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Existence Principles for Singular Vector Nonlocal Boundary Value Problems with ϕ-Laplacian and their Applications

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Abstract

Existence principles for solutions of singular differential systems $(\phi(u'))' = f(t, u, u')$ satisfying nonlocal boundary conditions are stated. Here ϕ is a homeomorphism $\mathbb{R}^N$ onto $\mathbb{R}^N$ and the Carathéodory function f may have singularities in its space variables. Applications of the existence principles are given.

Key words: singular boundary value problem, system of differential equations, nonlocal boundary condition, existence principle, positive solution, ϕ-Laplacian, Leray–Schauder degree

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1 Introduction

Nonlocal singular boundary value problems with ϕ-Laplacian are discussed in the literature above all for scalar second-order differential equations (see, e.g., [1, 2, 3, 6, 23, 24, 27, 28] and references therein). Existence principles for solving such problems were given in [3, 22, 23, 24].

Regular systems of differential equations with ϕ-Laplacian were investigated together with two-point boundary conditions (Dirichlet, Neumann, periodic and nonlinear) in many papers (see, e.g., [4, 8, 10, 11, 13, 15, 16, 17, 18, 19, 32] and references therein) and in [20] with nonlocal boundary conditions $u(0) = 0$,

$$|u'(T)|_*^{p-2}u'(T) = \int_{t_0}^{T} |u'(s)|_*^{p-2}u'(s)\,\mathrm{d}g(s),$$

where $|u|_*$ is the Euclidean norm in $\mathbb{R}^N$.

The ϕ-Laplacian like operator is discussed for p-Laplacian $\phi_p(u) = |u|_*^{p-2}u$ ([16, 17, 18, 19, 20, 32]), for $p(t)$-Laplacian $\phi_{p(t)}(u) = |u|_*^{p(t)-2}u$ ([10, 11]), for $(p_1,\dots,p_N)$-Laplacian $\phi_{(p_1,\dots,p_N)}(u) = (\phi_{p_1}(u_1),\dots,\phi_{p_N}(u_N))$ ([8, 13, 15, 32]), where ϕ_{p_i} is a one-dimensional p_i-Laplacian, and for a strictly monotone homeomorphism $\phi\colon \mathbb{R}^N \to \mathbb{R}^N$ ([4, 17]).

Singular problems for second-order differential systems have received less attention. We refer to [7, 14, 21, 33], where the solvability of singular periodic, Dirichlet and mixed boundary value problems was studied for systems of two second-order differential equations.

The aim of this paper is to give existence principles for solving regular and singular nonlocal boundary value problems for systems of differential equations with ϕ-Laplacian and show their applications. The existence principles are proved by the combination of the Leray–Schauder degree theory ([9]) with regularization and sequential techniques. We note that our existence principles are closely related to that given in [29] for n-order differential equations and in [3, 22, 23, 24] for second-order differential equations.

The following notation is used throughout the paper. $N \in \mathbb{N}$, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$, $T \in \mathbb{R}_+$.

$\mathbb{R}^N$ is the space of N-dimensional vectors $x = (x_1,\dots,x_N)$ with $x_j \in \mathbb{R}$ and the norm $\|x\| = \max\{|x_j|\colon 1 \le j \le N\}$.

If $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$, then

$$y > x \iff y - x \in \mathbb{R}_+^N, \qquad y \ge x \iff y - x \in [0,\infty)^N$$

and

$$\lambda x = (\lambda x_1,\dots,\lambda x_N), \quad |x| = (|x_1|,\dots,|x_N|).$$

$C([0,T];\mathbb{R}^N)$ is the space of continuous vector-functions $x\colon [0,T] \to \mathbb{R}^N$ with the norm $\|x\|_\infty = \max\{\|x(t)\|\colon t \in [0,T]\}$, and if $N = 1$, then $\|x\|_\infty = \max\{|x(t)|\colon t \in [0,T]\}$.

$L^1([0,T];\mathbb{R}^N)$ is the space of Lebesgue integrable vector-functions $x\colon [0,T] \to \mathbb{R}^N$ with the norm $\|x\|_L = \int_0^T \|x(t)\|\,\mathrm{d}t$, and if $N = 1$, then $\|x\|_L = \int_0^T |x(t)|\,\mathrm{d}t$.

$AC([0,T];\mathbb{R}^N)$ is the space of vector-functions $x\colon [0,T] \to \mathbb{R}^N$ which are absolutely continuous on $[0,T]$.

$C^1([0,T];\mathbb{R}^N)$ and $AC^1([0,T];\mathbb{R}^N)$ is the space of vector-functions $x\colon [0,T] \to \mathbb{R}^N$ having continuous derivatives on $[0,T]$ and absolutely continuous derivatives on $[0,T]$, respectively.

We note that a vector-function is said to be continuous, integrable, etc., if such are all its elements. Similarly, sup, max for vectors functions is understood componentwise.

In order to precede a confusion, if we work with sequences of N-dimensional vector-functions, then we denote such sequences by $\{x_{(n)}\}$, $\{f_{(n)}\}\dots$, where $x_{(n)} = (x_{n1},\dots,x_{nN})$, $f_{(n)} = (f_{n1},\dots,f_{nN})$.

We recall that a function $h\colon [0,T]\times\mathcal{D}\to\mathbb{R}^N$, $\mathcal{D}\subset\mathbb{R}^N\times\mathbb{R}^N$, satisfies *the local Carathéodory conditions on the set* $[0,T]\times\mathcal{D}$ if

(i) $h(\cdot,x,y)\colon [0,T]\to\mathbb{R}^N$ is measurable for each $(x,y)\in\mathcal{D}$,

(ii) $h(t,\cdot,\cdot)\colon \mathcal{D}\to\mathbb{R}^N$ is continuous for a.e. $t\in[0,T]$,

(iii) for each compact set $\mathcal{M}\subset\mathcal{D}$ there is a $\rho_{\mathcal{M}}\in L^1([0,T];\mathbb{R}_+)$ such that

$$\|h(t,x,y)\|\le\rho_{\mathcal{M}}(t)\quad\text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in\mathcal{M}.$$

In this case we write $h\in\mathrm{Car}([0,T]\times\mathcal{D};\mathbb{R}^N)$.

Let $\mathcal{D}\subset\mathbb{R}^N\times\mathbb{R}^N$ is not closed and let $(x_0,y_0)\in\overline{\mathcal{D}}$. We say that $h\colon [0,T]\times\mathcal{D}\to\mathbb{R}^N$ has *a singularity at* (x_0,y_0) if

$$\lim_{\mathcal{D}\ni(x,y)\to(x_0,y_0)}\|h(t,x,y)\|=\infty\quad\text{for a.e. } t\in[0,T]. \tag{1}$$

Here $\overline{\mathcal{D}}$ denotes the closure of $\mathcal{D}$. In this case we also say that h has a singularity at the point (x_0,y_0) of its space variables x,y. If (1) holds only for such $(x_0,y_0)\in\overline{\mathcal{D}}$, $x_0=(x_{01},\dots,x_{0N})$, $y_0=(y_{01},\dots,y_{0N})$ that $\min\{|x_{0j}|,|y_{0j}|\colon 1\le j\le N\}=0$, then we say that h has a singularity only at zero value of components its space variables.

In order to give nonlocal boundary conditions, we introduce a set $\mathcal{A}$. $\mathcal{A}$ is the set of operators $\alpha\colon C^1([0,T];\mathbb{R}^N)\to\mathbb{R}^N$ which are

(j) continuous,

(jj) bounded, that is, $\alpha(\Omega)$ is bounded for any bounded $\Omega\subset C^1([0,T];\mathbb{R}^N)$.

Example 1.1 Let $a\in\mathbb{R}^N$, $p\in\mathbb{R}_+$, $0\le t_1<\dots<t_m\le T$ and let $\xi_j\in[0,T]$ $(j=1,\dots,N)$. Then the operators

$$\alpha_1(x)=(\max\{x_1(t)\colon t\in[0,T]\},\dots,\max\{x_N(t)\colon t\in[0,T]\})-a,$$
$$\alpha_2(x)=(\min\{x_1(t)\colon t\in[0,T]\},\dots,\min\{x_N(t)\colon t\in[0,T]\})-a,$$
$$\alpha_3(x)=\sum_{j=1}^m a_jx(t_j),\quad \alpha_4(x)=\int_0^T|x'(t)|^p\mathrm{d}t,\quad \alpha_5(x)=(x_1'(\xi_1),\dots,x_N'(\xi_N))$$

belong to the set $\mathcal{A}$.

Let $f\in\mathrm{Car}([0,T]\times\mathcal{D};\mathbb{R}^N)$, where $\mathcal{D}\subset\mathbb{R}^N\times\mathbb{R}^N$ is not necessary closed, and let $f(t,x,y)$ admit singularities in its space variables x,y. We consider the system of differential equations

$$(\phi(u'))'=f(t,u,u'), \tag{2}$$

where ϕ satisfies the condition

(H_1) $\phi\in C(\mathbb{R}^N;\mathbb{R}^N)$ is injective and such that $\lim_{\|x\|\to\infty}\|\phi(x)\|=\infty$.

It is known that under condition (H_1) the function ϕ is a homeomorphism $\mathbb{R}^N$ onto $\mathbb{R}^N$ and that $\lim_{\|x\|\to\infty}\|\phi^{-1}(x)\|=\infty$ (see [9, 30, 31]).

Together with system (2) we investigate the (generally nonlinear and nonlocal) boundary conditions

$$\alpha(u) = 0, \quad \beta(u) = 0, \quad \alpha, \beta \in \mathcal{A}. \tag{3}$$

We say that $\alpha, \beta \in \mathcal{A}$ satisfy *the compatibility condition* if for each $\mu \in [0,1]$ the system

$$\left.\begin{aligned} \alpha(a+tb) - \mu\alpha(-a-tb) = 0, \\ \beta(a+tb) - \mu\beta(-a-tb) = 0, \end{aligned}\right\} \tag{4}$$

has a solution $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$.

A function $u \in C^1([0,T];\mathbb{R}^N)$ is called *a solution of problem* (2), (3) if $\phi(u') \in AC([0,T];\mathbb{R}^N)$, u satisfies the boundary conditions (3) and $(\phi(u'(t)))' = f(t,u(t),u'(t))$ holds almost everywhere on $[0,T]$.

The paper is organized as follows. Section 2 contains two existence principles. The first principle concerns with the solvability of regular functional-differential problems $(\phi(u'(t)))' = (Fu)(t)$, (3), and the second one deals with the solvability of singular problem (2), (3). Applications of both the principles are given in Section 3. Here we discuss system (2) together with nonlocal boundary conditions, which include as the special cases the Dirichlet conditions and multipoint boundary conditions.

2 Existence principles

We first investigate the regular functional-differential system

$$(\phi(u'(t)))' = (Fu)(t), \tag{5}$$

where the function ϕ satisfies condition (H_1) and the operator F satisfies the condition

$$(H_2) \quad \left\{\begin{aligned} & F\colon C^1([0,T];\mathbb{R}^N) \to L^1([0,T];\mathbb{R}^N) \text{ is continuous, and for any } r > 0, \\ & \sup\left\{\|(Fx)(t)\| \colon x \in C^1([0,T];\mathbb{R}^N), \|x\|_\infty \le r, \|x'\|_\infty \le r\right\} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in L^1([0,T];\mathbb{R}). \end{aligned}\right.$$

Remark 2.1 If $f \in \mathrm{Car}([0,T] \times (\mathbb{R}^N \times \mathbb{R}^N);\mathbb{R}^N)$ and $(Fx)(t) = f(t,x(t),x'(t))$ for $x \in C^1([0,T];\mathbb{R}^N)$, then F fulfils condition (H_2).

By a *solution of problem* (5), (3) we mean a function $u \in C^1([0,T];\mathbb{R}^N)$ such that $\phi(u') \in AC([0,T];\mathbb{R}^N)$, u satisfies the boundary conditions (3) and (5) holds for almost all $t \in [0,T]$.

The existence result for problem (5), (3) is given in the following theorem.

Theorem 2.2 (existence principle for regular problems). *Let* (H_1) *and* (H_2) *hold. Let* $\varphi \in L^1([0,T];\mathbb{R}^N)$. *Suppose that there exist positive constants* S_0 *and* S_1 *such that*

$$\|u\|_\infty < S_0, \qquad \|u'\|_\infty < S_1 \tag{6}$$

for all solutions u of the equations

$$\left.\begin{aligned}(\phi(u'(t)))' = \lambda\varphi(t), \quad \lambda \in [0,1],\\ (\phi(u'(t)))' = \lambda(Fu)(t) + (1-\lambda)\varphi(t), \quad \lambda \in [0,1],\end{aligned}\right\} \tag{7}$$

satisfying the boundary conditions (3). *Also assume that α, β in* (3) *satisfy the compatibility condition and there exist positive constants Λ_0 and Λ_1 such that the estimates*

$$\|a\| < \Lambda_0, \qquad \|b\| < \Lambda_1,$$

are fulfilled for each $\mu \in [0,1]$ and all solutions $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$ of system (4).

Then problem (5), (3) *has a solution u satisfying inequality* (6).

Proof Let

$$\Omega = \big\{x \in C^1([0,T];\mathbb{R}^N)\colon \|x\|_\infty < \max\{S_0, \Lambda_0 + \Lambda_1 T\},\ \|x'\|_\infty < \max\{S_1, \Lambda_1\}\big\}.$$

Then Ω is an open, bounded and symmetric with respect to $0 \in C^1([0,T];\mathbb{R}^N)$ subset of the Banach space $C^1([0,T];\mathbb{R}^N)$. Define operators

$$\mathbb{Q}\colon [0,1] \times \overline{\Omega} \to L^1([0,T];\mathbb{R}^N), \qquad \mathcal{F}\colon [0,1] \times \overline{\Omega} \to C^1([0,T];\mathbb{R}^N)$$

by the formulas

$$\mathbb{Q}(\lambda, x)(t) = \lambda(Fx)(t) + (1-\lambda)\varphi(t),$$

$$\mathcal{F}(\lambda, x)(t) = x(0) + \alpha(x) + \int_0^t \phi^{-1}\Big(\phi(x'(0) + \beta(x)) + \int_0^s \mathbb{Q}(\lambda, x)(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s.$$

The fact that $\mathcal{F}$ is a continuous operator follows from condition (H_2), $\varphi \in L^1([0,T];\mathbb{R}^N)$, the continuity of ϕ, α, β, and from the Lebesgue dominated convergence theorem. In order to prove that $\mathcal{F}$ is a compact operator it remains to verify that $\mathcal{F}([0,1] \times \overline{\Omega})$ is relatively compact in $C^1([0,T];\mathbb{R}^N)$. Since $\overline{\Omega}$ is bounded in $C^1([0,T];\mathbb{R}^N)$ we have

$$\|\alpha(x)\| \le r, \quad \|\beta(x)\| \le r, \quad \|(Fx)(t)\| \le \varrho(t)$$

for a.e. $t \in [0,T]$ and all $x \in \overline{\Omega}$, where r is a positive constant and $\varrho \in L^1([0,T];\mathbb{R}_+)$. Set $K = \max\{S_1, \Lambda_1\} + r$. Then, for $x \in \overline{\Omega}$, the inequalities $\|x'(0) + \beta(x)\| \le K$ and

$$\|\phi(x'(0) + \beta(x))\| \le \sup\{\|\phi(y)\|\colon \|y\| \le K\} =: M$$

hold. Hence

$$\begin{aligned}&\Big\|\phi^{-1}\Big(\phi(x'(0) + \beta(x)) + \lambda\int_0^t \mathbb{Q}(\lambda, x)(s)\,\mathrm{d}s\Big)\Big\|\\ &\qquad \le \sup\{\|\phi^{-1}(y)\|\colon \|y\| \le M + \|\varrho\|_L + \|\varphi\|_L\} =: V,\end{aligned}$$

and consequently,

$$\|\mathcal{F}(\lambda,x)(t)\| \le \max\{S_0, \Lambda_0 + \Lambda_1 T\} + r + VT, \quad \|\mathcal{F}(\lambda,x)'(t)\| \le V,$$
$$\|\phi(\mathcal{F}(\lambda,x)'(t_2)) - \phi(\mathcal{F}(\lambda,x)'(t_1))\| \le \Big|\int_{t_1}^{t_2} (\varrho(t) + \|\varphi(t)\|)\,\mathrm{d}t\Big|$$

for $t, t_1, t_2 \in [0,T]$ and $(\lambda, x) \in [0,1]\times\overline{\Omega}$. Here $\mathcal{F}(\lambda,x)'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\lambda,x)(t)$. Therefore $\mathcal{F}([0,1]\times\overline{\Omega})$ is bounded in $C^1([0,T];\mathbb{R}^N)$ and the set

$$\big\{\phi(\mathcal{F}(\lambda,x)')\colon (\lambda,x)\in[0,1]\times\overline{\Omega}\big\}$$

is equicontinuous on $[0,T]$. Using the fact that ϕ^{-1} is uniformly continuous on compact subsets of $\mathbb{R}^N$ and

$$\|\mathcal{F}(\lambda,x)'(t_2) - \mathcal{F}(\lambda,x)'(t_1)\| = \big\|\phi^{-1}\big(\phi(\mathcal{F}(\lambda,x)'(t_2))\big) - \phi^{-1}\big(\phi(\mathcal{F}(\lambda,x)'(t_1))\big)\big\|,$$

we conclude that the set $\{\mathcal{F}(\lambda,x)'\colon (\lambda,x)\in[0,1]\times\overline{\Omega}\}$ is equicontinuous on $[0,T]$. Now the Arzelà–Ascoli theorem gives that $\mathcal{F}([0,1]\times\overline{\Omega})$ is relatively compact in $C^1([0,T];\mathbb{R}^N)$. Hence $\mathcal{F}$ is a compact operator.

Suppose that x_0 is a fixed point of the operator $\mathcal{F}(1,\cdot)$. Then

$$x_0(t) = x_0(0) + \alpha(x_0) + \int_0^t \phi^{-1}\Big(\phi(x_0'(0) + \beta(x_0)) + \int_0^s (Fx_0)(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s.$$

Hence $\alpha(x_0) = 0$, $\beta(x_0) = 0$ and x_0 is a solution of (5). Therefore, x_0 is a solution of problem (5), (3). In order to prove our theorem it suffices to show that

$$\deg(\mathcal{I} - \mathcal{F}(1,\cdot), \Omega, 0) \ne 0, \tag{8}$$

where "deg" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identity operator on $C^1([0,T];\mathbb{R}^N)$. For this end, let us define a compact operator $\mathcal{K}\colon [0,2]\times\overline{\Omega}\to C^1([0,T];\mathbb{R}^N)$ as

$$\mathcal{K}(\mu,x)(t) = \begin{cases} x(0) + \alpha(x) - (1-\mu)\alpha(-x) + t[x'(0) + \beta(x) - (1-\mu)\beta(-x)] \\ \hfill \text{if } \mu\in[0,1], \\ x(0) + \alpha(x) + \displaystyle\int_0^t \phi^{-1}\Big(\phi(x'(0) + \beta(x)) + (\mu-1)\int_0^s \varphi(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ \hfill \text{if } \mu\in(1,2]. \end{cases}$$

Then $\mathcal{K}(0,\cdot)$ is odd (that is, $\mathcal{K}(0,-x) = -\mathcal{K}(0,x)$ for $x\in\overline{\Omega}$) and

$$\mathcal{K}(2,\cdot) = \mathcal{F}(0,\cdot). \tag{9}$$

If $\mathcal{K}(\mu_1,\hat{x}) = \hat{x}$ for some $(\mu_1,\hat{x})\in[0,1]\times\overline{\Omega}$, then

$$\hat{x}(t) = \hat{x}(0) + \alpha(\hat{x}) - (1-\mu_1)\alpha(-\hat{x}) + t\big[\hat{x}'(0) + \beta(\hat{x}) - (1-\mu_1)\beta(-\hat{x})\big]$$

for $t \in [0,T]$. Hence $\hat{x}(t) = \hat{a} + t\hat{b}$, where $\hat{a} = \hat{x}(0) + \alpha(\hat{x}) - (1-\mu_1)\alpha(-\hat{x})$ and $\hat{b} = \hat{x}'(0) + \beta(\hat{x}) - (1-\mu_1)\beta(-\hat{x})$. Since $\hat{x}(0) = \hat{a}$ and $\hat{x}'(0) = \hat{b}$, we have $\alpha(\hat{x}) - (1-\mu_1)\alpha(-\hat{x}) = 0$, $\beta(\hat{x}) - (1-\mu_1)\beta(-\hat{x}) = 0$, which means that

$$\begin{aligned}\alpha(\hat{a} + t\hat{b}) - (1-\mu_1)\alpha(-\hat{a} - t\hat{b}) = 0,\\ \beta(\hat{a} + t\hat{b}) - (1-\mu_1)\beta(-\hat{a} - t\hat{b}) = 0.\end{aligned}$$

Therefore, $(\hat{a}, \hat{b}) \in \mathbb{R}^N \times \mathbb{R}^N$ is a solution of system (4) (with μ replaced by $1-\mu_1$), and consequently, $\|\hat{a}\| < \Lambda_0$, $\|\hat{b}\| < \Lambda_1$ by the assumption. Then $\|\hat{x}\|_\infty < \Lambda_0 + \Lambda_1 T$ and $\|\hat{x}'\|_\infty < \Lambda_1$, which gives $\hat{x} \notin \partial\Omega$. If $\mathcal{K}(\mu_2, \tilde{x}) = \tilde{x}$ for some $(\mu_2, \tilde{x}) \in (1,2] \times \overline{\Omega}$, then

$$\tilde{x}(t) = \tilde{x}(0) + \alpha(\tilde{x}) + \int_0^t \phi^{-1}\Big(\phi(\tilde{x}'(0) + \beta(\tilde{x})) + (\mu_2 - 1)\int_0^s \varphi(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s$$

for $t \in [0,T]$. Hence $\tilde{x}$ fulfils the boundary conditions (3) and $\tilde{x}$ is a solution of the equation $(\phi(u'))' = (\mu_2 - 1)\varphi(t)$. So $\|\tilde{x}\|_\infty < S_0$, $\|\tilde{x}'\|_\infty < S_1$, which implies $\tilde{x} \notin \partial\Omega$. We have shown that $\mathcal{K}(\mu, x) \neq x$ for all $(\mu, x) \in [0,2] \times \overline{\Omega}$. Now, by the Borsuk antipodal theorem and the homotopy property (see, e.g., [9]),

$$\left.\begin{aligned}\deg(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) \neq 0,\\ \deg(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{K}(2,\cdot), \Omega, 0).\end{aligned}\right\} \tag{10}$$

Finally, assume that $\mathcal{F}(\lambda, \overline{x}) = \overline{x}$ for some $(\lambda, \overline{x}) \in [0,1] \times \overline{\Omega}$. Then $\overline{x}$ fulfils (3) and $\overline{x}$ is a solution of the second equation in (7). By assumption (6), $\|\overline{x}\|_\infty < S_0$ and $\|\overline{x}'\|_\infty < S_1$. Hence $\overline{x} \notin \partial\Omega$, and the homotopy property gives

$$\deg(\mathcal{I} - \mathcal{F}(0,\cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{F}(1,\cdot), \Omega, 0).$$

This equality together with (9) and (10) show that (8) holds. We have proved that problem (5), (3) has a solution u. The fact that u fulfils (6) follows from our a priori estimate (6) for solutions of problem (7), (3). $\square$

In order to obtain an existence result for singular problem (2), (3), we use regularization and sequential techniques. For this end we discuss the regular differential system associated to (2)

$$(\phi(u'(t)))' = f_{(n)}(t, u(t), u'(t)), \tag{11}$$

where $f_{(n)} \in \mathrm{Car}([0,T] \times (\mathbb{R}^N \times \mathbb{R}^N); \mathbb{R}^N)$ and $n \in \mathbb{N}$. The following result is an existence principle for problem (2), (3).

Theorem 2.3 (existence principle for singular problems). *Let* (H_1) *hold. Let* $f \in \mathrm{Car}([0,T] \times \mathcal{D}; \mathbb{R}^N)$, *where* $\mathcal{D} \subset \mathbb{R}^N \times \mathbb{R}^N$, *and* $f(t,x,y)$ *have singularities only at zero value of components its space variables* x, y. *Let* $\nu \in \{-1, 1\}$, $f_{(n)} \in \mathrm{Car}([0,T] \times (\mathbb{R}^N \times \mathbb{R}^N); \mathbb{R}^N)$ *and* $f_{(n)}$ *satisfy the inequality*

$$\left.\begin{aligned}0 \le \nu f_{(n)}(t,x,y) \le Q(t, |x|, |y|)\\ \textit{for a.e. } t \in [0,T] \textit{ and all } x, y \in \mathbb{R}_0^N \times \mathbb{R}_0^N,\ n \in \mathbb{N},\end{aligned}\right\} \tag{12}$$

where $Q \in \mathrm{Car}([0,T] \times (\mathbb{R}_+^N \times \mathbb{R}_+^N); \mathbb{R}_+^N)$. *Suppose that the regular problem* (11), (3) *has a solution* $u_{(n)}$ *for each* $n \in \mathbb{N}$, *and there exists a subsequence* $\{u_{(k_n)}\}$ *of* $\{u_{(n)}\}$ *converging in* $C^1([0,T]; \mathbb{R}^N)$ *to some* $u = (u_1, \dots, u_N)$.

Then u *is a solution of the singular problem* (2), (3) *if* u_j *and* u'_j *have a finite number of zeros,* $j = 1, \dots, N$, *and*

$$\lim_{n\to\infty} f_{(k_n)}(t, u_{(k_n)}(t), u'_{(k_n)}(t)) = f(t, u(t), u'(t)) \quad \textit{for a.e. } t \in [0,T]. \tag{13}$$

Proof Assume that (13) holds and that $0 \le \xi_1 < \dots < \xi_m \le T$ are all zeros of u_j and u'_j, $j = 1, \dots, N$. Since $\{u_{(k_n)}\}$ is convergent in $C^1([0,T]; \mathbb{R}^N)$ there exists $L > 0$ such that $\|u_{(k_n)}\|_\infty \le L$, $\|u'_{(k_n)}\|_\infty \le L$ for $n \in \mathbb{N}$. It follows from the equality

$$\phi(u'_{(k_n)}(T)) - \phi(u'_{(k_n)}(0)) = \int_0^T f_{(k_n)}(t, u_{(k_n)}(t), u'_{(k_n)}(t))\,\mathrm{d}t$$

that, cf. (12),

$$\nu \int_0^T f_{(k_n)}(t, u_{(k_n)}(t), u'_{(k_n)}(t))\,\mathrm{d}t \le S \quad \text{for } n \in \mathbb{N}, \tag{14}$$

where $S = 2\sup\{|\phi(z)|\colon \|z\| \le L\} \in \mathbb{R}_+^N$. Now, by (12)–(14) and the Fatou lemma (see, e.g., [5, 12]), $f(t, u(t), u'(t)) \in L^1([0,T]; \mathbb{R}^N)$.

Set $\xi_0 = 0$ and $\xi_{m+1} = T$. We claim that for all $j \in \{0, 1, \dots, m\}$ such that $\xi_j < \xi_{j+1}$, the equality

$$\phi(u'(t)) = \phi(u'(\tau_j)) + \int_{\tau_j}^t f(s, u(s), u'(s))\,\mathrm{d}s \tag{15}$$

is satisfied for $t \in [\xi_j, \xi_{j+1}]$, where $\tau_j = \frac{\xi_j + \xi_{j+1}}{2}$. Indeed, let $j \in \{0, 1, \dots, m\}$ and $\xi_j < \xi_{j+1}$. Let us take an arbitrary $\delta \in (0, \frac{\xi_{j+1} - \xi_j}{2})$ and look at the interval $[\xi_j + \delta, \xi_{j+1} - \delta]$. We know that $|u| > 0$ and $|u'| > 0$ on (ξ_j, ξ_{j+1}), and therefore, $|u(t)| \ge \rho$, $|u'(t)| \ge \rho$ for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$, where $\rho \in \mathbb{R}_+^N$. Hence there exists $n_0 \in \mathbb{N}$ such that $|u_{(k_n)}(t)| \ge \frac{\rho}{2}$, $|u'_{(k_n)}(t)| \ge \frac{\rho}{2}$ for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ and $n \ge n_0$. Taking into account (12) we have

$$\nu f_{(k_n)}(t, u_{(k_n)}(t), u'_{(k_n)}(t)) \le \psi(t) \ \text{ for a.e. } t \in [\xi_j + \delta, \xi_{j+1} - \delta] \text{ and all } n \ge n_0,$$

where $\psi \in L^1([\xi_j + \delta, \xi_{j+1} - \delta]; \mathbb{R}_+^N)$,

$$\psi(t) = \sup\left\{Q(t, x, y)\colon \frac{\rho}{2} \le x \le L_*,\ \frac{\rho}{2} \le y \le L_*\right\}$$

with $L_* = (L, \dots, L) \in \mathbb{R}_+^N$. Letting $n \to \infty$ in

$$\phi(u'_{(k_n)}(t)) = \phi(u'_{(k_n)}(\tau_j)) + \int_{\tau_j}^t f_{(k_n)}(s, u_{(k_n)}(s), u'_{(k_n)}(s))\,\mathrm{d}s$$

yields (15) for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ by the Lebesgue dominated convergence theorem. Since $\delta \in (0, \frac{\xi_{j+1}-\xi_j}{2})$ is arbitrary, (15) holds on the interval (ξ_j, ξ_{j+1}). Now, using the fact that $f(t, u(t), u'(t)) \in L^1([0,T];\mathbb{R}^N)$, we conclude that (15) is true also at $t = \xi_j$ and $t = \xi_{j+1}$. Consequently, $\phi(u') \in AC([0,T];\mathbb{R}^N)$ and $(\phi(u'(t)))' = f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$. Finally, $\alpha(u) = 0$ and $\beta(u) = 0$, which follows from the continuity of α and β, from $\lim_{n\to\infty} u_{(k_n)} = u$ in $C^1([0,T];\mathbb{R}^N)$ and from $\alpha(u_{(k_n)}) = 0$, $\beta(u_{(k_n)}) = 0$ for $n \in \mathbb{N}$. Hence u is a solution of problem (2), (3). $\square$

3 An application of existence principles

In this section we investigate a nonlocal singular problem. This problem is solved by regularization and sequential techniques. We construct a sequence of auxiliary regular problems and prove their solvability by Theorem 2.2. The existence of a solution to the nonlocal singular problem is proved by Theorem 2.3.

3.1 Formulation of nonlocal singular problems

We first define a set $\mathcal{B}$ of operators γ which is used in the formulation of our nonlocal boundary conditions. We say that $\gamma \in \mathcal{B}$ if $\gamma\colon C^1([0,T];\mathbb{R}^N) \to \mathbb{R}^N$ and if there exists a continuous, even functional $\tau_j\colon C^1([0,T];\mathbb{R}^N) \to [0,T)$, $j = 1,\dots,N$, such that

$$\gamma(x) = (x_1(\tau_1(x)),\dots,x_N(\tau_N(x))) \quad \text{for } x \in C^1([0,T];\mathbb{R}^N).$$

It is easy to check that $\mathcal{B} \subset \mathcal{A}$. We note that the set $\mathcal{A}$ is introduced in Section 1.

Example 3.1 Let $t_j \in [0,T)$, $j = 1,\dots,N$. Then the operators

$$\begin{aligned}
\gamma_1(x) &= x(0),\\
\gamma_2(x) &= (x_1(t_1),\dots,x_N(t_N)),\\
\gamma_3(x) &= \left(x_1\left(\frac{\|x'(t_1)\|}{1+\|x'\|_\infty}T\right),\dots,x_N\left(\frac{\|x'(t_N)\|}{1+\|x'\|_\infty}T\right)\right),\\
\gamma_4(x) &= \left(x_1\left(\frac{\|x\|_L}{1+\|x\|_\infty}\right),\dots,x_N\left(\frac{\|x\|_L}{1+\|x\|_\infty}\right)\right)
\end{aligned}$$

belong to the set $\mathcal{B}$.

Consider system (2) together with the boundary conditions

$$u(0) = 0, \quad u(T) = \gamma(u), \quad \gamma \in \mathcal{B}. \tag{16}$$

We see that (16) is the special case of (3) with $\alpha(x) = x(0)$ and $\beta(x) = x(T) - \gamma(x)$. The boundary conditions (16) include as special cases the Dirichlet conditions (if $\gamma(x) = x(0)$) and multipoint boundary conditions (if, e.g., $\gamma(x) = (x_1(t_1),\dots,x_N(t_N))$).

Throughout this section we work with the following conditions on the functions ϕ and $f = (f_1, \dots, f_N)$ in (2).

(S_1) $\phi\colon \mathbb{R}^N \to \mathbb{R}^N$, $\phi(x) = (\phi_1(x_1), \dots, \phi_N(x_N))$, where $\phi_j\colon \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi_j(\mathbb{R}) = \mathbb{R}$, $j = 1, \dots, N$.

(S_2) $f \in \mathrm{Car}([0,T] \times \mathcal{D}; \mathbb{R}^N)$, $\mathcal{D} = \mathbb{R}_+^N \times \mathbb{R}_0^N$, and there exists a positive constant c such that

$$f_j(t,x,y) \le -c \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathcal{D},\ j = 1, \dots, N.$$

(S_3) f fulfils the estimate

$$-f_j(t,x,y) \le h_j(t, \|x\|, \|y\|) + \sum_{k=1}^{N} (a_{jk}(x_k) + b_{jk}(|y_k|))$$

for a.e. $t \in [0,T]$ and all $(x,y) \in \mathcal{D}$, $j = 1, \dots, N$, where

$$h_j \in \mathrm{Car}([0,T] \times ([0,\infty) \times [0,\infty)); \mathbb{R}_+)$$

is nondecreasing with respect to the second and third variable, $a_{jk}, b_{jk}\colon \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing and such that

$$\int_0^d a_{jk}(s)\,\mathrm{d}s < \infty, \qquad \int_0^d b_{jk}(\phi_k^{-1}(s))\,\mathrm{d}s < \infty$$

for each $d \in \mathbb{R}_+$, $j,k = 1, \dots, N$, and

$$\limsup_{v \to \infty} \frac{1}{\phi_j(v)} \int_0^T h_j(t, 1 + Tv, 1 + v)\,\mathrm{d}t < 1 \quad \text{for } j = 1, \dots, N. \tag{17}$$

We say that a function $u \in C^1([0,T]; \mathbb{R}^N)$ is a *solution of problem* (2), (16) if $u(t) > 0$ for $t \in (0,T)$, $\phi(u') \in AC([0,T]; \mathbb{R}^N)$, u fulfils (16) and

$$(\phi(u'(t)))' = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [0,T].$$

3.2 Auxiliary regular problems

For $n \in \mathbb{N}$, define $\varphi_n \in C^0(\mathbb{R}; \mathbb{R}_+)$, $\chi_n \in C^0(\mathbb{R}^N; \mathbb{R}_+^N)$ and $\mathbb{R}_n \subset \mathbb{R}$ by

$$\varphi_n(s) = \begin{cases} |s| & \text{for } |s| \ge \frac{1}{n}, \\ \frac{1}{n} & \text{for } |s| < \frac{1}{n}, \end{cases} \qquad \chi_n(x) = (\varphi_n(x_1), \dots, \varphi_n(x_N)),$$

$$\mathbb{R}_n = \mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n}\right].$$

Let f satisfy (S_2) and let $f_{(n)}\colon [0,T] \times (\mathbb{R}^N \times \mathbb{R}^N) \to \mathbb{R}^N$ be defined by the following procedure

$$f_{(n)}^0(t,x,y) = f(t, \chi_n(x), y) \quad \text{for } (x,y) \in \mathbb{R}^N \times \mathbb{R}_n^N,$$

$$f^1_{(n)}(t,x,y) = \begin{cases} f^0_{(n)}(t,x,y) \quad \text{for } y \in \mathbb{R}^N_n, \\ \dfrac{n}{2}\left[\left(y_1 + \dfrac{1}{n}\right) f^0_{(n)}\left(t,x,\left(\dfrac{1}{n}, y_2, \dots, y_N\right)\right)\right. \\ \qquad \left. - \left(y_1 - \dfrac{1}{n}\right) f^0_{(n)}\left(t,x,\left(-\dfrac{1}{n}, y_2, \dots, y_N\right)\right)\right] \\ \qquad\qquad \text{for } y \in \left[-\frac{1}{n}, \frac{1}{n}\right] \times \mathbb{R}^{N-1}_n, \end{cases}$$

$$f^2_{(n)}(t,x,y) = \begin{cases} f^1_{(n)}(t,x,y) \quad \text{for } y \in \mathbb{R} \times \mathbb{R}^{N-1}_n, \\ \dfrac{n}{2}\left[\left(y_2 + \dfrac{1}{n}\right) f^1_{(n)}\left(t,x,\left(y_1, \dfrac{1}{n}, y_3, \dots, y_N\right)\right)\right. \\ \qquad \left. - \left(y_2 - \dfrac{1}{n}\right) f^1_{(n)}\left(t,x,\left(y_1, -\dfrac{1}{n}, y_3, \dots, y_N\right)\right)\right] \\ \qquad\qquad \text{for } y \in \mathbb{R} \times \left[-\frac{1}{n}, \frac{1}{n}\right] \times \mathbb{R}^{N-2}_n, \end{cases}$$

$$\vdots$$

$$f^N_{(n)}(t,x,y) = \begin{cases} f^{N-1}_{(n)}(t,x,y) \quad \text{for } y \in \mathbb{R}^{N-1} \times \mathbb{R}_n, \\ \dfrac{n}{2}\left[\left(y_N + \dfrac{1}{n}\right) f^{N-1}_{(n)}\left(t,x,\left(y_1,, \dots, y_{N-1}, \dfrac{1}{n}\right)\right)\right. \\ \qquad \left. - \left(y_N - \dfrac{1}{n}\right) f^{N-1}_{(n)}\left(t,x,\left(y_1, \dots, y_{N-1}, -\dfrac{1}{n}\right)\right)\right] \\ \qquad\qquad \text{for } y \in \mathbb{R}^{N-1} \times \left[-\frac{1}{n}, \frac{1}{n}\right]. \end{cases}$$

Let

$$f_{(n)}(t,x,y) = f^N_{(n)}(t,x,y) \quad \text{for } t \in [0,T],\ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Then $f_{(n)} = (f_{n1}, \dots, f_{nN}) \in \mathrm{Car}([0,T] \times (\mathbb{R}^N \times \mathbb{R}^N); \mathbb{R}^N)$ and

$$\left.\begin{array}{c} f_{nj}(t,x,y) \le -c \\ \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,\ j = 1, \dots, N,\ n \in \mathbb{N}. \end{array}\right\} \tag{18}$$

It follows from (S_3) that

$$\left.\begin{array}{c} -f_{nj}(t,x,y) \le h_j(t, 1 + \|x\|, 1 + \|y\|) + \displaystyle\sum_{k=1}^{N} (a_{jk}(|x_k|) + b_{jk}(|y_k|)) \\ \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}^N_0 \times \mathbb{R}^N_0,\ j = 1, \dots, N,\ n \in \mathbb{N}. \end{array}\right\} \tag{19}$$

We discuss the auxiliary regular system

$$(\phi(u'))' = f_{(n)}(t, u, u'), \quad n \in \mathbb{N}. \tag{20}$$

In order to prove that problem (20), (16) has a solution we use Theorem 2.2. We need a priori bounds for solutions of the systems

$$(\phi_j(u_j'))' = -\lambda c, \quad \lambda \in [0,1],\ j = 1,\dots,N, \tag{21}$$

$$(\phi_j(u_j'))' = \lambda f_{nj}(t,u,u') - (1-\lambda)c, \quad \lambda \in [0,1],\ j = 1,\dots,N, \tag{22}$$

satisfying the boundary condition (16), where c is from (S_2).

We give a priori bounds for solutions of problems (21), (16) and (22), (16) in Lemma 3.3, beginning to the useful lemma.

Lemma 3.2 *Let (S_1) hold. Let $u \in C^1([0,T];\mathbb{R}^N)$, $\phi(u') \in AC([0,T];\mathbb{R}^N)$ and let $u = (u_1,\dots,u_N)$ fulfil* (16). *If*

$$(\phi_j(u_j'(t)))' < 0 \quad \textit{for a.e. } t \in [0,T] \textit{ and } j = 1,\dots,N, \tag{23}$$

then

(i) *u_j' is decreasing on $[0,T]$ and there exist $t_1,\dots,t_N \in (0,T)$ such that*

$$u_j'(t_j) = 0 \quad \textit{for } j = 1,\dots,N,$$

(ii) *$u(t) > 0$ for $t \in (0,T)$.*

Proof Let (23) hold. Since ϕ_j is increasing and $\phi_j(0) = 0$ by (S_1), u_j' is decreasing on $[0,T]$. In addition, by (16), $u_j(T) = u_j(\tau_j(u))$, and consequently, $u_j'(t_j) = 0$ for some $t_j \in (\tau_j(u),T)$. This t_j is unique in $[0,T]$ since u_j' is decreasing. Therefore, $u_j' > 0$ on $[0,t_j)$ and $u_j' < 0$ on $(t_j,T]$. Using the fact that $u_j(0) = 0$ by (16), we have $u_j > 0$ on $(0,t_j]$ and u_j is decreasing on $[t_j,T]$. Consequently, $u_j(T) = u_j(\tau_j(u)) \ge u_j(0) = 0$, and so $u_j(T) \ge 0$. We have $u_j(t) > 0$ for $t \in (0,T)$ and $j = 1,\dots,N$, which means that $u > 0$ on $(0,T)$. □

Lemma 3.3 *Let (S_1)–(S_3) hold. Then there exists a positive constant S independent of λ and n such that any solution u of problems* (21), (16) *and* (22), (16) *fulfils the estimates*

$$\|u\|_\infty < ST, \qquad \|u'\|_\infty < S. \tag{24}$$

Proof Let u be a solution of problem (21), (16). If $\lambda = 0$, then $u = 0$. Let $\lambda \in (0,1]$. Then $u_j'(t_j) = 0$, $j = 1,\dots,N$, by Lemma 3.2, where $t_j \in (0,T)$. Hence $\phi_j(u_j'(t)) = -\lambda c(t-t_j)$, and consequently, $u_j'(t) = -\phi_j^{-1}(\lambda c(t-t_j))$ for $t \in [0,T]$. Therefore $|u_j'(t)| < \phi_j^{-1}(cT)$ for $t \in [0,T]$, $j = 1,\dots,N$, and so $|u_j(t)| < \phi_j^{-1}(cT)T$ for these t and j since $u_j(0) = 0$. Put $S_1 = \max\{\phi_j^{-1}(cT) \colon 1 \le j \le N\}$. Then $\|u\|_\infty < S_1T$ and $\|u'\|_\infty < S_1$ for all solutions u of problem (21), (16).

Now, let u be a solution of problem (22), (16). By (18), $(\phi_j(u_j'(t)))' \le -c$ for a.e. $t \in [0,T]$ and for $j = 1,\dots,N$. Besides, Lemma 3.2 guarantees that u_j' is decreasing on $[0,T]$, $u_j'(t_j) = 0$ for a $t_j \in (0,T)$ and $u_j > 0$ on $(0,T)$. Integrating $(\phi_j(u_j'(t)))' \le -c$ over $[t,t_j]$ and $[t_j,t]$ we get

$$\left.\begin{aligned} u_j'(t) &\ge \phi_j^{-1}(c(t_j-t)) \quad \text{for } t \in [0,t_j],\\ u_j'(t) &\le -\phi_j^{-1}(c(t-t_j)) \text{ for } t \in (t_j,T]. \end{aligned}\right\} \tag{25}$$

Since $u_j(0) = 0$ by (16), we have

$$u_j(t) = \int_0^t u_j'(s)\,\mathrm{d}s \ge \int_0^t \phi_j^{-1}(c(t_j-s))\,\mathrm{d}s = \frac{1}{c}\int_{c(t_j-t)}^{ct_j} \phi_j^{-1}(s)\,\mathrm{d}s, \quad t \in [0,t_j],$$

$$\begin{aligned} u_j(T) - u_j(t) = \int_t^T u_j'(s)\,\mathrm{d}s &\le -\int_t^T \phi_j^{-1}(c(s-t_j))\,\mathrm{d}s\\ &= -\frac{1}{c}\int_{c(t-t_j)}^{c(T-t_j)} \phi_j^{-1}(s)\,\mathrm{d}s, \quad t \in [t_j,T]. \end{aligned}$$

In particular, $t = t_j$ gives

$$u_j(t_j) \ \ge \ \frac{1}{c}\int_0^{ct_j} \phi_j^{-1}(s)\,\mathrm{d}s, \tag{26}$$

$$u_j(T) - u_j(t_j) \ \le \ -\frac{1}{c}\int_0^{c(T-t_j)} \phi_j^{-1}(s)\,\mathrm{d}s. \tag{27}$$

Due to $u_j(T) \ge 0$, inequality (27) yields

$$u_j(t_j) \ge \frac{1}{c}\int_0^{c(T-t_j)} \phi_j^{-1}(s)\,\mathrm{d}s. \tag{28}$$

Now, we conclude from (26) and (28) that

$$u_j(t_j) \ge \frac{1}{c}\int_0^{(cT)/2} \phi_j^{-1}(s)\,\mathrm{d}s =: \alpha_j, \quad j = 1,\dots,N. \tag{29}$$

Let $\alpha = \min\{\alpha_j : 1 \le j \le N\}$. Keeping in mind that u_j' is decreasing, we see that u_j is concave on $[0,T]$, and therefore, it follows from $0 \le u_j(t) \le u_j(t_j)$ for $t \in [0,T]$ and from (29) that

$$u_j(t) \ge \begin{cases} \dfrac{\alpha t}{T} & \text{for } t \in \left[0, \dfrac{T}{2}\right],\\[2ex] \dfrac{\alpha(T-t)}{T} & \text{for } t \in \left(\dfrac{T}{2}, T\right], \end{cases} \qquad j = 1,\dots,N. \tag{30}$$

Since

$$\phi_j(u_j'(t)) = \int_{t_j}^t [\lambda f_{nj}(s,u(s),u'(s)) - (1-\lambda)c]\,\mathrm{d}s,$$

inequalities (18) and (19) imply that

$$\begin{aligned}|u_j'(t)| &= \left|\phi_j^{-1}\left(\int_{t_j}^{t}[\lambda f_{nj}(s,u(s),u'(s))-(1-\lambda)c]\,\mathrm{d}s\right)\right|\\ &\le \phi_j^{-1}\left(\int_0^T |f_{nj}(s,u(s),u'(s))|\,\mathrm{d}s\right)\\ &\le \phi_j^{-1}\left(\int_0^T [h_j(s,1+\|u\|_\infty,1+\|u'\|_\infty)+W_j(s)]\,\mathrm{d}s\right) \qquad (31)\end{aligned}$$

for $t\in[0,T]$ and $j=1,\dots,N$, where

$$W_j(t)=\sum_{k=1}^{N}(a_{jk}(u_k(t))+b_{jk}(|u_k'(t)|)).$$

By (S_3), the functions a_{jk} and b_{jk} are nonincreasing on $\mathbb{R}_+$. Therefore, by (25) and (30),

$$\left.\begin{aligned}\int_0^T a_{jk}(u_k(s))\,\mathrm{d}s &\le \int_0^{T/2} a_{jk}\left(\frac{\alpha s}{T}\right)\mathrm{d}s+\int_{T/2}^{T} a_{jk}\left(\frac{\alpha(T-s)}{T}\right)\mathrm{d}s\\ &= \frac{2T}{\alpha}\int_0^{\alpha/2} a_{jk}(s)\,\mathrm{d}s,\\ \int_0^T b_{jk}(|u_k'(s)|)\,\mathrm{d}s &\le \int_0^{t_k} b_{jk}(\phi_k^{-1}(c(t_k-s)))\,\mathrm{d}s\\ &\qquad+\int_{t_k}^{T} b_{jk}(\phi_k^{-1}(c(s-t_k)))\,\mathrm{d}s\\ &= \frac{1}{c}\left(\int_0^{ct_k} b_{jk}(\phi_k^{-1}(s))\,\mathrm{d}s+\int_0^{c(T-t_k)} b_{jk}(\phi_k^{-1}(s))\,\mathrm{d}s\right)\\ &< \frac{2}{c}\int_0^{cT} b_{jk}(\phi_k^{-1}(s))\,\mathrm{d}s\end{aligned}\right\} \quad (32)$$

for $j=1,\dots,N$. Put

$$\Lambda=\frac{2T}{\alpha}\sum_{j,k=1}^{N}\int_0^{\alpha/2} a_{jk}(s)\,\mathrm{d}s+\frac{2}{c}\sum_{j,k=1}^{N}\int_0^{cT} b_{jk}(\phi_k^{-1}(s))\,\mathrm{d}s.$$

Then $\Lambda<\infty$ by (S_3), and (31), (32) yield

$$|u_j'(t)|<\phi_j^{-1}\left(\int_0^T h_j(s,1+\|u\|_\infty,1+\|u'\|_\infty)\,\mathrm{d}s+\Lambda\right),\ t\in[0,T],\ j=1,\dots,N.$$

Since $\|u\|_\infty\le T\|u'\|_\infty$, we have

$$\|u'\|_\infty<\max\left\{\phi_j^{-1}\left(\int_0^T h_j(s,1+T\|u'\|_\infty,1+\|u'\|_\infty)\,\mathrm{d}s+\Lambda\right): 1\le j\le N\right\}. \qquad (33)$$

By condition (17), there exists $S_2 > 0$ such that

$$\phi_j^{-1}\left(\int_0^T h_j(s, 1+Tv, 1+v)\,\mathrm{d}s + \Lambda\right) < v \quad \text{for } v \geq S_2 \text{ and } j = 1, \dots, N.$$

Hence

$$\max\left\{\phi_j^{-1}\left(\int_0^T h_j(s, 1+Tv, 1+v)\,\mathrm{d}s + \Lambda\right) : 1 \leq j \leq N\right\} < v \quad \text{for } v \geq S_2.$$

This inequality together with (33) give $\|u'\|_\infty < S_2$, and then $\|u\|_\infty < S_2 T$. To summarize, the lemma is true for $S = \max\{S_1, S_2\}$. □

We are now in the position to give the existence result for problem (20), (16).

Lemma 3.4 *Let* (S_1)*–*(S_3) *hold. Then for each* $n \in \mathbb{N}$ *there exists a solution* u *of problem* (20), (16) *and*

$$\|u\|_\infty < ST, \qquad \|u'\|_\infty < S, \tag{34}$$

where S *is a positive constant independent of* n.

Proof We apply Theorem 2.2 with $\varphi(t)$ and $(Fu)(t)$ replaced by $-(c, \dots, c) \in \mathbb{R}^N$ and $f_{(n)}(t, u(t), u'(t))$, respectively. Let S be a positive constant in Lemma 3.3. Then any solution u of problems (21), (16) and (22), (16) fulfils estimate (24) by Lemma 3.3, and $u > 0$ on $(0, T)$ by Lemma 3.2. We note that the boundary conditions (16) can be written in the form (3) with $\alpha(x) = x(0)$ and $\beta(x) = x(T) - \gamma(x)$, and that $\alpha, \beta \in \mathcal{A}$. Since, by the definition, $\gamma(x) = (x_1(\tau_1(x)), \dots, x_N(\tau_N(x)))$, where $\tau_j \colon C^1([0,T]; \mathbb{R}^N) \to [0, T)$ is a continuous and even functional, we have

$$\gamma(a + tb) = (a_1 + \tau_1(a+tb)b_1, \dots, a_N + \tau_N(a+tb)b_N)$$

and $\tau_j(a+tb) = \tau_j(-a-tb)$ for $a = (a_1, \dots, a_N)$, $b = (b_1, \dots, b_N)$ in $\mathbb{R}^N$. Hence, we can write system (4) as

$$\left.\begin{aligned} (1+\mu)a &= 0, \\ (1+\mu)(T - \tau_j(a+tb))b_j &= 0, \quad j = 1, \dots, N. \end{aligned}\right\} \tag{35}$$

Then $a = 0$, and consequently, $(T - \tau_j(tb))b_j = 0$ for $j = 1, \dots, N$. In view of $\tau_j(tb) < T$, we have $b = 0$. Therefore, $(a, b) = (0, 0) \in \mathbb{R}^N \times \mathbb{R}^N$ is the unique solution of system (35) for each $\mu \in [0, 1]$. We have shown that the assumptions of Theorem 2.2 are satisfied. Hence problem (20), (16) has a solution u satisfying inequality (34). □

For the solvability of the singular problem (2), (16) we need the following result concerning solutions of problem (20), (16).

Lemma 3.5 *Let* (S_1)–(S_3) *hold. Let* $u_{(n)}$ *be a solution of problem* (20), (16). *Then* $\{u'_{(n)}\}$ *is equicontinuous on* $[0,T]$.

Proof By Lemmas 3.3 and 3.4, there exists a positive constant S such that

$$\|u_{(n)}\|_\infty < ST, \quad \|u'_{(n)}\|_\infty < S \quad \text{for } n \in \mathbb{N}. \tag{36}$$

Keeping in mind $u_{(n)} = (u_{n1}, \dots, u_{nN})$ by our notation, it follows from the proof of Lemma 3.3 that (cf. (30)),

$$u_{nj}(t) \geq \begin{cases} \dfrac{\alpha t}{T} & \text{for } t \in \left[0, \dfrac{T}{2}\right], \\ \dfrac{\alpha(T-t)}{T} & \text{for } t \in \left(\dfrac{T}{2}, T\right], \end{cases} \qquad n \in \mathbb{N},\ j = 1, \dots, N, \tag{37}$$

where α is a positive constant. Next, by Lemma 3.2, there exists $t_{nj} \in (0,T)$ such that

$$u'_{nj}(t_{nj}) = 0 \quad \text{for } n \in \mathbb{N} \text{ and } j = 1, \dots, \mathbb{N},$$

and it follows from the proof od Lemma 3.3 (cf. (25)) that

$$\left.\begin{array}{ll} u'_{nj}(t) \geq \phi_j^{-1}(c(t_{nj}-t)) & \text{if } t \in [0, t_{nj}], \\ u'_{nj}(t) \leq -\phi_j^{-1}(c(t-t_{nj})) & \text{if } t \in [t_{nj}, T], \\ \multicolumn{2}{c}{\text{for } n \in \mathbb{N},\ j = 1, \dots, n.} \end{array}\right\} \tag{38}$$

Let $0 \leq t_1 < t_2 \leq T$. Then

$$\int_{t_1}^{t_2} a_{jk}(u_{nk}(s))\,\mathrm{d}s \leq \begin{cases} \displaystyle\int_{t_1}^{t_2} a_{jk}\left(\frac{\alpha s}{T}\right)\mathrm{d}s = \frac{T}{\alpha}\int_{(\alpha t_1)/T}^{(\alpha t_2)/T} a_{jk}(s)\,\mathrm{d}s & \\ \hfill \text{if } t_2 \leq \frac{T}{2}, \\ \displaystyle\int_{t_1}^{t_2} a_{jk}\left(\frac{\alpha(T-s)}{T}\right)\mathrm{d}s = \frac{T}{\alpha}\int_{(\alpha(T-t_2))/T}^{(\alpha(T-t_1))/T} a_{jk}(s)\,\mathrm{d}s & \\ \hfill \text{if } t_1 > \frac{T}{2}, \\ \displaystyle\int_{t_1}^{T/2} a_{jk}\left(\frac{\alpha s}{T}\right)\mathrm{d}s + \int_{T/2}^{t_2} a_{jk}\left(\frac{\alpha(T-s)}{T}\right)\mathrm{d}s & \\ \displaystyle\quad = \frac{T}{\alpha}\left(\int_{(\alpha t_1)/T}^{\alpha/2} a_{jk}(s)\,\mathrm{d}s + \int_{(\alpha(T-t_2))/T}^{\alpha/2} a_{jk}(s)\,\mathrm{d}s\right) & \\ \hfill \text{if } t_1 < \frac{T}{2} < t_2, \end{cases}$$

$$
\int_{t_1}^{t_2} b_{jk}(|u'_{nk}(s)|)\,\mathrm{d}s \le
$$

$$
\le \begin{cases}
\displaystyle\int_{t_1}^{t_2} b_{jk}(\phi_j^{-1}(c(t_{nk}-s)))\,\mathrm{d}s = \frac{1}{c}\int_{c(t_{nk}-t_2)}^{c(t_{nk}-t_1)} b_{jk}(\phi_j^{-1}(s))\,\mathrm{d}s & \text{if } t_2 \le t_{nk}, \\[2ex]
\displaystyle\int_{t_1}^{t_2} b_{jk}(\phi_j^{-1}(c(s-t_{nk})))\,\mathrm{d}s = \frac{1}{c}\int_{c(t_1-t_{nk})}^{c(t_2-t_{nk})} b_{jk}(\phi_j^{-1}(s))\,\mathrm{d}s & \text{if } t_1 \ge t_{nk}, \\[2ex]
\displaystyle\int_{t_1}^{t_{nk}} b_{jk}(\phi_j^{-1}(c(t_{nk}-s)))\,\mathrm{d}s + \int_{t_{nk}}^{t_2} b_{jk}(\phi_j^{-1}(c(s-t_{nk})))\,\mathrm{d}s \\
\displaystyle\quad = \frac{1}{c}\left(\int_0^{c(t_{nk}-t_1)} b_{jk}(\phi_j^{-1}(s))\,\mathrm{d}s + \int_0^{c(t_2-t_{nk})} b_{jk}(\phi_j^{-1}(s))\,\mathrm{d}s\right) & \text{if } t_1 < t_{nk} < t_2.
\end{cases}
$$

Hence, for $j,k = 1,\dots,N$ and $n \in \mathbb{N}$, the inequalities

$$
\left.\begin{aligned}
&\int_{t_1}^{t_2} a_{jk}(u_{nk}(s))\,\mathrm{d}s < \\
&< \frac{2T}{\alpha}\sup\left\{\int_{\xi_1}^{\xi_2} a_{jk}(s)\,\mathrm{d}s \colon 0 \le \xi_1 < \xi_2 \le T,\ \xi_2-\xi_1 \le \frac{\alpha}{T}(t_2-t_1)\right\}, \\
&\int_{t_1}^{t_2} b_{jk}(|u'_{nk}(s)|)\,\mathrm{d}s < \\
&< \frac{2}{c}\sup\left\{\int_{\nu_1}^{\nu_2} b_{jk}(\phi^{-1}(s))\,\mathrm{d}s \colon 0 \le \nu_1 < \nu_2 \le T,\ \nu_2-\nu_1 \le c(t_2-t_1)\right\}
\end{aligned}\right\} \tag{39}
$$

are satisfied. Integrating $(\phi_j(u'_{nj}(t)))' = f_{nj}(t, u_{(n)}(t), u'_{(n)}(t))$ over $[t_1,t_2] \subset [0,T]$ and using (19) and (36), we get

$$
|\phi_j(u'_{nj}(t_2)) - \phi_j(u'_{nj}(t_1))| \le \int_{t_1}^{t_2} \Big[h_j(s, 1+TS, 1+S)) + V_j(s)\Big]\,\mathrm{d}s \tag{40}
$$

for $j = 1,\dots,N$ and $n \in \mathbb{N}$, where $V_j(t) = \sum_{k=1}^N (a_{jk}(u_{nk}(t)) + b_{jk}(|u'_{nk}(t)|))$. Since $h_j(t, 1+TS, 1+S) \in L^1([0,T];\mathbb{R}_+)$ and a_{jk}, $b_{jk}(\phi^{-1})$ are locally integrable on $[0,\infty)$ by (S_3), we conclude from inequalities (39) and (40) that $\{\phi_j(u'_{nj})\}$ is equicontinuous on $[0,T]$ for $j = 1,\dots,N$. In view of the equality $|u'_{nj}(t_2) - u'_{nj}(t_1)| = |\phi_j^{-1}(\phi_j(u'_{nj}(t_2))) - \phi_j^{-1}(\phi_j(u'_{nj}(t_1)))|$ and the fact that ϕ_j^{-1} is continuous and increasing, we see that $\{u'_{nj}\}$ is equicontinuous on $[0,T]$ for $j = 1,\dots,N$, which means that $\{u'_{(n)}\}$ is equicontinuous on $[0,T]$. $\square$

3.3 The main result and an example

Theorem 3.6 *Let (S_1)–(S_3) hold. Then problem* (2), (16) *has a solution u, $u \in C^1([0,T];\mathbb{R}^N)$, $u(t) > 0$ for $t \in (0,T)$, and $\phi(u') \in AC([0,T];\mathbb{R}^N)$.*

Proof By Lemma 3.4, for each $n \in \mathbb{N}$, there is a solution $u_{(n)} = (u_{n1}, \dots, u_{nN})$ of problem (20), (16) satisfying inequality (36), where S is a positive constant. Lemma 3.5 guarantees that $\{u'_{(n)}\}$ is equicontinuous on $[0,T]$ and it follows from its proof that inequalities (37) and (38) are true, where α is a positive constant and $t_{nj} \in (0,T)$ is the unique zero of u'_{nj}. Hence, by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem, there exist convergent subsequences $\{u_{(k_n)}\}$, $\{t_{k_n j}\}$, and $u \in C^1([0,T];\mathbb{R}^N)$, $\xi_j \in [0,T]$ such that $\lim_{n\to\infty} u_{(k_n)} = u$ in $C^1([0,T];\mathbb{R}^N)$ and $\lim_{n\to\infty} t_{k_n j} = \xi_j$ in $\mathbb{R}$, $j = 1, \dots, N$. Letting $n \to \infty$ in (37) and (38) (with u_{nj} and t_{nj} replaced by $u_{k_n j}$ and $t_{k_n j}$) yields

$$u_j(t) \ge \begin{cases} \dfrac{\alpha t}{T} & \text{for } t \in \left[0, \dfrac{T}{2}\right], \\[2mm] \dfrac{\alpha(T-t)}{T} & \text{for } t \in \left(\dfrac{T}{2}, T\right], \end{cases} \tag{41}$$

$$\left.\begin{aligned} u'_j(t) &\ge \phi_j^{-1}(c(\xi_j - t)) && \text{for } t \in [0, \xi_j], \\ u'_j(t) &\le -\phi_j^{-1}(c(t - \xi_j)) && \text{for } t \in [\xi_j, T], \end{aligned}\right\} \tag{42}$$

for $j = 1, \dots, N$. Hence, u_j has at most two zeros on $[0,T]$, u'_j vanishes only at ξ_j and equality (13) holds. Besides, by (19),

$$\left.\begin{gathered} 0 \le -f_{(n)}(t,x,y) \le Q(t,|x|,|y|) \\ \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in \mathbb{R}_0^N \times \mathbb{R}_0^N,\ n \in \mathbb{N}, \end{gathered}\right\}$$

where $Q(t,x,y) = (Q_1(t,x,y), \dots, Q_N(t,x,y)) \in \mathrm{Car}([0,T] \times (\mathbb{R}_+^N \times \mathbb{R}_+^N); \mathbb{R}_+^N)$,

$$Q_j(t,x,y) = h_j(t, 1 + \|x\|, 1 + \|y\|) + \sum_{j=1}^{N} (a_{jk}(x_k) + b_{jk}(y_k)).$$

Hence the assumptions of Theorem 2.3 are satisfied, and consequently, u is a solution of problem (2), (16), $u \in C^1([0,T];\mathbb{R}^N)$ and $\phi(u') \in AC([0,T];\mathbb{R}^N)$. The inequality $u > 0$ on $(0,T)$ follows from (41). □

Example 3.7 Let $p_1, p_2 \in (1,\infty)$, $\alpha_j, \beta_j, \rho_j, \nu_j$ be positive constants, $j = 1, 2$, $\max\{\alpha_1, \alpha_2\} < p_1 - 1$, $\max\{\rho_1, \rho_2\} < p_2 - 1$, $\beta_1 + \beta_2 < \min\{1, p_1 - 1\}$, $\nu_1 < 1$ and $\nu_2 < p_2 - 1$. Consider the problem

$$\left.\begin{aligned} (|u'_1|^{p_1-2}u'_1)' + \frac{1}{\sqrt{t}}\,(1 + u_1^{\alpha_1} + |u'_2|^{\alpha_2}) + \frac{1}{|u'_1|^{\beta_1} u_2^{\beta_2}} &= 0, \\ (|u'_2|^{p_2-2}u'_2)' + \frac{1}{\sqrt{t(T-t)}}\,(1 + (u'_1)^{\rho_1} + |u_2|^{\rho_2}) + \frac{1}{u_1^{\nu_1}} + \frac{1}{|u'_2|^{\nu_2}} &= 0, \end{aligned}\right\} \tag{43}$$

$$u(0) = 0, \qquad u(T) = (u_1(t_1), u(t_2)), \tag{44}$$

where $t_1, t_2 \in [0, T)$. System (43) satisfies conditions (S_1)–(S_3) with

$$\phi(v) = (|v|^{p_1-2}v, |v|^{p_2-2}v), \qquad c = \min\left\{\frac{1}{\sqrt{T}}, \frac{2}{T}\right\},$$

$$h_1(t,x,y) = \frac{1}{\sqrt{t}}(1 + x^{\alpha_1} + y^{\alpha_2}),$$

$$h_2(t,x,y) = \frac{1}{\sqrt{t(T-t)}}(1 + x^{\rho_2} + y^{\rho_1}),$$

$a_{11} = a_{22} = b_{12} = b_{21} = 0,$

$$a_{12}(x) = \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{x^{\beta_1+\beta_2}}, \qquad b_{11}(x) = \frac{\beta_1}{\beta_1+\beta_2}\frac{1}{x^{\beta_1+\beta_2}},$$

$$a_{21}(x) = \frac{1}{x^{\nu_1}}, \qquad b_{22}(x) = \frac{1}{x^{\nu_2}}$$

since

$$\frac{1}{|u_1'|^{\beta_1}u_2^{\beta_2}} \le \frac{\beta_1}{\beta_1+\beta_2}\frac{1}{|u_1'|^{\beta_1+\beta_2}} + \frac{\beta_2}{\beta_1+\beta_2}\frac{1}{u_2^{\beta_1+\beta_2}}$$

by the Young inequality. Therefore, Theorem 3.6 guarantees that there exists a solution $u \in C^1([0,T];\mathbb{R}^2)$ of problem (43), (44), $u > 0$ on $(0,T)$ and $\phi(u') \in AC([0,T];\mathbb{R}^2)$.

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