J_h$, we define the kth-order interpolation operator π_k of projection type in the same way as in Section 2.

Lemma 4.1. *For an arbitrary element $e = (x_e - h_e, x_e + h_e) \times (y_e - \hbar_e, y_e + \hbar_e)$, the interpolation operator π_k possesses the following properties:*

$$\int_e (u - \pi_k u) q \,\mathrm{d}x\,\mathrm{d}y = 0 \quad \forall\, q \in Q_{k-2}(e), \ e \in J_h, \ k \geqslant 2, \tag{4.1}$$

$$\int_l (u - \pi_k u) p \,\mathrm{d}s = 0 \quad \forall\, p \in P_{k-2}(l), \ \textit{edge } l \subset \partial e, \ k \geqslant 2. \tag{4.2}$$

P r o o f. First, it follows from integration by parts, the orthogonality of the Legendre polynomials, and $\omega_{k+1}(x_e \pm h_e) = 0$ that

$$\int_{x_e - h_e}^{x_e + h_e} \omega_{k+1}(x) q_s(x) \,\mathrm{d}x = -\int_{x_e - h_e}^{x_e + h_e} l_k(x) q_{s+1}(x) \,\mathrm{d}x = 0 \quad \forall\, q_s \in P_s(e_1), \ s \leqslant k-2, \ k \geqslant 2,$$

where $q_s = D_1 q_{s+1}$. A similar equality holds for $\tilde{\omega}_{k+1}(y)$. Thus, from (2.9)–(2.10) we immediately obtain the conclusions of Lemma 4.1. □

Relations (2.7) and (4.2) can ensure that $\pi_k u \in C(\overline{\Omega})$ or $\pi_k\colon\ H_0^1(\Omega) \cap H^2(\Omega) \to S_h$, $k \geqslant 1$. Let A be the second-order partial differential operator given by (1.2). Introduce the corresponding bilinear form:

$$A(u,v) = \sum_{i,j=1}^{2} (a_{ij}\, D_i u, D_j v) + \sum_{i=1}^{2} (a_i\, D_i u, v) + (a_0\, u, v), \tag{4.3}$$

where $(\cdot,\cdot)$ stands for the inner product in the $L_2(\Omega)$ space, $a_{ij}(x,y)$, $a_i(x,y)$, and $a_0(x,y)$ are properly smooth functions. It is well known that the *interpolation elementary estimates* (also called interpolation weak estimates) play an important role in the study of superconvergence. Some elementary estimates have been given for the interpolation of Lagrange type [18]. For the interpolation of projection type, by using the properties of π_k and the Bilinear Lemma [2], we can also prove the following results (a detailed proof can be found in [16, Theorems 7.5–7.6]).

Theorem 4.1. *Assume that the bilinear form $A(u,v)$ is defined by (4.3), $u \in H_0^1(\Omega) \cap W_p^{k+2}(\Omega)$. Then, the interpolation operator π_k satisfies the following superapproximation elementary estimate:*

$$|A(u - \pi_k u, v_h)| \leqslant C h^{k+1} \|u\|_{k+2,p} \|v_h\|_{1,q}, \quad k \geqslant 1. \tag{4.4}$$

Furthermore, when $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$, we have the following ultraapproximation elementary estimate:

$$|A(u - \pi_k u, v_h)| \leqslant C h^{k+2} \|u\|_{k+3,p} \|v_h\|_{1,q}, \quad k \geqslant 2, \tag{4.5}$$

where $v_h \in S_h$, $2 \leqslant p \leqslant \infty$, $1/p + 1/q = 1$.

Now, we consider the weak form of the problem (1.1): For given $f \in L_2(\Omega)$, find $u \in H_0^1(\Omega)$ satisfying

$$A(u,v) = (f,v) \quad \forall\, v \in H_0^1(\Omega). \tag{4.6}$$

As usual, we assume that $A(u,v)$ is a uniformly elliptic and bounded bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ such that the solution of the problem (4.6) uniquely exists. Define the finite element approximation of the problem (4.6) by finding $u_h \in S_h$ such that

$$A(u - u_h, v_h) = 0 \quad \forall\, v_h \in S_h. \tag{4.7}$$

Next we further assume that the regular family of partitions J_h is quasi-uniform, that is, it is regular and $\max\limits_{e \in J_h} h/h(e) \leqslant \sigma$ with a positive constant σ. Below we shall employ the *Green function* method for our superconvergence analysis. For the detailed discussion on this kind of Green function, the reader is referred to the monograph [18].

For any given point $z \in \Omega$, there exists a unique $\delta_h^z(x) \in S_h$, the discrete δ-function at point $x = z$, which satisfies

$$(\delta_h^z, v_h) = v_h(z) \quad \forall\, v_h \in S_h.$$

Let A^* be the adjoint operator of A, and define the *smooth Green function* by setting $G^z(x) \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$A^* G^z(x) = \delta_h^z(x) \quad \forall\, x \in \Omega. \tag{4.8}$$

Denote by L any indicated direction, and define the directional derivative operator ∂_z by

$$\partial_z G^z = \lim_{\Delta z \to 0, \Delta z // L} (G^{z+\Delta z} - G^z)/|\Delta z|.$$

From (4.8) it is easy to see that $\partial_z G^z(x)$ belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ and satisfies

$$A^*(\partial_z G^z, v) = \partial_z P_h v(z) \quad \forall\, v \in H_0^1(\Omega), \tag{4.9}$$

where $A^*(u, v)$ is the bilinear form associated with A^* such that

$$A^*(u, v) = A(v, u) \quad \forall\, (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

and $P_h\colon\, L_2(\Omega) \to S_h$ is the L_2 projection operator defined by

$$(v - P_h v, v_h) = 0 \quad \forall\, v_h \in S_h, \;\; v \in L_2(\Omega).$$

Define the finite element approximation $\partial_z G_h^z \in S_h$ of $\partial_z G^z$ according to the following condition:

$$A^*(\partial_z G^z - \partial_z G_h^z, v_h) = 0 \quad \forall\, v_h \in S_h. \tag{4.10}$$

For $\partial_z G^z$ and its finite element approximation, we have the following estimate (see [18, Theorem 3.14]):

$$\|\partial_z G^z\|_{1,1} + \|\partial_z G_h^z\|_{1,1} \leqslant C|\ln h|, \tag{4.11}$$

where the constant C is independent of z and h.

Theorem 4.2. *Assume that $A(u,v)$ is a uniformly elliptic and bounded bilinear form. Let the family of partitions J_h be quasi-uniform, u and u_h satisfy (4.7), and $u\in H_0^1(\Omega)\cap W_\infty^{k+2}(\Omega)$. Then, we have the following super-approximation estimate:*

$$\|\pi_k u-u_h\|_{1,\infty}\leqslant Ch^{k+1}|\ln h|\|u\|_{k+2,\infty},\quad k\geqslant 1. \tag{4.12}$$

Moreover, for the special case that $A(u,v)=(\nabla u,\nabla v)+(a_0u,v)$, when $u\in H_0^1(\Omega)\cap W_\infty^{k+3}(\Omega)$, we have the following ultra-approximation estimate:

$$\|\pi_k u-u_h\|_{1,\infty}\leqslant Ch^{k+2}|\ln h|\|u\|_{k+3,\infty},\quad k\geqslant 2. \tag{4.13}$$

Proof. Set $e_h=\pi_k u-u_h$. Then, it follows from the equations (4.7), (4.9)–(4.10), the estimate (4.11), and the elementary estimate (4.4) that

$$\begin{aligned}\partial_z e_h(z)=A^*(\partial_z G_h^z,e_h)=A(e_h,\partial_z G_h^z)&=A(\pi_k u-u,\partial_z G_h^z)\\ &\leqslant Ch^{k+1}|\ln h|\|u\|_{k+2,\infty},\end{aligned}$$

from which (4.12) is derived. Similarly, (4.13) can be also obtained by virtue of (4.5). □

Now, we are in the position to prove the superconvergence and the ultraconvergence of the finite element approximation.

Theorem 4.3. *Assume that $A(u,v)$ is a uniformly elliptic and bounded bilinear form. Let the family of partitions J_h be quasi-uniform, u and u_h satisfy (4.7), $u\in H_0^1(\Omega)\cap W_\infty^{k+2}(\Omega)$, $k\geqslant 1$, $D_0=E_1\times E_2=(x_0-h_i,x_0+h_{i+1})\times(y_0-\hbar_j,y_0+\hbar_{j+1})$, and R the derivative recovery operator with the Gauss sample points. Then, in the recovery intervals E_l, we have the following superconvergence result:*

$$\|D_l u-RD_l u_h\|_{0,\infty,E_l}\leqslant Ch^{k+1}|\ln h|\|u\|_{k+2,\infty},\quad k\geqslant 1,\ \ l=1,2. \tag{4.14}$$

Furthermore, when $u\in H_0^1(\Omega)\cap W_\infty^{k+1+s}(\Omega)$, $s=1,2$, $k\geqslant s$ is odd, $D_0=(x_0-h_i,x_0+h_i)\times(y_0-\hbar_j,y_0+\hbar_j)$, and the sample points are the roots of equation (3.7) with β_k and β_{k+1} being given by (3.8), we have the following superconvergence and ultraconvergence results at point (x_0,y_0):

$$|\nabla u(x_0,y_0)-R\nabla u_h(x_0,y_0)|\leqslant Ch^{k+s}|\ln h|\|u\|_{k+1+s,\infty},\quad s=1,2, \tag{4.15}$$

where when $s=2$ (ultraconvergence), $A(u,v)=(\nabla u,\nabla v)+(a_0u,v)$.

P r o o f. From Lemma 3.1 we know that R is a linear bounded operator. Then, in terms of Theorem 3.1 and (4.12), we obtain

$$\begin{aligned}\|D_l u - RD_l u_h\|_{0,\infty,E_l} &\\ \leqslant \|D_l u - RD_l \pi_k u\|_{0,\infty,E_l} + \|R\|\|D_l \pi_k u - D_l u_h\|_{0,\infty,E_l} &\\ \leqslant Ch^{k+1}(|u|_{k+2,\infty,D_0} + |\ln h|\|u\|_{k+2,\infty}).\end{aligned}$$

Thus, the superconvergence estimate (4.14) is derived.

Similarly, the estimate (4.15) can be obtained by using Theorem 3.2 and (4.12)–(4.13). □

It is very important for a finite element method to have a computable a posteriori error estimator, such that we can assess the accuracy of the finite element solution and enhance the efficiency by the adaptive algorithm in practical applications. The results in Theorem 4.3 can be used to produce a reliable a posteriori error estimator for the finite element approximation in derivative. In fact, we have by (4.14) or (4.15) that

$$\begin{aligned}\nabla(u-u_h)(x_0,y_0) &= (\nabla u - R\nabla u_h)(x_0,y_0) + (R\nabla u_h - \nabla u_h)(x_0,y_0)\\ &= O(h^{k+s}) + (R\nabla u_h - \nabla u_h)(x_0,y_0), \quad s = 1 \text{ or } 2.\end{aligned} \tag{4.16}$$

We know that, generally speaking, the interior nodal point (x_0, y_0) is not the superconvergence point of the derivative of the finite element approximation, that is, $\nabla(u-u_h)(x_0,y_0) = O(h^k)$. Then, from (4.16) we see that

$$|(R\nabla u_h - \nabla u_h)(x_0,y_0)|/|\nabla(u-u_h)(x_0,y_0)| \to 1, \quad h \to 0. \tag{4.17}$$

Hence, the quantity $|(R\nabla u_h - \nabla u_h)(x_0,y_0)|$ provides an asymptotically exact a posteriori error estimator of $|\nabla(u-u_h)(x_0,y_0)|$ for the finite element approximation to the elliptic boundary value problem.

5. Numerical Experiments

In this section, we will illustrate the superconvergence property of our derivative recovery method by numerical examples.

Consider the model problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega = [0,1]\times[0,1],\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{5.1}$$

where the exact solution is taken as $u = (x,y) = 10x\sin\pi x\sin\pi y$, and f is the corresponding source term. Ω is partitioned into uniform rectangles with mesh size h in both x- and y-direction, and the third-order finite element space Q_3 is employed.

First, by means of Lemma 3.2, we determine the sample points. For this purpose, we consider the following equation:

$$\begin{aligned}\beta_k l_k(x) + \beta_{k+1} l_{k+1}(x) &= 0, \quad x \in (-1,1),\\ l_k(x) &= \sqrt{\frac{2k+1}{2}}\,\frac{1}{k!}\,\frac{1}{2^k}\,\frac{\mathrm{d}^k}{\mathrm{d}x^k}\,(x^2-1)^k,\\ \beta_i &= (k+2)\int_{-1}^{1}(x+1)^{k+1} l_i(x)\,\mathrm{d}x, \quad i = k, k+1.\end{aligned} \tag{5.2}$$

For $k = 3$, the equation (5.2) can be rearranged as follows:

$$105x^4 + 560x^3 - 90x^2 - 336x + 9 = 0, \quad x \in (-1,1). \tag{5.3}$$

The three roots of equation (5.3) in $(-1,1)$ are, successively,

$$\xi_1 = -0.76267982, \quad \xi_2 = 0.02662742, \quad \xi_3 = 0.78428386.$$

Then, for the patch recovery interval $E_1 = (x_0 - h, x_0 + h)$, the sample set is

$$G = \Big\{ g_{\pm i} = x_0 \pm \frac{h}{2}(\xi_i + 1),\ i = 1,2,3 \Big\}.$$

Now, according to the formulae (3.10) and (3.11), we can calculate the recovered derivative values of $D_1 u_h(x_0, y_0)$ and $D_2 u_h(x_0, y_0)$, where u_h is the third order finite element solution of the problem (5.1). We shall examine the computational accuracy at the interior mesh points of the partition with $h = 1/4$. Tab. 1 gives the recovered derivative values in x-direction with mesh size h successively being halved. Tab. 2 gives the L_∞-error of $D_1 u - RD_1 u_h$ for different mesh sizes h. We see that the recovered derivatives possess remarkably high accuracy, and the L_∞-error goes to zero rapidly as h gets smaller and smaller. The computational results confirm our theoretical analysis numerically.

(x_0, y_0)	$h = 1/4$	$h = 1/8$	$h = 1/16$	$h = 1/32$	exact values
$(0.25, 0.25)$	8.92755830	8.92702908	8.92699015	8.92699082	8.92699081
$(0.25, 0.50)$	12.62547403	12.62472560	12.62467178	12.62467145	12.62467148
$(0.25, 0.75)$	8.92755830	8.92702908	8.92699015	8.92699082	8.92699081
$(0.50, 0.25)$	7.07174660	7.07111447	7.07106771	7.07106782	7.07106781
$(0.50, 0.50)$	10.00095996	10.00006598	10.00000049	10.00000002	10.00000000
$(0.50, 0.75)$	7.07174660	7.07111447	7.07106771	7.07106782	7.07106781
$(0.75, 0.25)$	−6.78072434	−6.78095503	−6.78097225	−6.78097248	−6.78097245
$(0.75, 0.50)$	−9.58939233	−9.58971856	−9.58974350	−9.58974321	−9.58974320
$(0.75, 0.75)$	−6.78072434	−6.78095503	−6.78097225	−6.78097248	−6.78097245

Table 1. Recovered derivative values in x-direction.

h	1/4	1/8	1/16	1/32
errors	8.03E−04	6.59E−05	6.60E−07	3.00E−08

Table 2. Errors in L_∞-norm.

Acknowledgement. The authors would like to thank Professor Michal Křížek and the anonymous referees for their valuable suggestions, which significantly improved the presentation of this paper.

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