Zdeněk Dušek; Oldřich Kowalski; Zdeněk Vlášek

Homogeneous Geodesics in 3-dimensional Homogeneous Affine Manifolds

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 50 (2011), No. 1, 29--42

Persistent URL: `http://dml.cz/dmlcz/141718`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **50**, 1 (2011) 29–42

Homogeneous Geodesics in 3-dimensional Homogeneous Affine Manifolds*

Zdeněk DUŠEK [1], Oldřich KOWALSKI [2], Zdeněk VLÁŠEK [3]

[1] *Department of Algebra and Geometry, Faculty of Science, Palacký University*
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: zdenek.dusek@upol.cz

[2] [3] *Mathematical Institute, Faculty of Mathematics and Physics*
Charles University, Sokolovská 83, 186 75 Praha, Czech Republic
e-mail: kowalski@karlin.mff.cuni.cz
e-mail: vlasek@karlin.mff.cuni.cz

(Received June 16, 2010)

Abstract

For studying homogeneous geodesics in Riemannian and pseudo-Riemannian geometry (on reductive homogeneous spaces) there is a simple algebraic formula which works, at least potentially, in every given case. In the affine differential geometry, there is not such a universal formula. In the previous work, we proposed a simple method of investigation of homogeneous geodesics in homogeneous affine manifolds in dimension 2. In the present paper, we use this method on certain classes of homogeneous connections on the examples of 3-dimensional Lie groups.

Key words: affine connection, affine Killing vector field, homogeneous manifold, homogeneous geodesic

2000 Mathematics Subject Classification: 53B05, 53C30

1 Introduction

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold.* Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the *homogeneous space* G/H. In general, there may exist more than one

*The first author was supported by the research project MSM 6198959214, the second and the third author were supported by the reesearch project MSM 0021620839 of the Czech Ministry MŠMT

such group $G \subset I_0(M)$. For any fixed choice $M = G/H$, G acts effectively on G/H from the left. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *pseudo-Riemannian homogeneous space*.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If the metric g is indefinite, the reductive decomposition may not exist (see for instance [12] for an example of nonreductive pseudo-Riemannian homogeneous space). For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM we obtain a scalar product $\langle , \rangle$ on $\mathfrak{m}$. This scalar product is obviously $\mathrm{Ad}(H)$-invariant.

The definition of a homogeneous geodesic is well known in the Riemannian case. In the pseudo-Riemannian case it must be modified as follows:

Definition 1 A geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be *homogeneous* if there exists (1) a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J; (2) a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$. The vector X is then called a *geodesic vector*.

For results on homogeneous geodesics in homogeneous Riemannian manifolds we refer for example to [4], [15], [18], [20], [22]. A homogeneous Riemannian manifold all of whose geodesics are homogeneous is called a Riemannian *g.o. manifold*. For many results and further references on Riemannian g.o. manifolds see for example [7], [10], [14], [19], [2]. Homogeneous geodesics are interesting also in pseudo-Riemannian geometry and null homogeneous geodesics are of particular interest. In [12] and [24], the authors study plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics. In these papers, each geodesic vector X is characterized by the formula (1). See also [5], [8], [9]. A rigorous mathematical proof of this characterization is given in [8]:

Lemma 2 *Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \quad \textit{for all } Z \in \mathfrak{m}, \textit{ where } k \in \mathbb{R} \textit{ is some } \text{constant}. \tag{1}$$

Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

For the study of homogeneous geodesics in homogeneous affine manifolds, we cannot use the algebraic tools as in pseudo-Riemannian geometry. In [11], the present authors described a new, elementary method for studying homogeneous geodesics in homogeneous affine manifolds. Let us now recall this method.

Definition 3 Let ∇ be an affine connection on a manifold M. Then ∇, or also (M, ∇), is said to be *homogeneous*, if for each two points $x, y \in M$ there exists an affine transformation $\varphi\colon M \to M$ such that $\varphi(x) = y$. It means that φ is a diffeomorphism such that

$$\nabla_{\varphi_* X} \varphi_* Y = \varphi_* (\nabla_X Y)$$

holds for every vector fields X, Y defined on M.

Definition 4 A vector field X on an affine manifold (M, ∇) is called an *affine Killing vector field* if the Lie derivative $\mathcal{L}_X \nabla$ vanishes, or, equivalently, if X satisfies the equation

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0, \tag{2}$$

for all vector fields Y, Z.

Proposition 5 *Let (M, ∇) be a homogeneous affine manifold and $p \in M$. There exist $n = \dim(M)$ affine Killing vector fields which are linearly independent at each point of some neighbourhood $\mathcal{U}$ of p.*

Proof Let (M, ∇) be homogeneous, i.e., $M = G/H$, where ∇ is G-invariant. Let $X_1, \ldots, X_n$ be an n-tuplet of linearly independent vectors of $T_p(M)$. For each index $i = 1, \ldots, n$, there exists a vector $Z_i \in \mathfrak{g}$ such that

$$\frac{d}{dt} \exp(tZ_i)(p)|_{t=0} = X_i. \tag{3}$$

Let us define, for each $i = 1, \ldots, n$, a vector field K_i by

$$K_i(q) = \frac{d}{dt} \exp(tZ_i)(q)|_{t=0} \quad \text{for each } q \in M. \tag{4}$$

Then each K_i is called a fundamental vector field attached to Z_i ([13]). At the same time, $K_1, \ldots, K_n$ are affine Killing vector fields on (G/H). They are linearly independent at p and they must be linearly independent also at each point q of some neighbourhood $\mathcal{U}$ of p. $\square$

Definition 6 In a homogeneous affine manifold (M, ∇), by a *homogeneous geodesic* we mean a geodesic which is an orbit of an one-parameter group of affine diffeomorphisms. The canonical parameter of the group need not be the affine parameter of the geodesic.

Recall that a parametrized curve in a manifold M is said to be *regular* if $\gamma'(t) \neq 0$ for all values of t. The following proposition is well known:

Proposition 7 *Let $M = G/H$ (where G acts transitively and effectively on M) be a homogeneous space with a left-invariant affine connection ∇. Then each regular curve which is an orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.*

Definition 8 A nonvanishing smooth vector field Z on M is said to be *geodesic along its regular integral curve* γ if the curve $\gamma(t)$ is geodesic up to a possible reparametrization. If all regular integral curves of Z are geodesics up to a reparametrization, then the vector field Z is called *a geodesic vector field.*

Proposition 9 ([11]) *Let Z be a nonvanishing Killing vector field on (M, ∇).*

1) Z is geodesic along its integral curve γ if and only if

$$\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)} \tag{5}$$

holds along γ, where $k_\gamma \in \mathbb{R}$ is a constant. If $k_\gamma = 0$, then t is the affine parameter of geodesic γ. If $k_\gamma \neq 0$, then the affine parameter of this geodesic is $s = e^{-k_\gamma t}$.

2) Z is a geodesic vector field if and only if

$$\nabla_Z Z = k \cdot Z \tag{6}$$

holds on M. Here k is a smooth function on M, which is constant along integral curves of the vector field Z.

Let us now remind the classification of homogeneous affine connections in dimension 2. The following basic theorem from [3] is a generalization of the classification result by B. Opozda, [23], to connections with nonzero torsion. We only change slightly the notation:

Theorem 10 *Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold M. Then, in a neighborhood $\mathcal{U}$ of each point $m \in M$, either ∇ is locally a Levi-Civita connection of the unit sphere or, there is a system (u, v) of local coordinates and constants A, B, C, D, E, F, G, H such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

Type A

$$\nabla_{\partial_u}\partial_u = A\partial_u + B\partial_v, \quad \nabla_{\partial_u}\partial_v = C\partial_u + D\partial_v,$$
$$\nabla_{\partial_v}\partial_u = E\partial_u + F\partial_v, \quad \nabla_{\partial_v}\partial_v = G\partial_u + H\partial_v.$$

Type B

$$\nabla_{\partial_u}\partial_u = \tfrac{A}{u}\partial_u + \tfrac{B}{u}\partial_v, \quad \nabla_{\partial_u}\partial_v = \tfrac{C}{u}\partial_u + \tfrac{D}{u}\partial_v,$$
$$\nabla_{\partial_v}\partial_u = \tfrac{E}{u}\partial_u + \tfrac{F}{u}\partial_v, \quad \nabla_{\partial_v}\partial_v = \tfrac{G}{u}\partial_u + \tfrac{H}{u}\partial_v,$$

where not all A, B, C, D, E, F, G, H are zero.

In [11], the present authors studied in details the homogeneous affine connections of type A in the plane $\mathbb{R}^2$ and those of type B in the open half-plane. We proved that any connection of type A admits at least one geodesic Killing vector field. Connectios of type B do not admit geodesic Killing vector fields in general, but they admit at least one homogeneous geodesic through any point. In this paper, we are going to study some 3-dimensional examples.

2 Homogeneous connections in dimension three

Let now (M, ∇) be a 3-dimensional manifold with an arbitrary affine connection. Let the vector field X be an affine Killing vector field on (M, ∇). If there is a system (u^1, u^2, u^3) of global coordinates on M, then the formula (2) is satisfied for all Y, Z if, and only if, it is satisfied for all coordinate vector fields ($Y = \partial_j$, $Z = \partial_k$), where $1 \leq j, k \leq 3$. Moreover, the inequality $j \leq k$ can be added if the torsion is zero. If we denote by $\Gamma^l_{jk} = \Gamma^l_{jk}(u^1, u^2, u^3)$ the corresponding Christoffel symbols ($1 \leq j, k, l \leq 3$) and put $X = \sum_{i=1}^3 X^i(u^1, u^2, u^3)\partial_i$, then X is an affine Killing vector field if, and only if, the following system of 27 partial differential equations holds:

$$\frac{\partial^2}{\partial u^k \partial u^j} X^l + \sum_{i=1}^{n} \left(X^i \frac{\partial}{\partial u^i} \Gamma^l_{jk} - \Gamma^i_{jk} \frac{\partial}{\partial u^i} X^l + \Gamma^l_{ji} \frac{\partial}{\partial u^k} X^i + \Gamma^l_{ik} \frac{\partial}{\partial u^j} X^i \right) = 0. \quad (7)$$

As well known, the symmetrization of any affine connection ∇ has the same geodesics as ∇. Hence we can restrict our calculations to the case of zero torsion. We use the inequality $j \leq k$ and the number of equations is reduced to 18. We first simplify our notation as follows:

$$\begin{aligned}
&u^1, u^2, u^3 \to u, v, w, \\
&X^1(u,v,w), X^2(u,v,w), X^3(u,v,w) \to a(u,v,w), b(u,v,w), c(u,v,w), \\
&\Gamma^l_{11} \to A^l,\ \Gamma^l_{22} \to B^l,\ \Gamma^l_{33} \to C^l,\ \Gamma^l_{12} \to E^l,\ \Gamma^l_{13} \to F^l,\ \Gamma^l_{23} \to G^l,\ l = 1,2,3.
\end{aligned} \quad (8)$$

Then, an affine Killing vector field

$$X = a(u,v,w)\partial_u + b(u,v,w)\partial_v + c(u,v,w)\partial_w$$

is characterized by the system of PDEs as follows:

$$\begin{aligned}
&a_{uu} + a_u A^1 - a_v A^2 - a_w A^3 + 2b_u E^1 + 2c_u F^1 + aA^1_u + bA^1_v + cA^1_w = 0, \\
&b_{uu} + 2a_u A^2 + b_u(2E^2 - A^1) - b_w A^3 + 2c_u F^2 + aA^2_u + bA^2_v + cA^2_w = 0, \\
&c_{uu} + 2a_u A^3 + 2b_u E^3 + c_u(2F^3 - A^1) - c_v A^2 - c_w A^3 + aA^3_u + bA^3_v + cA^3_w = 0, \\
&a_{uv} + a_v(A^1 - E^2) - a_w E^3 + b_u B^1 + b_v E^1 + c_u G^1 + c_v F^1 \\
&\quad + aE^1_u + bE^1_v + cE^1_w = 0, \\
&b_{uv} + a_u E^2 + a_v A^2 + b_u(B^2 - E^1) - b_w E^3 + c_u G^2 + c_v F^2 \\
&\quad + aE^2_u + bE^2_v + cE^2_w = 0, \\
&c_{uv} + a_u E^3 + a_v A^3 + b_u B^3 + b_v E^3 + c_u(G^3 - E^1) + c_v(F^3 - E^2) - c_w E^3 \\
&\quad + aE^3_u + bE^3_v + cE^3_w = 0, \\
&a_{uw} - a_v F^2 + a_w(A^1 - F^3) + b_u G^1 + b_w E^1 + c_u C^1 + c_w F^1 \\
&\quad + aF^1_u + bF^1_v + cF^1_w = 0, \\
&b_{uw} + a_u F^2 + a_w A^2 + b_u(G^2 - F^1) - b_v F^2 + b_w(E^2 - F^3) + c_u C^2 + c_w F^2 \\
&\quad + aF^2_u + bF^2_v + cF^2_w = 0,
\end{aligned}$$

$$
\begin{aligned}
&c_{uw} + a_u F^3 + a_w A^3 + b_u G^3 + b_w E^3 + c_u(C^3 - F^1) - c_v F^2 \\
&\quad + aF^3_u + bF^3_v + cF^3_w = 0, \\
&a_{vv} - a_u B^1 + a_v(2E^1 - B^2) - a_w B^3 + 2b_v B^1 + 2c_v G^1 \\
&\quad + aB^1_u + bB^1_v + cB^1_w = 0, \\
&b_{vv} + 2a_v E^2 - b_u B^1 + b_v B^2 - b_w B^3 + 2c_v G^2 + aB^2_u + bB^2_v + cB^2_w = 0, \\
&c_{vv} + 2a_v E^3 + 2b_v B^3 - c_u B^1 + c_v(2G^3 - B^2) - c_w B^3 \\
&\quad + aB^3_u + bB^3_v + cB^3_w = 0, \\
&a_{vw} - a_u G^1 + a_v(F^1 - G^2) + a_w(E^1 - G^3) + b_v G^1 + b_w B^1 + c_v C^1 + c_w G^1 \\
&\quad + aG^1_u + bG^1_v + cG^1_w = 0, \\
&b_{vw} + a_v F^2 + a_w E^2 - b_u G^1 + b_w(B^2 - G^3) + c_v C^2 + c_w G^2 \\
&\quad + aG^2_u + bG^2_v + cG^2_w = 0, \\
&c_{vw} + a_v F^3 + a_w E^3 + b_v G^3 + b_w B^3 - c_u G^1 + c_v(C^3 - G^2) \\
&\quad + aG^3_u + bG^3_v + cG^3_w = 0, \\
&a_{ww} - a_u C^1 - a_v C^2 + a_w(2F^1 - C^3) + 2b_w G^1 + 2c_w C^1 \\
&\quad + aC^1_u + bC^1_v + cC^1_w = 0, \\
&b_{ww} + 2a_w F^2 - b_u C^1 - b_v C^2 + b_w(2G^2 - C^3) + 2c_w C^2 \\
&\quad + aC^2_u + bC^2_v + cC^2_w = 0, \\
&c_{ww} + 2a_w F^3 + 2b_w G^3 - c_u C^1 - c_v C^2 + c_w C^3 + aC^3_u + bC^3_v + cC^3_w = 0.
\end{aligned} \tag{9}
$$

We shall now consider some 3-dimensional groups and invariant affine connections on them. From now on, we will write the indices by $A, \dots, G$ down.

3 The abelian group $\mathbb{R}^3$

Let us consider the abelian group $\mathbb{R}^3(u, v, w)$ with the affine connection whose Christoffel symbols

$$A_1, \dots, G_3 \tag{10}$$

are constants and let this group act on itself by the left translations. The infinitesimal translations are the vector fields

$$\partial_u, \partial_v, \partial_w \tag{11}$$

and it is easy to verify that these are affine Killing vector fields. Hence the translations are affine diffeomorphisms and the group $\mathbb{R}^3$ with the given affine connection is an affine homogeneous manifold.

Let us now consider the general Killing vector field

$$X = x\partial_u + y\partial_v + z\partial_w \equiv (x, y, z)$$

and investigate the condition $\nabla_X X = kX$. We obtain, after splitting this equation with respect to the basic Killing fields $\partial_u, \partial_v, \partial_w$, the three equations

$$\begin{aligned} x^2A_1 + y^2B_1 + z^2C_1 + 2xyE_1 + 2xzF_1 + 2yzG_1 &= kx \\ x^2A_2 + y^2B_2 + z^2C_2 + 2xyE_2 + 2xzF_2 + 2yzG_2 &= ky \\ x^2A_3 + y^2B_3 + z^2C_3 + 2xyE_3 + 2xzF_3 + 2yzG_3 &= kz. \end{aligned} \tag{12}$$

Now we denote by $L_1(x,y,z), L_2(x,y,z)$ and $L_3(x,y,z)$ the left-hand sides of these equations and we eliminate k in the right-hand sides. We obtain three conditions

$$\begin{aligned} S_1(x,y,z) &= L_2(x,y,z)z - L_3(x,y,z)y = 0, \\ S_2(x,y,z) &= L_3(x,y,z)x - L_1(x,y,z)z = 0, \\ S_3(x,y,z) &= L_1(x,y,z)y - L_2(x,y,z)x = 0. \end{aligned} \tag{13}$$

Explicitly, we have

$$\begin{aligned} S_1(x,y,z) &= x^2zA_2 - x^2yA_3 + y^2zB_2 - y^3B_3 + z^3C_2 - yz^2C_3 \\ &\quad +2xyzE_2 - 2xy^2E_3 + 2xz^2F_2 - 2xyzF_3 + 2yz^2G_2 - 2y^2zG_3 = 0, \\ S_2(x,y,z) &= -x^2zA_1 + x^3A_3 - y^2zB_1 + xy^2B_3 - z^3C_1 + xz^2C_3 \\ &\quad -2xyzE_1 + 2x^2yE_3 - 2xz^2F_1 + 2x^2zF_3 - 2yz^2G_1 + 2xyzG_3 = 0, \\ S_3(x,y,z) &= x^2yA_1 - x^3A_2 + y^3B_1 - xy^2B_2 + yz^2C_1 - xz^2C_2 \\ &\quad +2xy^2E_1 - 2x^2yE_2 + 2xyzF_1 - 2x^2zF_2 + 2y^2zG_1 - 2xyzG_2 = 0. \end{aligned} \tag{14}$$

Here we have the algebraical dependency of S_1, S_2, S_3,

$$xS_1(x,y,z) + yS_2(x,y,z) + zS_3(x,y,z) = 0. \tag{15}$$

It is easy to see that just these three (algebraically dependent) equations (13) together give the full information, i.e., these three equations are equivalent with the conditions (12), and two of them are *not enough* for such equivalence.

For the later use, we shall need the following

Lemma 11 *Two cubic curves in the real projective plane $P_2(\mathbb{R})$ have always a non-empty real intersection.*

This statement must be known from some literature. But, instead of giving references, we present here a short proof which was kindly offered us by Professor Thomas Friedrich. We start with

Theorem 12 (Borsuk–Ulam, [1]) *Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. Then there exists a point $\mathbf{x}^* \in S^n$ such that $f(\mathbf{x}^*) = f(-\mathbf{x}^*)$.*

In particular, if $f\colon S^n \to \mathbb{R}^n$ is an odd map (i.e. $f(\mathbf{x}) = -f(-\mathbf{x})$), then there exist always a point $\mathbf{x}^* \in S^n$ such that $f(\mathbf{x}^*) = o$ (the origin).

Put $n = 2$ and consider two cubic curves in $P_2(\mathbb{R})$, or, equivalently, two homogeneous polynomials $U(x,y,z)$ and $V(x,y,z)$ of degree 3 defined on $\mathbb{R}^3 \setminus o$. Then we define a map $f\colon S^2 \to \mathbb{R}^2$ by the formula

$$f(x,y,z) = (U(x,y,z), V(x,y,z)) \quad \text{for each } (x,y,z) \in S^2 \subset \mathbb{R}^3.$$

Here f is continuous and it satisfies the condition $f(-\mathbf{x}) = -f(\mathbf{x})$, $\mathbf{x} \in S^2$. Consequently, by the Borsuk-Ulam theorem, there exist a point $\mathbf{x}^* \in S^2$ such that $f(\mathbf{x}^*) = (0,0)$, i.e. $U(\mathbf{x}^*) = 0$ and $V(\mathbf{x}^*) = 0$. This concludes the proof of Lemma 11. □

We get the following main result:

Theorem 13 *Let one of the following conditions holds:*

a) $(G_1)^2 - B_1C_1 < 0$,

b) $(F_2)^2 - A_2C_2 < 0$,

c) $(E_3)^2 - A_3B_3 < 0$.

Then the invariant affine connection on $\mathbb{R}^3$ corresponding to the given parameters $A_1, A_2, \dots, G_3$ admits at least one geodesic Killing vector field.

Proof Let a) be satisfied. Put $x = 0$ in the equations $S_2 = 0, S_3 = 0$. Then we obtain the equations

$$-z(B_1y^2 + 2G_1yz + C_1z^2) = 0, \qquad y(B_1y^2 + 2G_1yz + C_1z^2) = 0.$$

Because the quadratic form $B_1y^2 + 2G_1yz + C_1z^2$ is strictly definite, the only consequence is $y = 0, z = 0$. Hence we see that, if (x_0, y_0, z_0) is a nontrivial common solution of the equations S_2, S_3, which always exists according to Lemma 11, then $x_0 \neq 0$. From the formula (15) we see that (x_0, y_0, z_0) is also a solution of the equation $S_1 = 0$. Thus all conditions for existence of a solution of the equations (14) are satisfied and a geodesic Killing vector field exists.

Now, the cases b) and c) are treated analogously: we substitute first $y = 0$ in the equations $S_1 = 0, S_3 = 0$ and then $z = 0$ in the equations $S_1 = 0, S_2 = 0$. □

Corollary 14 *On an unbounded open domain of the parameter space $\mathbb{R}^{18}[A_1, A_2, \dots, G_3]$, the corresponding invariant connections admit at least one homogeneous geodesic through any point $p \in \mathbb{R}^3$.*

4 Other examples

In this section we will consider other examples of 3-dimensional groups with invariant affine connections. For these groups, we will not consider geodesic Killing vector fields, because there are no such vector fields in general. For a Killing vector field, we will investigate just the integral curve through the origin. Because the connection is homogeneous, we obtain immediately the same properties at other points.

4.1 The Heisenberg group H_3

The Heisenberg group H_3 can be represented by the matrices of the form

$$\begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}, \tag{16}$$

hence H_3 can be identified with the 3-space $\mathbb{R}^3(u,v,w)$ equipped with the Riemannian metric $ds^2 = du^2 + (dv - u dw)^2 + dw^2$. Then H_3 acts on itself from the left by isometries with respect to this metric and corresponding Killing vector fields (infinitesimal isometries) are $\partial_v, \partial_w, \partial_u + w\partial_v$. Now, an affine connection ∇ on H_3 is left-invariant if, and only if, the Lie algebra $\mathrm{span}\{\partial_u + w\partial_v, \partial_v, \partial_w\}$ consists of affine Killing vector fields of ∇. We write down the equations (9) for the three choices

1) $a(u,v,w) = 1,\ b(u,v,w) = w,\ c(u,v,w) = 0;$
2) $a(u,v,w) = 0,\ b(u,v,w) = 1,\ c(u,v,w) = 0;$
3) $a(u,v,w) = 0,\ b(u,v,w) = 0,\ c(u,v,w) = 1.$

By solving this system of 54 differential equations, we obtain the Christoffel symbols in the form

$$\begin{aligned}
A_1(u,v,w) &= A_1,\\
A_2(u,v,w) &= A_2 + A_3 u,\\
A_3(u,v,w) &= A_3,\\
B_1(u,v,w) &= B_1,\\
B_2(u,v,w) &= B_2 + B_3 u,\\
B_3(u,v,w) &= B_3,\\
C_1(u,v,w) &= C_1 - 2G_1 u + B_1 u^2,\\
C_2(u,v,w) &= C_2 + (C_3 - 2G_2)u + (B_2 - 2G_3)u^2 + B_3 u^3,\\
C_3(u,v,w) &= C_3 - 2G_3 u + B_3 u^2,\\
E_1(u,v,w) &= E_1,\\
E_2(u,v,w) &= E_2 + E_3 u,\\
E_3(u,v,w) &= E_3,\\
F_1(u,v,w) &= F_1 - E_1 u,\\
F_2(u,v,w) &= F_2 + (F_3 - E_2)u - E_3 u^2,\\
F_3(u,v,w) &= F_3 - E_3 u,\\
G_1(u,v,w) &= G_1 - B_1 u,\\
G_2(u,v,w) &= G_2 + (G_3 - B_2)u - B_3 u^2,\\
G_3(u,v,w) &= G_3 - B_3 u,
\end{aligned} \tag{17}$$

where $A_1, \dots, G_3$ are constant parameters. We will investigate existence of homogeneous geodesics for this class of connections.

Our next step will be more subtle than for the group $\mathbb{R}^3$. Consider a general affine Killing vector field from $\mathrm{span}\{\partial_v, \partial_w, \partial_u + w\partial_v\}$, i.e. $X = x(\partial_u + w\partial_v) + y\partial_v + z\partial_w$, where x, y, z are arbitrary parameters. Now we shall look at Killing vector field for ∇ which is geodesic along an integral curve. If such a vector field exists, then (H_3, ∇) admits affine homogeneous geodesics.

We see easily that the general trajectories of the vector field

$$X = x(\partial_u + w\partial_v) + y\partial_v + z\partial_w$$

are given by

$$u(t) = xt + c_1, \quad v(t) = xzt^2/2 + (y + c_3 x)t + c_2, \quad w(t) = zt + c_3. \tag{18}$$

Now, we shall study only the trajectory starting at the origin of $H_3(u,v,w)$, i.e. we put the integration constants c_1, c_2, c_3 equal to zero. Its tangent vector $\gamma'(t)$ satisfies

$$u'(t) = x, \quad v'(t) = xzt + y, \quad w'(t) = z. \tag{19}$$

We express the equation $\nabla_{\gamma'(t)}\gamma'(t) = k_\gamma \gamma'(t)$ along $\gamma(t)$ and we evaluate it at the origin $(t=0)$. We obtain the equations

$$\begin{aligned} x^2A_1 + y^2B_1 + z^2C_1 + 2xyE_1 + 2xzF_1 + 2yzG_1 &= kx \\ x^2A_2 + y^2B_2 + z^2C_2 + 2xyE_2 + xz(2F_2+1) + 2yzG_2 &= ky \\ x^2A_3 + y^2B_3 + z^2C_3 + 2xyE_3 + 2xzF_3 + 2yzG_3 &= kz. \end{aligned} \tag{20}$$

Again, by the elimination of the factor k from the right-hand sides, we obtain the cubic equations. In this case, we have

$$\begin{aligned} S_1(x,y,z) &= x^2zA_2 - x^2yA_3 + y^2zB_2 - y^3B_3 + z^3C_2 - yz^2C_3 \\ &\quad + 2xyzE_2 - 2xy^2E_3 + xz^2(2F_2+1) - 2xyzF_3 + 2yz^2G_2 - 2y^2zG_3 = 0, \\ S_2(x,y,z) &= -x^2zA_1 + x^3A_3 - y^2zB_1 + xy^2B_3 - z^3C_1 + xz^2C_3 \\ &\quad - 2xyzE_1 + 2x^2yE_3 - 2xz^2F_1 + 2x^2zF_3 - 2yz^2G_1 + 2xyzG_3 = 0, \\ S_3(x,y,z) &= x^2yA_1 - x^3A_2 + y^3B_1 - xy^2B_2 + yz^2C_1 - xz^2C_2 \\ &\quad + 2xy^2E_1 - 2x^2yE_2 + 2xyzF_1 - x^2z(2F_2+1) + 2y^2zG_1 - 2xyzG_2 = 0. \end{aligned} \tag{21}$$

In contrast with the equations (12) and (14) which characterized the existence of geodesic Killing vector field, equations (20) and (21) characterize existence of Killing vector field geodesic along its integral curve through the origin $e \in H_3$. Nevertheless, we can prove in the same way as for $\mathbb{R}^3$ the existence of a solution of the system (21). This solution implies existence of a homogeneous geodesic through the origin $e \in H_3$. Because considered connections are homogeneous, the same property holds at any point $p \in H_3$. Hence, we obtain the following:

Theorem 15 *Let one of the following conditions holds:*

a) $(G_1)^2 - B_1C_1 < 0$,

b) $(F_2 + 1/2)^2 - A_2C_2 < 0$,

c) $(E_3)^2 - A_3B_3 < 0$.

Then the invariant affine connection corresponding to the given parameters $A_1, A_2, \dots, G_3$ admits at least one homogeneous geodesic through any point $p \in H_3$.

4.2 The group $E(1,1)$

The group $E(1,1)$ can be represented by the matrices of the form

$$\begin{pmatrix} e^{-w} & 0 & u \\ 0 & e^{w} & v \\ 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

Hence, $E(1,1)$ can be identified with the 3-space $\mathbb{R}^3(u,v,w)$ equipped with the Riemannian metric $ds^2 = e^{2w}du^2 + e^{-2w}dv^2 + dw^2$. The right-invariant vector fields (the Killing vector fields) are $\partial_u, \partial_v, \partial_w - u\partial_u + v\partial_v$. For the Christoffel symbols $A_1, \dots, G_3$, we obtain by solving the equations (9) the conditions

$$\begin{aligned}
A_1(u,v,w) &= A_1 e^{w},\\
A_2(u,v,w) &= A_2 e^{3w},\\
A_3(u,v,w) &= A_3 e^{2w},\\
B_1(u,v,w) &= B_1 e^{-3w},\\
B_2(u,v,w) &= B_2 e^{-w},\\
B_3(u,v,w) &= B_3 e^{-2w},\\
C_1(u,v,w) &= C_1 e^{-w},\\
C_2(u,v,w) &= C_2 e^{w},\\
C_3(u,v,w) &= C_3,\\
E_1(u,v,w) &= E_1 e^{-w},\\
E_2(u,v,w) &= E_2 e^{w},\\
E_3(u,v,w) &= E_3,\\
F_1(u,v,w) &= F_1,\\
F_2(u,v,w) &= F_2 e^{2w},\\
F_3(u,v,w) &= F_3 e^{w},\\
G_1(u,v,w) &= G_1 e^{-2w},\\
G_2(u,v,w) &= G_2,\\
G_3(u,v,w) &= G_3 e^{-w},
\end{aligned} \tag{23}$$

where $A_1, \dots, G_3$ are constant parameters. Again, we will consider this class of connections and investigate existence of homogeneous geodesics.

Now we shall consider the Killing vector field

$$X = x\partial_u + y\partial_v + z(\partial_w - u\partial_u + v\partial_v),$$

where x, y, z are arbitrary parameters. We see easily that the general trajectories of the vector field $X = x\partial_u + y\partial_v + z(\partial_w - u\partial_u + v\partial_v)$ are given for $z \neq 0$ by

$$u(t) = \frac{x}{z} + c_1 e^{-tz}, \quad v(t) = -\frac{y}{z} + c_2 e^{tz}, \quad w(t) = zt + c_3 \tag{24}$$

and for $z = 0$ by

$$u(t) = xt + c_1, \quad v(t) = yt + c_2, \quad w(t) = c_3. \tag{25}$$

We shall study again the trajectories starting at the origin of $E_{(1,1)}(u,v,w)$. Hence, for $z \neq 0$, we choose the integration constants

$$c_1 = -\frac{x}{z}, \quad c_2 = \frac{y}{z}, \quad c_3 = 0. \tag{26}$$

The tangent vector $\gamma'(t)$ satisfies now

$$u'(t) = xe^{-tz}, \qquad v'(t) = ye^{tz}, \qquad w'(t) = z. \tag{27}$$

We calculate again the equation $\nabla_{\gamma'(t)}\gamma'(t) = k_\gamma \gamma'(t)$ along $\gamma(t)$ and we evaluate it at the origin ($t = 0$). We obtain the equations

$$\begin{aligned} x^2A_1 + y^2B_1 + z^2C_1 + 2xyE_1 + xz(2F_1 - 1) + 2yzG_1 &= kx \\ x^2A_2 + y^2B_2 + z^2C_2 + 2xyE_2 + 2xzF_2 + yz(2G_2 + 1) &= ky \\ x^2A_3 + y^2B_3 + z^2C_3 + 2xyE_3 + 2xzF_3 + 2yzG_3 &= kz. \end{aligned} \tag{28}$$

In an analogous way as for the previous groups, we deduce the following:

Theorem 16 *Let one of the following conditions holds:*

a) $(G_1)^2 - B_1C_1 < 0$,

b) $(F_2)^2 - A_2C_2 < 0$,

c) $(E_3)^2 - A_3B_3 < 0$.

Then the invariant affine connection on $E(1,1)$ corresponding to the given parameters $A_1, A_2, \dots, G_3$ admits at least one homogeneous geodesic through any point $p \in E(1,1)$.

4.3 The product group

Let us consider the group G of matrices of the form

$$\begin{pmatrix} e^w & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, \tag{29}$$

which is a semidirect product of a nonabelian 2-dimensional group and the real line. The right-invariant vector fields (the Killing vector fields) are $\partial_u, \partial_v, \partial_w + u\partial_u$. For the Christoffel symbols $A_1, A_2, \dots, G_3$, we obtain by solving the equations (9) the conditions

$$\begin{aligned} A_1(u,v,w) &= A_1e^{-w}, \\ A_2(u,v,w) &= A_2e^{-2w}, \\ A_3(u,v,w) &= A_3e^{-2w}, \\ B_1(u,v,w) &= B_1e^{w}, \\ B_2(u,v,w) &= B_2, \\ B_3(u,v,w) &= B_3, \\ C_1(u,v,w) &= C_1e^{w}, \\ C_2(u,v,w) &= C_2, \\ C_3(u,v,w) &= C_3, \\ E_1(u,v,w) &= E_1, \\ E_2(u,v,w) &= E_2e^{-w}, \\ E_3(u,v,w) &= E_3e^{-w}, \\ F_1(u,v,w) &= F_1, \\ F_2(u,v,w) &= F_2e^{-w}, \\ F_3(u,v,w) &= F_3e^{-w}, \\ G_1(u,v,w) &= G_1e^{w}, \\ G_2(u,v,w) &= G_2, \\ G_3(u,v,w) &= G_3 \end{aligned} \tag{30}$$

where $A_1, \ldots, G_3$ are constant parameters. We continue with the same procedure as with the group $E(1,1)$. Now, the integral curves of the Killing vector field $X = x\partial_u + y\partial_v + z(u\partial_u + \partial_w)$ and passing through the origin $e \in G$ are for $z \neq 0$

$$u(t) = \frac{x}{z}(-1 + \exp(tz)), \qquad v(t) = yt, \qquad w(t) = zt \tag{31}$$

and for $z = 0$

$$u(t) = xt, \qquad v(t) = yt, \qquad w(t) = 0. \tag{32}$$

The tangent vector $\gamma'(t)$ satisfies now

$$u'(t) = xe^{tz}, \qquad v'(t) = y, \qquad w'(t) = z. \tag{33}$$

From the equation $\nabla_{\gamma'(t)}\gamma'(t) = k_\gamma \gamma'(t)$ at the origin ($t = 0$), we obtain now the equations

$$\begin{aligned} x^2A_1 + y^2B_1 + z^2C_1 + 2xyE_1 + xz(2F_1 + 1) + 2yzG_1 &= kx \\ x^2A_2 + y^2B_2 + z^2C_2 + 2xyE_2 + 2xzF_2 + 2yzG_2 &= ky \\ x^2A_3 + y^2B_3 + z^2C_3 + 2xyE_3 + 2xzF_3 + 2yzG_3 &= kz. \end{aligned} \tag{34}$$

In an analogous way as with previous groups, we obtain the following result:

Theorem 17 *Let one of the following conditions holds:*

a) $(G_1)^2 - B_1C_1 < 0$,

b) $(F_2)^2 - A_2C_2 < 0$,

c) $(E_3)^2 - A_3B_3 < 0$.

Then the invariant affine connection on G corresponding to the given parameters $A_1, A_2, \ldots, G_3$ admits at least one homogeneous geodesic through any point $p \in G$.

5 Conclusions

Let us remark that the question about existence of homogeneous geodesics in homogeneous manifolds with an invariant affine connection was solved affirmatively in general during the preparation of this manuscript and the result is published in the short form in [6]. The examples in the present paper were the crucial step for understandnig the general situation.

References

[1] Alexandroff, P. S., Hopf, H.: Topologie, Band I. *Springer*, 1935.

[2] Alekseevsky, D., Arvanitoyeorgos, A.: *Riemannian flag manifolds with homogeneous geodesics.* Trans. Am. Math. Soc. **359**, 8 (2007), 3769–3789 (electronic).

[3] Arias-Marco, T., Kowalski, O.: *Classification of locally homogeneous affine connections with arbitrary torsion on 2-dimensional manifolds.* Monaths. Math. **153** (2008), 1–18.

[4] Calvaruso, G., Kowalski, O., Marinosci, R.A.: *Homogeneous geodesics in solvable Lie groups*. Acta Math. Hungarica **101**, 4 (2003), 313–322.

[5] Calvaruso, G., Marinosci, R. A.: *Homogeneous geodesics of three-dimensional unimodular Lorentzian Lie groups*. Mediterr. J. Math. **3** (2006), 467–481.

[6] Dušek, Z.: *The existence of homogeneous geodesics in homogeneous affine manifolds*. J. Geom. Phys. **60** (2010), 687–689.

[7] Dušek, Z., Kowalski, O.: *Geodesic graphs on the 13-dimensional group of Heisenberg type*. Math. Nachr. **254-255** (2003), 87–96.

[8] Dušek, Z., Kowalski, O.: *Light-like homogeneous geodesics and the Geodesic Lemma for any signature*. Publ. Math. Debrecen **71**, 1-2 (2007), 245–252.

[9] Dušek, Z., Kowalski, O.: *On six-dimensional pseudo-Riemannian almost g.o. spaces*. Journal of Geometry and Physics **57** (2007), 2014–2023.

[10] Dušek, Z., Kowalski, O., Nikčević, S. Ž.: *New examples of Riemannian g.o. manifolds in dimension 7*. Differential Geom. Appl. **21** (2004), 65–78.

[11] Dušek, Z., Kowalski, O., Vlášek, Z.: *Homogeneous geodesics in homogeneous affine manifolds*. Result. Math. **54** (2009), 273–288.

[12] Figueroa-O'Farrill, J., Meessen, P., Philip, S.: *Homogeneity and plane-wave limits*. J. High Energy Physics **05** (2005), 050–050.

[13] Kobayashi, S., Nomizu, N.: Foundations of differential geometry I, II. *Wiley Classics Library*, 1996.

[14] Kowalski, O., Nikčević, S.Ž.: *On geodesic graphs of Riemannian g.o. spaces*. Archiv der Math. **73** (1999), 223–234; Appendix: Archiv der Math. **79** (2002), 158–160.

[15] Kowalski, O., Nikčević, S.Ž., Vlášek, Z.: *Homogeneous geodesics in homogeneous Riemannian manifolds – Examples*. Geometry and Topology of Submanifolds, *World Sci. Publishing co.*, River Edge, NJ (2000), 104–112.

[16] Kowalski, O., Opozda, B., Vlášek, Z.: *On locally nonhomogeneous pseudo-Riemannian manifolds with locally homogeneous Levi-Civita connections*. International J.Math. **14**, 6 (2003), 1–14.

[17] Kowalski, O., Opozda, B., Vlášek, Z.: *A classification of locally homogenous connections on 2-dimensional manifolds via group-theoretical approach*. Central European J. Math. **2**, 1 (2004), 87–102.

[18] Kowalski, O., Szenthe, J.: *On the existence of homogeneous geodesics in homogeneous Riemannian manifolds*. Geom. Dedicata **81** (2000), 209–214; Erratum: Geom. Dedicata **84** (2001), 331–332.

[19] Kowalski, O., Vanhecke, L.: *Riemannian manifolds with homogeneous geodesics*. Boll. Un. Math. Ital. **5**, B(7) (1991), 189–246.

[20] Kowalski, O., Vlášek, Z.: *Homogeneous Riemannian manifolds with only one homogeneous geodesic*. Publ. Math. Debrecen **62**, 3-4 (2003), 437–446.

[21] Kowalski, O., Vlášek, Z.: *On the moduli space of locally homogeneous affine connections in plane domains*. Comment. Math. Univ. Carolinae **44**, 2 (2003), 229–234.

[22] Marinosci, R.A.: *Homogeneous geodesics in a three-dimensional Lie group*. Comm. Math. Univ. Carolinae **43**, 2 (2002), 261–270.

[23] Opozda, B.: *A classification of locally homogeneous connections on 2-dimensional manifolds*. Differential Geometry and its Applications **21**, 2 (2004), 173–198.

[24] Philip, S.: *Penrose limits of homogeneous spaces*. J. Geom. Phys. **56** (2006), 1516–1533.