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PERFECT COMPACTIFICATIONS OF FRAMES

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Abstract. Perfect compactifications of frames are introduced. It is shown that the Stone-Čech compactification is an example of such a compactification. We also introduce rim-compact frames and for such frames we define its Freudenthal compactification, another example of a perfect compactification. The remainder of a rim-compact frame in its Freudenthal compactification is shown to be zero-dimensional. It is shown that with the assumption of the Boolean Ultrafilter Theorem the Freudenthal compactification for spaces, as well as the Freudenthal-Morita Theorem for spaces, can be obtained from our frame constructions.

Keywords: perfect compactifications, rim-compact frame

MSC 2010: 06D20, 06B35, 54D35

1. Introduction

Perfect compactifications of topological spaces were introduced by Sklyarenko (see [11]) as the compactifications Y of a space X having the property that $\mathrm{Fr}_Y\, O\langle U\rangle = \mathrm{Cl}_Y\, \mathrm{Fr}_X\, U$ for every open subset U of X. Here Fr is the frontier (or boundary) operator, and $O\langle U\rangle = Y \setminus \mathrm{Cl}_Y(X \setminus U)$ which is the largest open subset of Y whose intersection with X gives the set U. Examples of such compactifications are the Stone-Čech compactification of a Tychonoff space and the Freudenthal compactification of a rim-compact Hausdorff space [11].

The purpose of this paper is to define such compactifications for frames with the main aim of defining the Freudenthal compactification for a class of frames which we call rim-compact. We shall show that the Freudenthal compactification for spaces can be obtained from the frame construction provided we assume the Boolean Ultrafilter Theorem. It is a well known fact that the classical construction of the Freudenthal compactification for spaces rests on the Boolean Ultrafilter Theorem. However, in

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the frame context we believe it is noteworthy to mention that our construction of the Freudenthal compactification does not depend on any choice principles.

We also obtain the frame analog of the Freudenthal-Morita theorem for spaces [6], [9] appearing in [11], namely: Every peripherally bicompact space X may be imbedded in a bicompactum with zero-dimensional (in the sense of *ind*) annex. We then show, and this is our final result, that this theorem for spaces follows from the frame version if we again assume the Boolean Ultrafilter Theorem.

2. Preliminaries

A *frame* L is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s \colon s \in S\}$$

for all $x \in L$, $S \subseteq L$. The top element of L is denoted by e and the bottom by 0. A *frame homomorphism* is a map $h \colon L \to M$ between frames that preserve finitary meets (including the top e) and arbitrary joins (including the bottom 0).

We thus have the category of frames and frame homomorphisms, which we denote by **Frm**. A frame map $h \colon L \to M$ is called *dense* if $x = 0$ whenever $h(x) = 0$. A frame L is called *compact* if whenever $e = \bigvee S$, then there exists a finite $F \subseteq S$ such that $e = \bigvee F$. For elements $a, b \in L$, we say that a is *rather below* b, written $a \prec b$, if there exists an element $c \in L$ such that $a \wedge c = 0$ and $c \vee b = e$. This is equivalent to the condition that $a^* \vee b = e$, where a^* is the *pseudocomplement* of a, i.e. the largest element in L whose meet with a is 0. A frame L is called *regular* if for each $a \in L$, $a = \bigvee \{x \in L \colon x \prec a\}$. A *compactification* of a frame L is a compact regular frame M together with a dense onto homomorphism $h \colon M \to L$.

The prototypical example of a frame is the frame $\mathscr{O}X$ of open sets of a topological space X, and of a frame homomorphism that is determined by any continuous map $f \colon X \to Y$ between topological spaces, namely, $\mathscr{O}f \colon \mathscr{O}Y \to \mathscr{O}X$ taking $U \in \mathscr{O}Y$ to $f^{-1}(U) \in \mathscr{O}X$. In fact, this determines a contravariant functor $\mathscr{O} \colon$ **Top** $\to$ **Frm** from the category **Top** of topological spaces and continuous maps to the category **Frm** of frames and their homomorphisms. There is also a contravariant functor $\Sigma \colon$ **Frm** $\to$ **Top**. This is described as follows: For each frame L, $\Sigma L = \{\xi \colon L \to \mathbf{2} \colon \xi \text{ is a frame homomorphism}\}$ (where $\mathbf{2}$ is the two-element frame), with open sets $\Sigma_a = \{\xi \in \Sigma L \colon \xi(a) = 1\}$ for $a \in L$. Also, for any frame homomorphism $h \colon L \to M$, $\Sigma h \colon \Sigma M \to \Sigma L$ is the map that acts by composition with h. The functors $\mathscr{O}$ and Σ are adjoint on the right with adjunction maps

$$\eta_L \colon L \to \mathscr{O}\Sigma L, \quad a \rightsquigarrow \Sigma_a$$

and

$$\varepsilon_X\colon\ X \to \Sigma\mathscr{O}X, \quad x \rightsquigarrow \tilde{x}, \quad \tilde{x}(U) = 1 \Leftrightarrow x \in U.$$

The space ΣL is called the *spectrum* of L, and L is said to be *spatial* if the map η_L is an isomorphism.

A congruence on a frame L is an equivalence relation on L which is also a subframe of $L \times L$. The *congruence lattice* $\mathscr{C}L$ of L consists of all the congruences on L. It is a frame with the bottom element $\Delta = \{(x, x)\colon\ x \in L\}$ and the top element $\nabla = L \times L$. Two particular congruences associated with each $a \in L$ are $\Delta_a = \{(x, y) \in L \times L\colon\ x \wedge a = y \wedge a\}$ and $\nabla_a = \{(x, y) \in L \times L\colon\ x \vee a = y \vee a\}$. These members of $\mathscr{C}L$ are complementary to each other in the sense that their meet is the bottom and their join is the top element. In general a congruence Θ does not necessarily have a complement in $\mathscr{C}L$ though of course it must have a pseudocomplement which we will denote by Θ^*.

A frame L is called *completely regular* if for each $a \in L$, $a = \bigvee\{x \in L\colon\ x \prec\!\prec a\}$ where $x \prec\!\prec a$ means that there exists a doubly indexed sequence of elements in L, $(x_{nk})_{n=0,1,\dots;\ k=0,1,\dots,2^n}$ such that

$$x = x_{n0},\ x_{nk} \prec x_{nk+1},\ x_{n2^n} = a,\ x_{nk} = x_{n+1\,2k}$$

for all $n = 0, 1, \dots$ and $k = 0, 1, \dots, 2^n$.

For general background on compactifications of frames the reader is referred to Banaschewski (see [3]), and for frames in general to the book by Johnstone (see [8]). Concerning spaces the central source from which the frame concepts and ideas were developed was the paper by Sklyarenko [11], to which we are greatly indebted.

3. Perfect compactifications

Definition 3.1. Let $h\colon M \to L$ be a compactification of L, $r\colon L \to M$ its right adjoint. Then (M, h) is said to be perfect with respect to an element $u \in L$ if $r(u \vee u^*) = r(u) \vee r(u^*)$. The compactification (M, h) is said to be a perfect compactification of L if it is perfect with respect to every element of L.

The point-free Stone-Čech compactification was introduced by Banaschewski and Mulvey in [5]. As an immediate consequence we have the following:

Theorem 3.2. *The Stone-Čech compactification of a completely regular frame is perfect.*

Proof. This follows from the corollary to Lemma 5 in the paper [1] where it is shown that the right adjoint of the compactification map preserves disjoint binary joins. □

Remark 3.3. The purpose of this remark is to provide a motivation for the above definition of perfect compactification of a frame. We see that this arises quite naturally from the work of Sklyarenko. We also show how, under certain circumstances, we can recover the perfect compactification for spaces from the frame counterpart. To this end let $i\colon X \hookrightarrow Y$ be a compactification (more generally an extension) of the space X. For any $U \in \mathscr{O}X$ (the open subsets of X), $O\langle U\rangle$ (in the notation of Sklyarenko) $= Y \setminus \mathrm{Cl}_Y(X \setminus U)$, and is the largest open subset of Y whose meet with X is U.

Now the frame map $\mathscr{O}i\colon \mathscr{O}Y \to \mathscr{O}X$ has the right adjoint $(\mathscr{O}i)_*$ given by $(\mathscr{O}i)_*(U) = \bigcup(V \in \mathscr{O}Y)((\mathscr{O}i)V = U) = \bigcup(V \in \mathscr{O}Y)(V \cap X = U)$ and therefore $(\mathscr{O}i)_*(U) = O\langle U\rangle$ for any $U \in \mathscr{O}X$. Let U^* represent the pseudocomplement of U in $\mathscr{O}X$, i.e. $X \setminus \mathrm{Cl}_X U$.

Then

$$\begin{aligned}
\mathrm{Cl}_Y(\mathrm{Fr}_X U) &= \mathrm{Fr}_Y O\langle U\rangle \\
&\Leftrightarrow \mathrm{Cl}_Y(\mathrm{Cl}_X U \setminus U) = \mathrm{Cl}_Y O\langle U\rangle \setminus O\langle U\rangle \\
&\Leftrightarrow Y \setminus \mathrm{Cl}_Y(\mathrm{Cl}_X U \setminus U) = Y \setminus (\mathrm{Cl}_Y O\langle U\rangle \setminus O\langle U\rangle) \\
&= (Y \setminus \mathrm{Cl}_Y O\langle U\rangle) \cup O\langle U\rangle \\
&= (Y \setminus \mathrm{Cl}_Y(O\langle U\rangle \cap X)) \cup O\langle U\rangle \\
&\quad \text{(since } X \text{ is dense in } Y) \\
&= (Y \setminus \mathrm{Cl}_Y U) \cup O\langle U\rangle \\
&= (Y \setminus \mathrm{Cl}_Y \mathrm{Cl}_X U) \cup O\langle U\rangle \\
&\quad \text{(since } \mathrm{Cl}_Y U = \mathrm{Cl}_Y(\mathrm{Cl}_X U)) \\
&= [Y \setminus \mathrm{Cl}_Y(X \setminus (X \setminus \mathrm{Cl}_X U))] \cup O\langle U\rangle \\
&= O\langle U^*\rangle \cup O\langle U\rangle \\
&\Leftrightarrow Y \setminus \mathrm{Cl}_Y(X \setminus (X \setminus (\mathrm{Cl}_X U \setminus U))) = O\langle U^*\rangle \cup O\langle U\rangle \\
&\Leftrightarrow O\langle X \setminus (\mathrm{Cl}_X U \setminus U)\rangle = O\langle U^*\rangle \cup O\langle U\rangle \\
&\Leftrightarrow O\langle U \cup (X \setminus \mathrm{Cl}_X U)\rangle = O\langle U^*\rangle \cup O\langle U\rangle \\
&\Leftrightarrow O\langle U \cup U^*\rangle = O\langle U^*\rangle \cup O\langle U\rangle \\
&\Leftrightarrow (\mathscr{O}i)_*(U \cup U^*) = (\mathscr{O}i)_*(U) \cup (\mathscr{O}i)_*(U^*).
\end{aligned}$$

We can therefore conclude from the above calculation that $i\colon X \hookrightarrow Y$ is a perfect compactification of the topological space X iff $\mathscr{O}i\colon \mathscr{O}Y \to \mathscr{O}X$ is a perfect compactification of the frame $\mathscr{O}X$.

Thus, given just the data that $i\colon X \hookrightarrow Y$ is an extension with Y Hausdorff and $\mathscr{O}i\colon \mathscr{O}Y \to \mathscr{O}X$ a perfect frame compactification, we can conclude that $i\colon X \hookrightarrow Y$ is a perfect compactification of X.

We can then recover the perfect compactification $i\colon X \hookrightarrow Y$ by noting that $i\colon X \hookrightarrow Y$ factorizes as

$$X \xrightarrow{\varepsilon_X} \Sigma\mathscr{O}X \xrightarrow{\Sigma\mathscr{O}i} \Sigma\mathscr{O}Y \cong Y$$

where we note that $Y \xrightarrow{\varepsilon_Y} \Sigma\mathscr{O}Y$ is a homeomorphism since Y is Hausdorff and therefore sober.

We have the following lemma which comes from Banaschewski [2], the proof of which is recorded here for the sake of completeness.

Lemma 3.4. *Let $h\colon M \to L$ be dense and onto, with r its right adjoint. Then*

(1) $r(a^*) = r(a)^*$ *for all* $a \in L$,
(2) $h(x^*) = h(x)^*$ *for all* $x \in M$.

Proof. (1) $h(r(a) \wedge r(a^*)) = a \wedge a^* = 0$ implies $r(a) \wedge r(a^*) = 0$ since h is dense, and hence $r(a^*) \leqslant r(a)^*$. Furthermore, $r(a) \wedge r(a)^* = 0$ implies $a \wedge h(r(a)^*) = 0$ which implies $h(r(a)^*) \leqslant a^*$ and hence $r(a)^* \leqslant r(a^*)$.

(2) $0 = x \wedge x^* \Rightarrow 0 = h(0) = h(x) \wedge h(x^*) \Rightarrow h(x^*) \leqslant h(x)^*$. Furthermore, $r(h(x)^*) = r(h(x))^* \leqslant x^*$ since $x \leqslant r(h(x))$. Thus $hr(h(x)^*) \leqslant h(x^*)$, i.e. $h(x)^* \leqslant h(x^*)$. □

We shall say that in a frame L, the pair (u, v) *disconnects* w in L if $w = u \vee v$, $u \wedge v = 0$ and $u \neq 0$, $v \neq 0$. We then have:

Theorem 3.5. *The following conditions are equivalent for a compactification $h\colon M \to L$ of L, r being the right adjoint of h.*

(1) *$h\colon M \to L$ is a perfect compactification.*
(2) *If a pair (u, v) disconnects w in L, then the pair $(r(u), r(v))$ disconnects $r(w)$ in M.*
(3) *r preserves disjoint binary joins.*

Proof. (1) ⇒ (3): Take any $u, v \in L$, $u \wedge v = 0$. We shall show the non-trivial inequality $r(u \vee v) \leqslant r(u) \vee r(v)$. Now $u \wedge v = 0 \Rightarrow v \leqslant u^* \Rightarrow u \vee v \leqslant u \vee u^* \Rightarrow$

$r(u \vee v) \leqslant r(u \vee u^*) = r(u) \vee r(u^*)$. Similarly $r(u \vee v) \leqslant r(v) \vee r(v^*)$. Hence

$$\begin{aligned} r(u \vee v) &\leqslant (r(u) \vee r(u^*)) \wedge (r(v) \vee r(v^*)) \\ &= [(r(u) \vee r(u^*)) \wedge r(v)] \vee [(r(u) \vee r(u^*)) \wedge r(v^*)] \\ &= (r(u) \wedge r(v)) \vee (r(u^*) \wedge r(v)) \vee (r(u) \wedge r(v^*)) \vee (r(u^*) \wedge r(v^*)) \\ &= r(u \wedge v) \vee r(u^* \wedge v) \vee r(u \wedge v^*) \vee r(u^* \wedge v^*) \\ &= r(0) \vee r(v) \vee r(u) \vee r((u \vee v)^*) \\ &= 0 \vee r(v) \vee r(u) \vee (r(u \vee v))^* \end{aligned}$$

since h is dense and by virtue of Lemma 3.4. Thus $r(u \vee v) \leqslant r(u) \vee r(v)$ as required.

(3) $\Rightarrow$ (2): Suppose $w = u \vee v$, with $u \wedge v = 0$, $u \neq 0$, $v \neq 0$ in L. Then $r(w) = r(u) \vee r(v)$, with $r(u) \neq 0$, $r(v) \neq 0$ and $r(u) \wedge r(v) = r(u \wedge v) = r(0) = 0$. Thus $(r(u), r(v))$ disconnects $r(w)$ in M.

(2) $\Rightarrow$ (1): Take any $u \in L$. Let $w = u \vee u^*$. If either $u = 0$ or $u^* = 0$, then $r(u) = 0$ or $r(u^*) = 0$ by denseness of h, and the equality $r(u \vee u^*) = r(u) \vee r(u^*)$ must certainly hold. If $u \neq 0$ and $u^* \neq 0$, then (u, u^*) disconnects w, and thus the pair $(r(u), r(u^*))$ disconnects $r(w)$. Hence $r(u \vee u^*) = r(u) \vee r(u^*)$. □

In [3] Banaschewski introduces the concept of a strong inclusion on a frame L. We recall this:

Definition 3.6. A strong inclusion on a frame L is a binary relation $\triangleleft$ on L such that

(1) if $x \leqslant a \triangleleft b \leqslant y$ then $x \triangleleft y$;
(2) $\triangleleft$ is a sublattice of $L \times L$;
(3) $a \triangleleft b \Rightarrow a \prec b$;
(4) $a \triangleleft b \Rightarrow a \triangleleft c \triangleleft b$ for some c in L;
(5) $a \triangleleft b \Rightarrow b^* \triangleleft a^*$;
(6) for each $a \in L$, $a = \bigvee x (x \triangleleft a)$.

Let $K(L)$ be the set of all compactifications of L, partially ordered by $(M, h) \leqslant (N, f)$ iff there exists a frame homomorphism $g \colon M \to N$ making the following diagram commute:

$$\begin{array}{ccc} M & \xrightarrow{\;g\;} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ L & = & L \end{array}$$

Also, let $S(L)$ be the set of all strong inclusions on L, partially ordered by set inclusion. Banaschewski [3] shows that $K(L)$ is isomorphic to $S(L)$ by exhibiting maps $K(L) \to S(L)$ and $S(L) \to K(L)$ which are order preserving and inverses of each other. The map $K(L) \to S(L)$ is given as follows: For a compactification (M, h) of L, let $r\colon L \to M$ be the right adjoint of h. Then for any $x, y \in L$ define $x \triangleleft y$ to mean that $r(x) \prec r(y)$. Then $\triangleleft$ turns out to be a strong inclusion on L. For the map $S(L) \to K(L)$, let $\triangleleft$ be any strong inclusion on L. Let γL be the set of all strongly regular ideals of L relative to $\triangleleft$, i.e. those ideals J of L for which $x \in J$ implies there exists $y \in J$ such that $x \triangleleft y$. Then $\bigvee\colon \gamma L \to L$ is dense, onto, and γL is a regular subframe of $\mathrm{Idl}(L)$, the frame of ideals of L, so that $(\gamma L, \bigvee)$ is a compactification of L. This is the compactification associated with the given $\triangleleft$.

If (M, h) is a compactification of a frame L, it is of interest then to know what additional properties the associated strong inclusion must satisfy if (M, h) is already a perfect compactification. This is given in the next

Proposition 3.7. *Let $h\colon M \to L$ be a compactification of L, $\triangleleft$ the associated strong inclusion. If (M, h) is a perfect compactification, then $\triangleleft$ satisfies*

$$\text{For all } x, y \in L,\ x \leqslant y,\ x \triangleleft y \vee y^* \text{ implies } x \triangleleft y.$$

Proof. Suppose $x \leqslant y$, $x \triangleleft y \vee y^*$. Then $r(x) \prec r(y \vee y^*) = r(y) \vee r(y^*)$, since (M, h) is a perfect compactification. Let $t \in L$ such that $r(x) \wedge t = 0$, $t \vee r(y) \vee r(y^*) = e$. Then $r(x) \wedge (t \vee r(y^*) = (r(x) \wedge t) \vee (r(x) \wedge r(y^*)) = 0 \vee (r(x) \wedge r(y^*)) = r(x) \wedge r(y^*) \leqslant r(x) \wedge r(x^*) = r(x \wedge x^*) = r(0) = 0$, by the denseness of h. Thus $r(x) \prec r(y)$, with the separating element $t \vee r(y^*)$. Hence $x \triangleleft y$. $\square$

If, on the other hand, a strong inclusion $\triangleleft$ on L satisfies that $x \leqslant y$, $x \triangleleft y \vee y^*$ implies $x \triangleleft y$, then the associated compactification $(\gamma L, \bigvee)$ must be perfect, as the following result shows.

Proposition 3.8. *Let $\triangleleft$ be a strong inclusion on L, and $(\gamma L, \bigvee)$ the compactification associated with $\triangleleft$. If $\triangleleft$ satisfies*

$$x \leqslant y,\ x \triangleleft y \vee y^* \text{ implies } x \triangleleft y \text{ for all } x, y \in L$$

then $(\gamma L, \bigvee)$ is a perfect compactification of L.

Proof. We recall first from Banaschewski [3] that the right adjoint $k\colon L \to \gamma L$ of $\bigvee\colon \gamma L \to L$ is given by: $k(a) = \{x \in L\colon x \triangleleft a\}$. We have to show that $k(a \vee a^*) = k(a) \vee k(a^*)$ for any $a \in L$. Suppose that $x \in k(a \vee a^*)$. Then $x \triangleleft a \vee a^*$. Further $x = (x \wedge a) \vee (x \wedge a^*)$. Now $x \wedge a \leqslant a$, $x \wedge a \leqslant x \triangleleft a \vee a^*$ implies $x \wedge a \triangleleft a \vee a^*$, which by virtue of the condition satisfied by $\triangleleft$ implies $x \wedge a \triangleleft a$. Furthermore $x \triangleleft a \vee a^*$ implies $x \triangleleft a^* \vee a^{**}$, since $a \leqslant a^{**}$. Hence $x \wedge a^* \triangleleft a^* \vee a^{**}$. Since $x \wedge a^* \leqslant a^*$, by the condition satisfied by $\triangleleft$ again, we have $x \wedge a^* \triangleleft a^*$. Thus $x \in k(a) \vee k(a^*)$. The reverse inclusion being clear, this proves that $k(a \vee a^*) = k(a) \vee k(a^*)$. $\square$

In view of the isomorphism between $K(L)$ and $S(L)$ mentioned above, the two propositions above imply

Proposition 3.9. *A compactification* (M, h) *of a frame* L *is perfect iff its associated strong inclusion* $\triangleleft$ *satisfies*

$$x \leqslant y,\ x \triangleleft y \vee y^* \Rightarrow x \triangleleft y \text{ for all } x, y \in L.$$

Remark 3.10. The above proof shows in effect that (M, h) is perfect with respect to $y \in L$ iff whenever $x \leqslant y$, $x \triangleleft y \vee y^*$ then $x \triangleleft y$.

Given an arbitrary compactification (M, h) of L, we do not in general expect its right adjoint r to preserve disjoint binary joins. However, if elements $u, v \in L$ are not just disjoint but such that $u \triangleleft v^*$ then $r(u \vee v) = r(u) \vee r(v)$ always holds as we show below.

Proposition 3.11. *Let* $h\colon M \to L$ *be a compactification of* L, $\triangleleft$ *the induced strong inclusion. If* $u, v \in L$ *and* $u \triangleleft v^*$ *then* $r(u \vee v) = r(u) \vee r(v)$

Proof. Since $u \triangleleft v^*$, we have $r(u) \prec r(v^*)$ and hence $r(u)^* \vee r(v)^* = e$. Thus $r(u^*) \vee r(v^*) = e$, since $r(u)^* = r(u^*)$ by Lemma 3.4. Hence

$$\begin{aligned} r(u \vee v) &= r(u \vee v) \wedge [r(u^*) \vee r(v^*)] \\ &= [r(u \vee v) \wedge r(u^*)] \vee [r(u \vee v) \wedge r(v^*)] \\ &= r((u \vee v) \wedge u^*) \vee r((u \vee v) \wedge v^*) \\ &= r(v \wedge u^*) \vee r(u \wedge v^*) \\ &\leqslant r(u) \vee r(v) \end{aligned}$$

proving the non-trivial inequality. □

4. Rim-compact frames and the Freudenthal compactification

We now introduce the concept of a rim-compact frame. This is the frame analogue of the well-known concept of rim-compactness for topological spaces which is that the space possesses a basis for its topology consisting of open sets with compact frontiers.

Definition 4.1. A regular frame L is called rim-compact if each $a \in L$ is a join of elements u such that $\uparrow(u \vee u^*)$ is compact

Remark 4.2. Rim-compact spaces (also called peripherally (bi)compact spaces) are the Hausdorff topological spaces having a basis for the topology consisting of open sets with compact frontier (see [11]). Let X be a topological space. For $U \in \mathscr{O}X$, $\operatorname{Fr}_X U = \operatorname{Cl}_X U \setminus U$. Now for any $V \in \mathscr{O}X$, $\uparrow V \cong \mathscr{O}(X \setminus V)$ as frames. Thus we have that $\uparrow U \cup U^*$ is compact $\Leftrightarrow \mathscr{O}(X \setminus (U \cup U^*)$ is compact $\Leftrightarrow \mathscr{O}((X \setminus U) \cap (X \setminus U^*))$ is compact $\Leftrightarrow \mathscr{O}((X \setminus U) \cap \operatorname{Cl}_X U)$ is compact $\Leftrightarrow \mathscr{O}(\operatorname{Fr}_X U)$ is compact $\Leftrightarrow \operatorname{Fr}_X U$ is compact. Therefore X is rim-compact as a topological space $\Leftrightarrow \mathscr{O}X$ is rim-compact as a frame.

We recall that the functors Σ and $\mathscr{O}$ induce a dual equivalence between the category of spatial frames and the category of sober topological spaces. Since every rim-compact space is sober and $\mathscr{O}X$ is rim-compact for such spaces, the functor $\mathscr{O}$ embeds the category of rim-compact spaces into the category of rim-compact frames. Therefore we may view the category of rim-compact frames (strictly speaking the rim-compact locales) as a generalization of the category of rim-compact spaces.

Definition 4.3. Let L be a rim-compact frame. A π-compact basis B for L is a basis B for L such that

(1) $a \in B \Rightarrow \uparrow(a \vee a^*)$ is compact,
(2) $a \in B \Rightarrow a^* \in B$,
(3) $a, b \in B \Rightarrow a \wedge b \in B$ and $a \vee b \in B$.

Remark 4.4. Let L be a rim-compact frame. Observe that L always has at least one π-compact basis: Indeed, let B be the basis for L consisting of all elements b such that $\uparrow(b \vee b^*)$ is compact. We have to show (2) and (3) in the above definition. Let $a \in B$. Since $a \vee a^* \leqslant a^* \vee a^{**}$ we have $\uparrow(a^* \vee a^{**})$ compact since $\uparrow(a \vee a^*)$ is, proving (2).

For (3) let $a, b \in B$. We have to show $\uparrow((a\wedge b)\vee(a\wedge b)^*), \uparrow((a\vee b)\vee(a\vee b)^*)$ compact. First we note the easily proved fact that $\uparrow c$ and $\uparrow d$ are compact if and only if $\uparrow(c\wedge d)$ is compact. Now $(a \wedge b) \vee (a \wedge b)^* = (a \vee (a \wedge b)^* \wedge (b \vee (a \wedge b)^*) \geqslant (a \vee a^*) \wedge (b \vee b^*)$. Hence $\uparrow((a \wedge b) \vee (a \wedge b)^*)$ is compact since $\uparrow(a \vee a^*) \wedge (b \vee b^*)$ is compact by the above note.

Also $((a\vee b)\vee(a\vee b)^*) = (a\vee b)\vee(a^*\wedge b^*) = (a\vee b\vee a^*)\wedge(a\vee b\vee b^*) \geqslant (a\vee a^*)\wedge(b\vee b^*)$ and so $\uparrow((a \vee b) \vee (a \vee b)^*)$ is compact as well.

Lemma 4.5. *Let L be rim-compact and let B be a π-compact basis for L. If $w \in L$ and $u \in B$ with $w \vee u = e$, then there exists $v \in B$ such that $v \prec u$ and $w \vee v = e$.*

Proof. Using regularity and the fact that B is a basis for L, we have $w = \bigvee x(x \prec w, x \in B)$. Then $u \vee \bigvee x(x \prec w, x \in B) = e$ and hence $u \vee u^* \vee \bigvee x(x \prec$

$w, x \in B) = e$. Since $\uparrow u \vee u^*$ is compact, we can find $x_i \in B$, $x_i \prec w$ for $i = 1, 2, \ldots, n$ (say) such that $u \vee u^* \vee \bigvee_{i=1}^{n} x_i = e$. Putting $x = \bigvee x_i$ $(i = 1, 2, \ldots, n)$, we have $x \in B$, $x \prec w$ and $u \vee u^* \vee x = e$. Let $v = u \wedge x^*$. Then $v \in B$, and furthermore $w \vee v = w \vee (u \wedge x^*) = (w \vee u) \wedge (w \vee x^*) = e \wedge e = e$. Also, $v \prec u$: $v \wedge (u^* \vee x) = (v \wedge u^*) \vee (v \wedge x) = (u \wedge x^* \wedge u^*) \vee (v \wedge x) = 0$ and $u \vee (u^* \vee x) = e$. □

Proposition 4.6. *Let B be a π-compact basis for a rim-compact frame L. Define $\lhd$ on L by: $a \lhd b \Leftrightarrow$ there exists $u \in B$ such that $a \prec u \prec b$. Then $\lhd$ is a strong inclusion on L.*

Proof. (i) $x \leqslant a \lhd b \leqslant y \Rightarrow x \lhd y$: Find $u \in B$ such that $a \prec u \prec b$. Then, of course, $x \prec u \prec y$ and so $x \lhd y$.

(ii) $\lhd$ is a sublattice of $L \times L$: Condition 3 together with 2 of the definition of a π-compact basis gives us $0, e \in B$, and then of course $0 \lhd 0$, $e \lhd e$. Furthermore, the implications $x \lhd a, b$ implies $x \lhd a \wedge b$, and $x, y \lhd a$ implies $x \vee y \lhd a$ follow from the properties of the rather below relation $\prec$ and the fact that B is closed under finite meet and finite joins.

(iii) $x \lhd a$ implies $x \prec a$ trivially.

(iv) Now suppose $x \lhd y$. Then there exists $u \in B$ such that $x \prec u \prec y$. Now $x^* \vee u = e$, and so by the above lemma, there exists $v \in B$, $v \prec u$ such that $x^* \vee v = e$. Hence $x \prec v \prec u \prec y$. Similarly we can get $w \in B$ such that $x \prec v \prec w \prec u \prec y$. Thus $x \lhd w \lhd y$.

(v) Also, $x \lhd a$ implies $a^* \lhd x^*$ follows from the properties of $\prec$ and the fact that B is closed under pseudocomplementation.

(vi) Now for any $a \in L$, $a = \bigvee x(x \prec a, x \in B)$. For $x \in B$ and $x \prec a$ we have $x^* \in B$ and $a \vee x^* = e$. By the above lemma there exists $v \in B$, $v \prec x^*$ and $a \vee v = e$. Hence $x \prec v^* \prec a$ with $v^* \in B$. Thus $x \lhd a$. Thus $a = \bigvee x(x \lhd a)$. □

Let L be any rim-compact frame, and let B be any π-compact basis for L. Let $\gamma_B L$ denote the compactification of L associated with the strong inclusion $\lhd_B$ given as in the above proposition, i.e. $a \lhd_B b \Leftrightarrow$ there exists $u \in B$ such that $a \prec u \prec b$. We then have the following:

Proposition 4.7. *Let $\gamma_B L$ be the compactification associated with the π-compact basis B of a rim-compact frame L, and let (M, h) be any compactification of L such that $(\gamma_B L, \vee) \leqslant (M, h)$. Then (M, h) is perfect with respect to every element of B.*

Proof. By Remark 3.10 we have to show that for each $u \in B$, whenever $x \leqslant u$ and $x \lhd u \vee u^*$, then $x \lhd u$. Here $\lhd$ is induced by (M, h). For $x \leqslant u$, $x \lhd u \vee u^*$, we have $x \prec u \vee u^*$ and thus $x^* \vee u \vee u^* = e$. Hence $x^* \vee u = e$, since $x \leqslant u$. By Lemma 4.5

there exists $v \in B$, $v \prec u$ such that $x^* \vee v = e$. Then $x \prec v \prec u$ with $v \in B$. Thus $x \triangleleft_B u$, and hence $x \triangleleft u$, since $(\gamma_B L, \vee) \leqslant (M, h)$. $\square$

Remark 4.8. Note in particular that $(\gamma_B L, \bigvee)$ is a compactification of a rim-compact L which is perfect with respect to every element of B. It need not be perfect with respect to every element of L, and consequently need not be a perfect compactification.

Let us call a compactification (M, h) of a rim-compact L a π-compactification of L if there exists a π-compact basis B of L such that $(M, h) \cong (\gamma_B L, \vee)$. Of independent interest is that such compactifications of L possess a base intimately connected with the given π-compact base for L, as we show below.

Proposition 4.9. *Let $(\gamma_B L, \vee)$ be the π-compactification of the rim-compact frame L with π-compact basis B. Let $k\colon L \to \gamma_B L$ be the right adjoint of $\vee\colon \gamma_B L \to L$, i.e. $k(a) = \{x \in L\colon x \triangleleft_B a\}$. Then $k(B) = \{k(u)\colon u \in B\}$ is a basis for $\gamma_B L$.*

Proof. For any $J \in \gamma_B L$, $J = \bigcup k(a)(a \in J)$. Since B is a basis for L, we have for each $a \in J$, $a = \bigvee u(u \in B, u \leqslant a)$. We shall show $k(a) = \bigvee k(u)(u \in B, u \leqslant a)$. Let $x \in k(a)$. Then $x \triangleleft_B a$ and thus, since $\triangleleft_B$ interpolates, there exists $c \in L$ such that $x \triangleleft_B c \triangleleft_B a$. Hence we can find $u, v \in B$ such that $x \prec u \prec c \prec v \prec a$. Thus $k(a) = \bigvee k(u)(u \in B, u \leqslant a)$, and we are done. $\square$

Referring to an earlier remark, it would be nice if there were a π-compact basis B for a rim-compact L for which $\gamma_B L$ is perfect with respect to every element of L and not just to those elements in B. This is indeed the case, as we show below, if we take B to consist of the totality of all elements u of L such that $\uparrow u \vee u^*$ is compact. (Note that such a B is a π-compact basis of L as we showed earlier.)

Denote this compactification with the above mentioned basis B by $(\gamma L, \vee)$. We call this the *Freudenthal compactification* of the rim-compact frame L.

Proposition 4.10. *The Freudenthal compactification γL of L is perfect.*

Proof. Let $u \in L$ be arbitrary, $x \leqslant u$, $x \triangleleft u \vee u^*$, where $\triangleleft$ is the strong inclusion associated with B mentioned above. We must show $x \triangleleft u$. Now find v such that $\uparrow v \vee v^*$ is compact and $x \prec v \prec u \vee u^*$. Let $w = v \wedge u$.

Then $x \prec w$. Indeed $x \leqslant u$, $x \prec u \vee u^*$ implies $x^* \vee u \vee u^* = e$ and hence $x^* \vee u = e$. Thus $x \prec u$, and since $x \prec v$ as well, we have $x \prec u \wedge v = w$. Furthermore $w \prec u$: Find t such that $v \wedge t = 0$, $t \vee u \vee u^* = e$. Then $w \wedge (t \vee u^*) = (w \wedge t) \vee (w \wedge u^*) = (v \wedge u \wedge t) \vee (v \wedge u \wedge u^*) = 0$. Thus $w \prec u$, with a separating element $t \vee u^*$.

We claim that $v \vee v^* \leqslant w \vee w^*$: Clearly $v^* \leqslant w^*$ since $w \leqslant v$. Hence $v^* \leqslant w \vee w^*$. Also, $v \prec u \vee u^*$ implies $v = (v \wedge u) \vee (v \wedge u^*) = w \vee (v \wedge u^*) \leqslant w \vee u^* \leqslant w \vee w^*$. Thus $v \vee v^* \leqslant w \vee w^*$ and hence $\uparrow w \vee w^*$ is compact. $\square$

5. Recovering the Freudenthal compactification for spaces

1. We first show that if we assume the Boolean ultrafilter theorem (abbreviated by BUT) then we can recover the Samuel compactification of a uniform space X from the Samuel compactification of the corresponding uniform frame $\mathscr{O}X$. Let us first recall that the Samuel compactification of a uniform space X is the completion of the totally bounded reflection of X. Now for the details. A *uniformity* on L is a collection $\mathscr{U}$ of covers of L which is a filter relative to $\leqslant$, satisfies the star-refinement property and is admissible. $\mathscr{U} \subseteq \mathrm{Cov}(L)$ is said to have the *star-refinement property* if for each $A \in \mathscr{U}$, there exists a $B \in \mathscr{U}$ such that $B^* \leqslant A$, i.e. $\{Bb\colon\ b \in B\} \leqslant A$, where $Bb = \bigvee x(x \in b, x \wedge b \neq 0)$. The collection $\mathscr{U} \subseteq \mathrm{Cov}(L)$ is said to be *admissible* if for each $a \in L$, $a = \bigvee\{x \in L\colon\ x \triangleleft a\}$. The expression $x \triangleleft a$ is read as "x is strongly below a" and means that $Cx \leqslant a$ for some $C \in \mathscr{U}$. A *uniform frame* is a frame L together with a specified uniformity $\mathscr{U}$, members of which are called the uniform covers of L.

For a uniform frame $(L, \mathscr{U})$ the Samuel compactification of L is the frame $\mathscr{R}L$ together with the join map $\vee\colon\ \mathscr{R}L \to L$, where $\mathscr{R}L$ is the frame of all *regular* ideals of L (see [4]). An ideal J is said to be regular whenever $x \in J$ implies there exists $y \in J$ such that $x \triangleleft y$.

We recall also the construction of the *Cauchy spectrum* ΨL of a uniform frame $(L, \mathscr{U})$ due to [4], which is the uniform space whose points are the regular Cauchy filters of L endowed with the uniformity generated by the covers

$$\Psi_A = \{\Psi_a\colon\ a \in A\},\ \Psi_a = \{F \in \Psi L\colon\ a \in F\} \quad (A \in \mathscr{U}).$$

If (X, μ) is a uniform space then, as implicit in [4], the Cauchy spectrum of the corresponding uniform frame $\mathscr{O}X$ is just the uniform completion of the space X.

Now we come to recovering the Samuel compactification for spaces. Let X be a uniform space, and denote by X_* its totally bounded reflection (we do not indicate their uniformities). Let CX and CX_* denote their respective uniform space completions, and let $\mathscr{O}X$ and $\mathscr{O}X_*$ denote the respective uniform frames. The Samuel compactification of X is then

$$CX_* = \Psi\mathscr{O}X_* \cong \Sigma(C\mathscr{O}X_*) = \Sigma(\mathscr{R}\mathscr{O}X);$$

making use of the fact that for a uniform frame L with L_* its totally bounded coreflection and CL the completion of L, we have from [4] that $\Psi L \cong \Sigma(CL)$ and $\mathscr{R}L = CL_*$ where $\mathscr{R}L$ is the Samuel compactification of L. It should be pointed out that the above equation holds even without any assumptions of choice principles.

However, to infer the compactness of CX_* requires BUT, either for using the well-known result in frame theory that under BUT the spectrum of a compact regular frame is compact Hausdorff, or for using the, perhaps, less well-known result that BUT is equivalent to the result for uniform spaces X, namely that X is compact if and only if X is complete and totally bounded (see Schechter [10]).

2. Now let X be a rim-compact Hausdorff space. We now show that we can recover the Freudenthal compactification of such a space from the Freudenthal compactification of the frame $\mathscr{O}X$ as we have defined it. Note that if X is rim-compact then it is completely regular (hence regular), being a subspace of a compact Hausdorff space. Thus $\mathscr{O}X$ is rim-compact and regular as a frame, which are the conditions for the general L in our situation.

We recall the Freudenthal compactification of X as given in Isbell [7]. Let δ be the (Efremovič) proximity on X given by: $A \neg\delta B \Leftrightarrow \exists$ compact K such that $X \setminus K = G \cup H$; G, H non-empty disjoint open sets in X with $A \subseteq G$ and $B \subseteq H$. Then δ is induced by a unique totally bounded uniformity μ_δ on X. The completion of the uniform space (X, μ_δ) is then the Freudenthal compactification of X.

Any proximity relation δ can be described equivalently in terms of the relation $\ll$ of strong inclusion according to the formula: $A \neg\delta B \Leftrightarrow A \ll X \setminus B$. That this is a strong inclusion on $\mathscr{O}X$ (in the sense introduced by Banaschewski in [3]) is well known or may be deduced from the theorems on the binary relation $\ll$ discussed in, for example, [13] or [12].

We now show that if $\triangleleft$ is the strong inclusion on $\mathscr{O}X$ defining the Freudenthal compactification of $\mathscr{O}X$ then $\triangleleft = \ll$. For this let $A, B \in \mathscr{O}X$ and suppose $A \ll B$. Then there exists $C \in \mathscr{O}X$ such that $A \ll C \ll B$. Since $C \ll B$ we can find a compact K such that $X \setminus K = G \cup H$, with G, H non-empty open and disjoint in $\mathscr{O}X$ and $C \subseteq G$, $X \setminus B \subseteq H$. Now $G \ll B$ so $G \prec B$. Also $A \ll C \subseteq G$ implies $A \prec G$. Hence $A \prec G \prec B$. Also G disjoint from H implies that $\mathrm{Cl}(G) \setminus G \subseteq X \setminus (G \cup H) \subseteq K$, so $\mathrm{Fr}(G)$ is compact, proving that $A \triangleleft B$. On the other hand if $A \triangleleft B$ in $\mathscr{O}X$ then there exists $W \in \mathscr{O}X$ such that $A \prec W \prec B$ and $\mathrm{Fr}(W)$ is compact. Now $X \setminus \mathrm{Fr}(W) = X \setminus (\mathrm{Cl}(W) \setminus W) = W \cup (X \setminus \mathrm{Cl}(W))$ with $A \subseteq W$ and $X \setminus B \subseteq X \setminus \mathrm{Cl}(W)$. Hence $A \ll B$.

For the X we are considering, let μ_δ be the uniformity as above and let all the objects described below be given relative to this uniformity. Further, let FX denote the Freudenthal compactification. Then

$$FX = CX = \Psi \mathscr{O}X \cong \Sigma(C\mathscr{O}X) = \Sigma(\mathscr{R}\mathscr{O}X) = \Sigma(\gamma \mathscr{O}X),$$

which follows from (1) above except for the last equality, which holds because $\triangleleft = \ll$. Of course, as in (1), BUT has to be assumed here in order to guarantee that FX is compact.

6. Remainder of a frame compactification

For any compactification $h\colon M \to L$ we define the *remainder* of L in the compactification to be M/Θ where $\Theta = (\ker h)^*$, the pseudocomplement of $\ker h$ in the congruence lattice $\mathscr{C}M$ of M. We now show that for a rim-compact L the remainder of L in its Freudenthal compactification is zero-dimensional, i.e. it has a basis of complemented elements. We require first the following

Lemma 6.1. $\uparrow k(b \vee b^*) \overset{\vee_R}{\to} \uparrow b \vee b^*$ *is an isomorphism for all* $b \in B$*, where* $\vee_R$ *acts as* $\vee$ *on the sublocale* $\uparrow k(b \vee b^*)$ *of* $\gamma_B L$*.*

Proof. That $\vee_R$ is a dense onto frame homomorphism follows immediately since $\vee\colon \gamma_B L \to L$ is. Also $\uparrow k(b \vee b^*)$ and $\uparrow b \vee b^*$ are regular (being sublocales of regular locales), indeed compact regular ($\uparrow b \vee b^*$ from definition, whereas $\uparrow k(b \vee b^*)$ is a closed sublocale of compact $\gamma_B L$). Thus $\vee_R$ (being dense onto between compact regular frames) must be an isomorphism. □

Recall from Banaschewski [2] that for congruences Θ, Ψ of L we have $\Theta \to \Psi = \bigcap_{(a,b)\in\Theta^+} \nabla_a \vee \Delta_b \vee \Psi$. Of course $\Theta \to 0$ is Θ^* (in $\mathscr{C}L$), so that $\Theta^* = \bigcap_{(a,b)\in\Theta^+} \nabla_a \vee \Delta_b$. Here $\Theta^+ = \{(a,b) \in \Theta\colon\ a \leqslant b\}$.

Theorem 6.2. *The remainder of* L *in its Freudenthal compactification is zero-dimensional.*

Proof. We have to show that $\gamma L/(\ker \vee)^*$ is zero-dimensional. Let $\varphi\colon \gamma L \to \gamma L/(\ker \vee)^*$ be the quotient map. The collection $\{\varphi k(b)\colon\ b \in B\}$ is clearly a basis for $\gamma L/(\ker \vee)^*$ since, as we showed earlier, $k(B)$ is a basis for γL and φ is onto. We now show that each $\varphi k(b)$ is complemented.

Claim: $\varphi k(b) \vee \varphi k(b^*) = e$. In order to show this we need to show $\varphi k(b \vee b^*) = \varphi(L)$ since k preserves disjoint binary joins. Thus we have to show $(k(b \vee b^*), L) \in (\ker \vee)^*$. For this take $I, J\colon\ I \leqslant J$, $\bigvee I = \bigvee J$. We ought to show $(k(b \vee b^*), L) \in \nabla_I \vee \Delta_J$, i.e. to show $(k(b \vee b^*) \vee I) \cap J = (L \vee I) \cap J$, i.e. to show $(k(b \vee b^*) \vee I) \cap J = J$, i.e. to show $J \subseteq k(b \vee b^*) \vee I$. Now $\uparrow k(b \vee b^*) \overset{\vee_R}{\to} \uparrow b \vee b^*$ is an isomorphism, so is one-to-one. We have $\bigvee(k(b \vee b^*) \vee I) = b \vee b^* \vee \bigvee I = b \vee b^* \vee \bigvee J = \bigvee(k(b \vee b^*) \vee J)$ so by the fact that $\vee_R$ is one-to-one, we have $k(b \vee b^*) \vee I = k(b \vee b^*) \vee J$. Thus $J \subseteq k(b \vee b^*) \vee I$ as required.

Obviously $\varphi k(b) \wedge \varphi k(b^*) = \varphi(k(0)) = \varphi(0) = 0$. Hence the $\varphi k(b)$ are complemented, thus showing that the remainder is zero-dimensional. □

We can now obtain the analog in frames of the Freudenthal-Morita theorem for spaces [6], [9] appearing in [11], namely: Every peripherally bicompact space X may be imbedded in a bicompactum with zero-dimensional (in the sense of ind) annex.

Corollary 6.3. *Every rim-compact frame L has a compactification such that the remainder of L in it is zero-dimensional.*

We now show that the frame version of the Freudenthal-Morita Theorem implies the classical theorem for spaces. In order to see this consider the general situation of a space X as a subspace of the space Y. The embedding $i\colon X \hookrightarrow Y$ gives rise to the surjective frame homomorphism $\mathscr{O}i\colon \mathscr{O}Y \to \mathscr{O}X$ given by $U \rightsquigarrow U \cap X$. The kernel of this map is given by

$$\ker \mathscr{O}i = \{(U,V) \in \mathscr{O}Y \times \mathscr{O}Y\colon\ U \cap X = V \cap X\}.$$

A calculation of its pseudocomplement gives

$$\begin{aligned}(\ker \mathscr{O}i)^* &= \bigcap_{(U,V)\in\Phi^+} \nabla_U \vee \Delta_V \\ &= \{(G,H) \in \mathscr{O}Y \times \mathscr{O}Y\colon\ (G,H) \in \nabla_U \vee \Delta_V \\ &\quad\ \text{for all } U \subseteq V,\ U \cap X = V \cap X\} \\ &= \{(G,H) \in \mathscr{O}Y \times \mathscr{O}Y\colon\ (G \cap V) \cup U = (H \cap V) \cup U \\ &\quad\ \text{for all } U \subseteq V,\ U \cap X = V \cap X\} \\ &= \{(G,H) \in \mathscr{O}Y \times \mathscr{O}Y\colon\ (G \cap (V \setminus U) \cup U = (H \cap (V \setminus U) \cup U \\ &\quad\ \text{for all } U \subseteq V,\ U \cap X = V \cap X\} \\ &= \{(G,H) \in \mathscr{O}Y \times \mathscr{O}Y\colon\ (G \cap (V \setminus U) = (H \cap (V \setminus U) \\ &\quad\ \text{for all } U \subseteq V,\ U \cap X = V \cap X\}.\end{aligned}$$

We now show that in the case where Y is a T_1 space, the latter set is precisely the set of those $(G,H) \in \mathscr{O}Y \times \mathscr{O}Y$ such that $(G \cap (Y \setminus X) = H \cap (Y \setminus X)$. To this end assume $G \cap (V \setminus U) = H \cap (V \setminus U)$ for all $U \subseteq V$, $U \cap X = V \cap X$. To show $G \cap (Y \setminus X) = H \cap (Y \setminus X)$, take $p \in G \cap (Y \setminus X)$. Now put $U = Y \setminus \{p\}$, $V = Y$. Then U and V are open in Y, $U \subseteq V$ and $U \cap X = V \cap X$. Hence $G \cap (V \setminus U) = H \cap (V \setminus U)$, i.e. $G \cap \{p\} = H \cap \{p\}$. Since $p \in G$ we have $p \in H$ so that $p \in H \cap (Y \setminus X)$. By symmetry, we then have $G \cap (Y \setminus X) = H \cap (Y \setminus X)$. On the other hand suppose $G \cap (Y \setminus X) = H \cap (Y \setminus X)$. Take $U \subseteq V$, $U \cap X = V \cap X$. To show $G \cap (V \setminus U) = H \cap (V \setminus U)$, take $p \in G \cap (V \setminus U)$. Now since $U \cap X = V \cap X$ and $p \in V$, $p \notin U$ we cannot have $p \in X$. Thus $p \in Y \setminus X$, so $p \in G \cap (Y \setminus X)$ and hence $p \in H$. Thus $G \cap (V \setminus U) \subseteq H \cap (V \setminus U)$ and by symmetry we have equality. Hence for a T_1 space Y with $X \subseteq Y$ we have

$$(\ker \mathscr{O}i)^* = \{G,H) \in \mathscr{O}Y \times \mathscr{O}Y\colon\ G \cap (Y \setminus X) = H \cap (Y \setminus X)\}.$$

We have of course that

$$\mathscr{O}Y/(\ker \mathscr{O}i)^* \cong \mathscr{O}(Y \setminus X).$$

It follows from the above analysis that if Y is a T_1 space and $f\colon X \to Y$ is an embedding, then

$$\mathscr{O}Y/(\ker \mathscr{O}f)^* \cong \mathscr{O}(Y \setminus f(X)).$$

The Freudenthal-Morita Theorem for spaces can then be obtained in the following way. Consider a rim-compact Hausdorff space X. We have seen in Section 5 that assuming BUT the Freudenthal compactification FX can be given by $\Sigma\gamma(\mathscr{O}X)$. We now show

Proposition 6.4. *Under BUT the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{O}\Sigma\gamma(\mathscr{O}X) & \xrightarrow{\mathscr{O}\Sigma\vee} & \mathscr{O}\Sigma\mathscr{O}X \\ \uparrow{\scriptstyle \eta_{\gamma(\mathscr{O}X)}} & & \uparrow{\scriptstyle \eta_{\mathscr{O}X}} \\ \gamma(\mathscr{O}X) & \xrightarrow[\vee]{} & \mathscr{O}X \end{array}$$

Proof. $\gamma(\mathscr{O}X)$ is compact regular and hence spatial by BUT. Hence the map $\eta_{\gamma(\mathscr{O}X)}$ is an isomorphism. So is also $\eta_{\mathscr{O}X}$ of course. Also for any $J \in \gamma(\mathscr{O}X)$

$$(\mathscr{O}\Sigma\vee)\eta_{\gamma(\mathscr{O}X)}(J) = (\mathscr{O}\Sigma\vee)(\Sigma_J) = (\Sigma\vee)^{-1}(\Sigma_J) = \Sigma_{\bigcup J}$$

and

$$\eta_{(\mathscr{O}X)} \vee (J) = \eta_{(\mathscr{O}X)}\big(\bigcup J\big) = \Sigma_{\bigcup J}$$

so we are done. □

We also require the following result: If $h\colon L \to M$ is an onto frame homomorphism then $\Sigma h\colon \Sigma M \to \Sigma L$ is an embedding. To see this recall that Σh is continuous and one to one, and that Σh is an open map because for arbitrary $a \in M$, $(\Sigma h)(\Sigma_a) = \Sigma_{r(a)} \cap (\Sigma h)(\Sigma M)$, where r is the right adjoint of h.

In our situation the join map $\vee\colon \gamma(\mathscr{O}X) \to \mathscr{O}X$ is onto, so $\Sigma\vee\colon \Sigma\mathscr{O}X \to \Sigma\gamma(\mathscr{O}X)$ is an embedding. Since $FX \cong \Sigma\gamma(\mathscr{O}X)$ and $X \cong \Sigma\mathscr{O}X$ we have

$$\begin{aligned} \mathscr{O}(FX \setminus X) &\cong \mathscr{O}(\Sigma\gamma(\mathscr{O}X) \setminus (\Sigma\vee)(\Sigma\mathscr{O}X)) \\ &\cong \mathscr{O}\Sigma\gamma(\mathscr{O}X)/(\ker(\mathscr{O}\Sigma\vee))^* \\ &\cong \gamma(\mathscr{O}X)/(\ker \vee)^*, \end{aligned}$$

the last step following from the above proposition. Thus $\mathscr{O}(FX \setminus X)$ is zero-dimensional and thus $FX \setminus X$ is zero-dimensional.

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