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Sets Expressible as Unions of Staircase n-Convex Polygons

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Abstract

Let k and n be fixed, $k \geq 1$, $n \geq 1$, and let S be a simply connected orthogonal polygon in the plane. For $T \subseteq S, T$ lies in a staircase n-convex orthogonal polygon P in S if and only if every two points of T see each other via staircase n-paths in S. This leads to a characterization for those sets S expressible as a union of k staircase n-convex polygons P_i, $1 \leq i \leq k$.

Key words: orthogonal polygons, staircase n-convex polygons

2000 Mathematics Subject Classification: 52A35

1 Introduction

We begin with some definitions from [3] and [4]. Let S be a nonempty set in the plane. Set S is call an *orthogonal polygon* if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. Let λ be a simple polygonal path whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Path λ is called a *staircase path* if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. Edge $[v_{i-1}, v_i]$ will be called *north, south, east, or west* according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms *north, south, east, west, northeast, northwest, southeast, southwest* to describe the relative position of points. For $n \geq 1$, if the staircase path λ is a union of at most n edges, then λ is called a *staircase* n-path. If the staircase path λ is a union of exactly n edges, then n is the *length* of λ. For points x and y in set S, we say x *sees* y (x is *visible* from y) *via staircase n-paths* if and only if there is a staircase n-path in S which contains both x and y. Set S is called *staircase n-convex* if and only if for every x, y in S, x sees y via staircase n-paths. Parallel definitions hold for staircase paths. Set S is *horizontally convex* if and only if for each x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. Vertically convex is defined analogously. Finally, set S is *orthogonally convex* if and only if S is both horizontally convex and vertically convex. Using [11, Lemma 1], an orthogonal polygon S is orthogonally convex if and only if it is staircase convex.

Many results in convexity that involve the usual idea of visibility via segments have analogues that employ the notion of visibility via staircase paths or visibility via staircase n-paths. (See [1]–[4].) Here we obtain a staircase n-convex analogue of a result by Lawrence, Hare, and Kenelly [8] which concerns a decomposition of a set into convex subsets. It is interesting to notice that, while some staircase n-path results have conclusions which involve an increase in path length from n to $n + 1$ (see [1], [2]), no such adjustment is necessary here.

Throughout the paper, $\mathrm{cl}\, S$ and $\mathrm{int}\, S$ will denote the closure and interior, respectively, for set S. If λ is a path containing points x and $y, \lambda(x, y)$ will represent the subpath of λ from x to y. The reader may refer to Valentine [12], to Lay [9], to Danzer, Grünbaum, Klee [5], and to Eckhoff [6] for discussions on visibility via straight line segments.

2 The results

Theorem 1 is a staircase n-path analogue of [4, Lemmas 1 and 2].

Theorem 1 *Let n be fixed, $n \geq 1$, and let S be a simply connected orthogonal polygon in the plane with $T \subseteq S$. Set T lies in a staircase n-convex orthogonal polygon $P \subseteq S$ if and only if every two points of T see each other via staircase n-paths in S.*

Proof The sufficiency is immediate. To establish the necessity, assume that every two points of T see each other via staircase n-paths in S, to show that T lies in an appropriate subset of S. If $n = 1$, the result is obvious, so assume that $n \geq 2$.

For the moment, assume that T is finite. For every pair of points a, b in T, consider the length $k_{a,b}$ of a shortest $a - b$ path in S. If $k_{a,b} \neq 2$, select such a shortest path $\lambda(a, b)$. If $k_{a,b} = 2$ and if S contains only one $a - b$ 2-path, let $\lambda(a, b)$ denote this path. If S contains both of the $a - b$ 2-paths (clearly exactly two exist), then we must choose our associated path or paths carefully. Without loss of generality, say that a is southwest of b and that the associated 2-paths are $[a, c_1] \cup [c_1, b]$ and $[a, c_2] \cup [c_2, b]$, where c_1 is north of a and c_2 is east of a. If T contains a point northwest of c_1, let $\lambda_1(a, b) = [a, c_1] \cup [c_1, b]$. If T contains a point southeast of c_2, let $\lambda_2(a, b) = [a, c_2] \cup [c_2, b]$. If neither situation occurs, define $\lambda_1(a, b)$ and $\lambda_2(a, b)$ as above. Thus we will select two $a - b$ 2-paths $\lambda_1(a, b)$ and $\lambda_2(a, b)$ unless T contains a point northwest of c_1 but no point southeast of c_2, or vice versa. In these cases, we will select only one $a-b$ 2-path.

Let W denote the collection of all the selected n-paths. As in [4, Lemma 1], consider the maximal bounded subset S_W of the plane whose boundary lies in $\cup\{\lambda\colon \lambda \in W\}$. Clearly S_W is a simply connected subset of S and $\cup\{\lambda\colon \lambda \in W\} \subseteq S_W$. Moreover, by the proof of [4, Lemma 1], S_W is an orthogonally convex polygon.

We will show that S_W satisfies the theorem. Choose points x, y in S_W to show that x sees y via a staircase n-path in S_W. Without loss of generality, assume that x is northwest of y. Clearly there exists a point z' in T and north of (possibly on) the horizontal line at x. Likewise, there is a point z in T and east of (possibly on) the vertical line at y. There are three cases to consider.

Case 1. Assume that we may choose points z', z in T so that z' is northwest of x and z is southeast of y. Consider a $z' - z\, n$-path in W, say $\lambda(z', z)$. (See Figures 1a and 1b.) There exists a point x_1 on $\lambda(z', z)$ so that $[x, x_1]$ is horizontal (and east) or vertical (and south). Observe that if $x \notin \lambda(z', z)$, then x_1 is on at least the second segment of $\lambda(z', z)$. Then $[x, x_1] \cup \lambda(x_1, z) \equiv \lambda'(x, z)$ is an $x-z$ staircase n-path contained in the orthogonally convex set S_W. Similarly, choose y_1 on $\lambda'(x, z)$ so that $[y_1, y]$ is horizontal (and east) or vertical (and south). The path $\lambda'(x, y_1) \cup [y_1, y]$ will be an $x-y$ staircase n-path in S_W, the desired result.

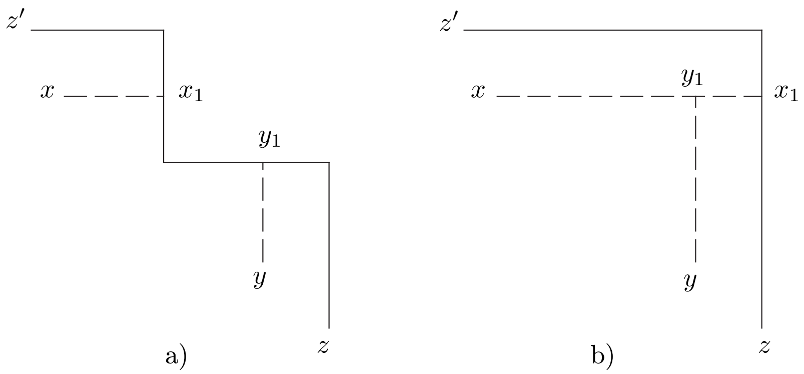

Fig. 1

Case 2. For an appropriate labeling, assume that point z' in T may be selected northwest of x but that there is no point of T southeast of y. This implies the existence in W of some minimal length path λ which contains points southeast of y (possibly y itself) and which joins a point southwest of y in T to a point northeast of y in T. Assume that $\lambda = \lambda(w, z)$, where w is southwest of y and z is northeast of y. (See Figure 2.)

Every two of the points z', z, w see each other via staircase n-paths in W. In case z' is south of (the horizontal line at) z, then x sees y via an east-south 2-path in S_W, the desired result. A parallel argument holds if z' is east of (the vertical line at) w. Hence we assume that neither situation occurs. That is, suppose that z' is northwest of w and northwest of z. Consider n-paths $\delta = \delta(z', z)$

and $\mu = \mu(z', w)$ in W from z' to z and from z' to w, respectively. (Again see Figure 2.) The simply connected subset of S_W bounded by $\lambda \cup \delta \cup \mu$ contains the $w - z$ 2-path $\lambda_1 = [w, u_1] \cup [u_1, z]$, where u_1 is north of w, west of z. Since $\lambda(w, z)$ has minimal length in S, it must be a 2-path as well. Moreover, since $\lambda(w, z)$ contains points southeast of y, either $\lambda = \lambda_2(w, z) = [w, u_2] \cup [u_2, z]$, where u_2 is east of w, south of z, or λ contains point y (or both). The former situation cannot occur, since it would imply the existence in T of some point southeast of u_2 and consequently southeast of y, contradicting our hypothesis for Case 2. Thus $\lambda \neq \lambda_2(w, z)$. Since λ is a 2-path from w to z, λ must be $\lambda_1(w, z)$. Furthermore, by our comment above, $\lambda = \lambda_1(w, z)$ must contain point y. But then one of the points z, w will be southeast of y, again contradicting our hypothesis. This situation cannot occur either, finishing the argument for Case 2.

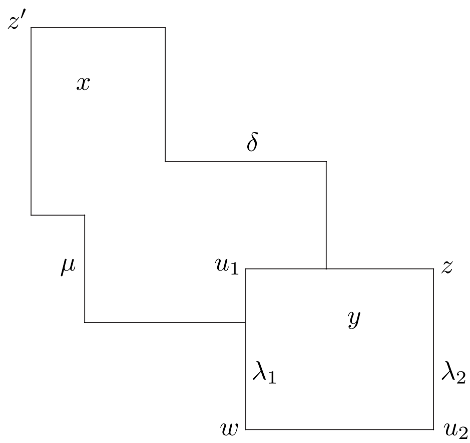

Fig. 2

Case 3. Assume that there is no point of T northwest of x and no point of T southeast of y. As in Case 2, this implies the existence in W of some minimal length path $\lambda = \lambda(z, w)$ which contains points southeast of y (possibly y itself) and which joins point z northeast of y in T to point w southwest of y in T. A parallel argument produces a minimal length path $\lambda' = \lambda'(z', w')$ which contains points northwest of x (possibly x itself) and which joins point z' northeast of x in T to point w' southwest of y in T. Every two of the points z, w, z', w' see each other via minimal length n-paths in W.

In case w' is south of (the horizontal line at) y, then x sees y via a (southeast) 2-path in S_W, finishing the argument. Parallel arguments hold if z is north of x, if z' is east of y, or if w is west of x. Assume that none of these situations occur. Then the region S_W contains the east-north 2-path from w' to z' and the north-east 2-path from w to z. An argument like the one in Case 2 can be applied to either of these 2-paths to obtain a contradiction. This finishes Case 3 and completes the argument for T finite.

When T is infinite, we use an approach from [4, Lemma 2]. Let $\mathcal{L}$ denote the family of lines determined by edges of S. In an obvious way, $\mathcal{L}$ gives rise to a collection $\mathcal{U}$ of nondegenerate closed rectangular regions such that:

1) No member of $\mathcal{U}$ contains any other nondegenerate closed rectangular region determined by $\mathcal{L}$, and

2) $\cup\{U\colon U \in \mathcal{U}\} = \mathrm{cl}(\mathrm{int}\, S)$. Let $\mathcal{A}$ be the family $\{\mathrm{int}\, U\colon U \in \mathcal{U}\} \cup \{(s,t)\colon [s,t]$ an edge of U, $U \in \mathcal{U}\} \cup \{(s,t)\colon [s,t]$ an edge of S and $(s,t) \cap \mathrm{cl}(\mathrm{int}, S) = \phi\}$. Notice that $\mathcal{A}$ is finite, $\mathcal{A}$ fails to cover at most a finite subset S_0 of S, and $\cup\{\mathrm{cl}\, A\colon A \in \mathcal{A}\} = S$. Moreover, by the proof of [3, Theorem 1], for $k \geq 2$, if N is a staircase k-convex polygon in S and $a \in N \cap A$ for some A in $\mathcal{A}$, then $N \cup \mathrm{cl}\, A$ lies in a staircase k-convex polygon in S.

Following the proof of [4, Lemma 2], when T is infinite, let T_0 denote the finite (possibly empty) subset of T not covered by any member of $\mathcal{A}$. That is, $T_0 = T \cap S_0$. Let $A_1, \dots, A_m$ denote the members of $\mathcal{A}$ which meet T, and for each $i, 1 \leq i \leq m$, choose $x_i \in T \cap A_i$. Since the set $T_0 \cup \{x_1, \dots, x_m\}$ is finite, by the first part of the proof, it lies in a staircase n-convex orthogonal polygon N_0 in S. Since $x_1 \in N_0 \cap A_1$, by the remark above, $N_0 \cup \mathrm{cl}\, A_1$ lies in a staircase n-convex orthogonal polygon N_1 in S. By an obvious induction, $N_0 \cup \mathrm{cl}\, A_1 \cup \dots \cup \mathrm{cl}\, A_m$ lies in a staircase n-convex orthogonal polygon N_m in S. Certainly $T \subseteq N_m$, and the set N_m satisfies the theorem. This completes the proof. □

Using Theorem 1, it is easy to establish the following corollaries. The first is an analogue of a result by Lawrence, Hare, and Kenelly [8] while the second is an analogue of results by McKinney [10] and Hare and Kenelly [7].

Corollary 1 *Let k and n be fixed, $k \geq 1$, $n \geq 1$, and let S be a simply connected orthogonal polygon in the plane. Set S is a union of k orthogonal polygons, each staircase n-convex, if and only if for every finite subset F of S, there is a k-partition of F into sets $F_1, \dots, F_k$ such that every pair in F_i can be joined by a staircase n-path in S, $1 \leq i \leq k$.*

Proof The sufficiency is obvious. The proof of the necessity follows from the proof of [4, Theorem 1], together with Theorem 1 above. □

Corollary 2 *Let n be fixed, $n \geq 1$, and let S be a simply connected orthogonal polygon in the plane. Set S is a union of two orthogonal polygons, each staircase n-convex, if and only if for every sequence $v_1, \dots, v_{j+1} = v_1$ in S, j odd, at least one consecutive pair of points v_i, v_{i+1} sees each other via staircase n-paths.*

Proof Again the sufficiency is clear. The argument for the necessity follows the proof of [4, Theorem 3] and uses Theorem 1 above. □

In conclusion, it is easy to see that the result in Theorem 1 fails without the requirement that set S be simply connected. Consider the following example.

Example 1 Let S denote the boundary of a rectangular region, with T the associated vertex set. Every two points of T see each other via staircase 2-paths in S. However, no staircase convex subset of S contains T.

References

[1] Breen, M.: *A Helly theorem for intersections of sets starshaped via staircase n-paths.* Ars Combinatoria **78** (2006), 47–63.

[2] Breen, M.: *Intersections and unions of orthogonal polygons starshaped via staircase n-paths.* Monatsh. Math. **148** (2006), 91–100.

[3] Breen, M.: *Staircase k-kernels for orthogonal polygons.* Arch. Math. **63** (1994), 182–190.

[4] Breen, M.: *Unions of orthogonally convex or orthogonally starshaped polygons.*. Geometriae Dedicata **53** (1994), 49–56.

[5] Danzer, L., Grünbaum, B., Klee, V.: *Helly's theorem and its relatives.* In: Convexity, Proc. Sympos. Pure Math., *Amer. Math. Soc.*, Providence, RI **7** (1962), 101–180.

[6] Eckhoff, J.: *Helly, Radon, and Carathéodory type theorems.* In: Gruber P. M., Wills J. M., (eds.) Handbook of Convex Geometry, vol. A, *North Holland*, New York (1993), 389–448.

[7] Hare, W. R., Jr., Kenelly, J. W.: *Sets expressible as unions of two convex sets.* Proc. Amer. Math. Soc. **25** (1970), 379–380.

[8] Lawrence, J. F., Hare, W. R., Jr., Kenelly, J. W.: *Finite unions of convex sets.* Proc. Amer. Math. Soc. **34** (1972), 225–228.

[9] Lay, S. R.: Convex Sets and Their Applications. *John Wiley*, New York, 1982.

[10] McKinney, R. L.: *On unions of two convex sets.* Canad. J. Math **18** (1966), 883–886.

[11] Motwani, R., Raghunathan, A., Saran, H.: *Covering orthogonal polygons with star polygons: the perfect graph approach.* J. Comput. Syst. Sci. **40** (1990), 19–48.

[12] Valentine, F. A.: Convex Sets. *McGraw-Hill*, New York, 1964.