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TWO VALUED MEASURE AND SOME NEW DOUBLE SEQUENCE SPACES IN 2-NORMED SPACES

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Abstract. The purpose of this paper is to introduce some new generalized double difference sequence spaces using summability with respect to a two valued measure and an Orlicz function in 2-normed spaces which have unique non-linear structure and to examine some of their properties. This approach has not been used in any context before.

Keywords: convergence, μ-statistical convergence, convergence in μ-density, condition (APO_2), 2-norm, 2-normed space, paranorm, paranormed space, Orlicz function, sequence space

MSC 2010: 40H05, 40C05

1. Introduction

The notion of summability of single sequences with respect to a two valued measure was introduced by Connor [3], [4] as a very interesting generalization of statistical convergence (see [9], [10], [21], [26], [30]). The notion of statistical convergence was further extended to double sequences independently by Moricz [19] and Mursaleen et al [20]. For more recent developments on double sequences one can consult the papers [5], [6], [7], [8], [1], [27] where more references can be found. In particular, very recently the first and third author investigated the summability of double sequences of real numbers with respect to a two valued measure and made many interesting observations [7] (see also [1] where the same has been investigated in an asymmetric metric space). The concept of 2-normed spaces was initially introduced by Gähler ([11], [12]) as a very interesting non-linear extension of the idea of usual normed

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linear spaces. Some initial studies on this structure can be seen from [11], [12], [13]. Recently a lot of interesting developments have occurred in 2-normed spaces in summability theory and related topics (see [14], [15], [25]).

In this article, in a natural way we first unite the approach of [7] with two norm and introduce the idea of summability of double sequences in 2-normed spaces using a two valued measure. Then using Orlicz functions, generalized double difference sequences and a two valued measure μ we introduce μ-statistical convergence of generalized double difference sequences with respect to an Orlicz function in 2-normed spaces. In this connection it should be mentioned that notable works involving the Orlicz function and the modulus function were done in [2], [17], [22], [24], [28]. We introduce and examine certain new double sequence spaces using the above tools as well as the 2-norm. This approach has not been considered in any context before.

2. Preliminaries

Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers, χ_A represents the characteristic function of $A \subseteq \mathbb{N}$ and $\mathbb{R}$ represents the set of all real numbers.

Recall that a set $A \subseteq \mathbb{N}$ is said to have the asymptotic density $d(A)$ if

$$d(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_A(j)$$

exists.

Definition 2.1 ([9], [30]). A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} \colon |x_n - \xi| \geqslant \varepsilon\}$.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense (see [23]):

A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be convergent to $\xi \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - \xi| < \varepsilon$ whenever $i, j \geqslant N_\varepsilon$. In this case we write $\lim\limits_{i,j\to\infty} x_{ij} = \xi$.

A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{ij}| < M$ for all $i, j \in \mathbb{N}$. That is, $\|x\|_{(\infty,2)} = \sup\limits_{i,j\in\mathbb{N}} |x_{ij}| < \infty$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and let $K(i,j)$ be the cardinality of the set $\{(m,n) \in K \colon m \leqslant i,\, n \leqslant j\}$. If the sequence $\{K(i,j)/(i\cdot j)\}_{i,j\in\mathbb{N}}$ has a limit in Pringsheim's sense then we say that K has double natural density, which is denoted by $d_2(K) = \lim\limits_{i,j\to\infty} K(i,j)/(i\cdot j)$.

Definition 2.2 ([19], [20]). A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ |x_{ij} - \xi| \geqslant \varepsilon\}$.

A statistically convergent double sequence of elements of a metric space (X, ϱ) is defined essentially in the same way ($\varrho(x_{ij}, \xi) \geqslant \varepsilon$ instead of $|x_{ij} - \xi| \geqslant \varepsilon$).

Throughout the paper μ will denote a complete $\{0,1\}$ valued finite additive measure defined on an algebra Γ of subsets of $\mathbb{N}\times\mathbb{N}$ that contains all subsets of $\mathbb{N}\times\mathbb{N}$ that are contained in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ and $\mu(A) = 0$ if A is contained in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ (see [7]).

Definition 2.3 ([7]). A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be μ-statistically convergent to $L \in \mathbb{R}$ if and only if for any $\varepsilon > 0$, $\mu(\{(i,j) \in \mathbb{N}\times\mathbb{N}\colon\ |x_{ij} - L| \geqslant \varepsilon\}) = 0$.

Definition 2.4 ([7]). A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in μ-density if there exists $A \in \Gamma$ with $\mu(A) = 1$ such that $\{x_{ij}\}_{(i,j)\in A}$ is convergent to L.

Definition 2.5 ([12]). Let X be a real vector space of dimension d, where $2 \leqslant d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\|\colon\ X\times X \to \mathbb{R}$ which satisfies

(i) $\|x,y\|$=0 if and only if x and y are linearly dependent;
(ii) $\|x,y\| = \|y,x\|$;
(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$, $\alpha \in \mathbb{R}$;
(iv) $\|x,y+z\| \leqslant \|x,y\|+\|x,z\|$. The ordered pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

As an example we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| =$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula $\|x,y\| = |x_1y_2 - x_2y_1|$, $x = (x_1,x_2)$, $y = (y_1,y_2)$. Recall that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x in X. Let $(X, \|\cdot,\cdot\|)$ be any 2-normed space and $S''(2-X)$ the set of all double sequences defined over the 2-normed space $(X, \|\cdot,\cdot\|)$. Clearly $S''(2-X)$ is a linear space under addition and scalar multiplication.

Recall ([16]) that an Orlicz function $M\colon\ [0,\infty) \to [0,\infty)$ is a continuous, convex and non decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently, the Orlicz function was used to define sequence spaces by Parashar and Choudhary ([22]) and others (see [2], [28]). An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x p(t)\,\mathrm{d}t$ where p is the known kernel of M, the right differential for $t \geqslant 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non decreasing and $p(t) \to \infty$ as $t \to \infty$. If convexity of the Orlicz function M is replaced by $M(x+y) \leqslant M(x)+M(y)$ then this function is called the modulus function, which was presented and discussed by Ruckle ([24]) and Maddox ([17]). Note that if M is an Orlicz function then $M(tx) \leqslant tM(x)$ for all t with $0 < t < 1$.

3. μ-statistical convergence and convergence in μ-density in 2-normed spaces

Definition 3.1. A double sequence $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to ξ in $(X, \|\cdot,\cdot\|)$ if for each $\varepsilon > 0$ and each $z \in X$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_{ij} - \xi, z\| < \varepsilon$ for all $i, j \geqslant n_\varepsilon$.

Definition 3.2. Let μ be a two valued measure on $\mathbb{N} \times \mathbb{N}$. A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be μ-statistically convergent to a point x in X if for each pre-assigned $\varepsilon > 0$ and for each $z \in X$, $\mu(A(z, \varepsilon)) = 0$ where $A(z, \varepsilon) = \{(i, j) \in \mathbb{N} \times \mathbb{N}\colon\ \|x_{ij} - x, z\| \geqslant \varepsilon\}$.

If a double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is μ-statistically convergent to a point x in a 2-normed space $(X, \|\cdot,\cdot\|)$ then we write

$$\mu\text{--}\lim_{i,j\to\infty} \|x_{ij} - x, z\| = 0$$

or

$$\mu\text{--}\lim_{i,j\to\infty} \|x_{ij}, z\| = \|x, z\|.$$

Here x is called the μ-statistical limit of the sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$.

Definition 3.3. Let μ be a two valued measure on $\mathbb{N} \times \mathbb{N}$. A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ of the points in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to $\xi \in X$ in μ-density if there exists a set $M \in \Gamma$ with $\mu(M) = 1$ such that $\{x_{ij}\}_{(i,j)\in M}$ is convergent to ξ in $(X, \|\cdot,\cdot\|)$.

We now give an example of a μ-statistically convergent double sequence in 2-normed spaces.

Example 3.1. Let μ be a two valued measure on $\mathbb{N} \times \mathbb{N}$ such that there is at least one $A \subseteq \mathbb{N} \times \mathbb{N}$ with $\mu(A) = 0$ which is not contained in any finite union of rows and columns of $\mathbb{N} \times \mathbb{N}$. Define the double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ in the 2-normed space $(X, \|\cdot,\cdot\|)$ by

$$x_{ij} = \begin{cases} (0, ij) & \text{if } (i, j) \in A, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Let $L = (0,0)$ and $z = (z_1, z_2)$. Then for every $\varepsilon > 0$ and $z \in X$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|x_{ij} - L, z\| \geqslant \varepsilon\} \subseteq A.$$

Thus

$$\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|x_{ij} - L, z\| \geqslant \varepsilon\}) = 0$$

for every $\varepsilon > 0$ and $z \in X$. This implies that

$$\mu\text{--}\lim_{i,j\to\infty} \|x_{ij}, z\| = \|L, z\|.$$

But it is noticeable that the double sequence is not convergent to L.

Similarly we can give non-trivial examples of double sequences which are convergent in μ-density in 2-normed spaces.

We next provide a proof of the fact that the μ-statistical limit operation for double sequences in a 2-normed space $(X, \|\cdot,\cdot\|)$ is linear with respect to summation and scalar multiplication.

Theorem 3.1. *Let μ be a two valued measure. For each $z \in X$,*

(i) *if $\mu\text{--}\lim_{i,j\to\infty} \|x_{ij}, z\| = \|x, z\|$ and $\mu\text{--}\lim_{i,j\to\infty} \|y_{ij}, z\| = \|y, z\|$ then*

$$\mu\text{--}\lim_{i,j\to\infty} \|x_{ij} + y_{ij}, z\| = \|x + y, z\|;$$

(ii) *if $\mu\text{--}\lim_{i,j\to\infty} \|x_{ij}, z\| = \|x, z\|$ then $\mu\text{--}\lim_{i,j\to\infty} \|ax_{ij}, z\| = \|ax, z\|$, $a \in \mathbb{R}$.*

Proof. (i) Let $\varepsilon > 0$ be given. Consider the following two sets: $A(\frac{1}{2}\varepsilon, z) = \{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \|x_{ij} - x, z\| \geqslant \frac{1}{2}\varepsilon\}$ and $B(\frac{1}{2}\varepsilon, z) = \{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \|y_{ij} - y, z\| \geqslant \frac{1}{2}\varepsilon\}$ for each $z \in X$. Then by hypothesis $\mu(A(\frac{1}{2}\varepsilon, z)) = 0$ and $\mu(B(\frac{1}{2}\varepsilon, z)) = 0$. Now $\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|x_{ij} + y_{ij} - (x+y), z\| \geqslant \varepsilon\} \subseteq \{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|x_{ij} - x, z\| \geqslant \frac{1}{2}\varepsilon\} \cup \{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|y_{ij} - y, z\| \geqslant \frac{1}{2}\varepsilon\}$. Therefore $\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|x_{ij} + y_{ij} - (x+y), z\| \geqslant \varepsilon\}) = 0$ and the result follows.

(ii) Let $\mu\text{--}\lim_{i,j\to\infty} \|x_{ij}, z\| = \|x, z\|$, $a \in \mathbb{R}$, $a \neq 0$. Now $\mu(\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \|x_{ij} - x, z\| \geqslant \varepsilon/|a|\}) = 0$ and from the definition of the 2-norm we have

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|ax_{ij} - ax, z\| \geqslant \varepsilon\} = \Big\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|x_{ij} - x, z\| \geqslant \frac{\varepsilon}{|a|}\Big\}$$

and so

$$\mu(\{(i,j) \in \mathbb{N} \times \mathbb{N} \colon \|ax_{ij} - ax, z\| \geqslant \varepsilon\} = 0.$$

Hence

$$\mu\text{-}\lim_{i,j\to\infty} \|ax_{ij}, z\| = \|ax, z\|$$

for every $z \in X$.

Similar observations are also true for μ-lim, i.e., the statistical limit operation in μ-density. $\square$

If $u = \{u_1, u_2, u_3, \ldots, u_d\}$ is a basis of the 2-normed space $(X, \|\cdot,\cdot,\|)$, then we have the following result.

Lemma 3.1. *Let μ be a two valued measure. A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is μ-statistically convergent to $x \in X$ if and only if $\mu\text{-}\lim\limits_{i,j\to\infty}\|x_{ij} - x, u_k\| = 0$ for every $k = 1,\ 2,\ 3,\ \ldots,\ d$.*

If C^2_μ and C^{*2}_μ denote respectively the sets of all double sequences in a 2-normed space $(X, \|\cdot,\cdot\|)$ which are μ-statistically convergent and convergent in μ-density in the 2-normed space $(X, \|\cdot,\cdot\|)$ then as in [7] we now consider the following condition.

(APO_2) (Additive property of null sets)

The measure μ is said to satisfy the condition (APO_2) if for every sequence $\{A_i\}_{i\in\mathbb{N}}$ of mutually disjoint μ-null sets (i.e. $\mu(A_i) = 0$ for all $i \in \mathbb{N}$) there exists a countable family of sets $\{B_i\}_{i\in\mathbb{N}}$ such that $A_i \Delta B_i$ is included in the union of a finite number of rows and columns of $\mathbb{N}\times\mathbb{N}$ for every $i \in \mathbb{N}$ and $\mu(B) = 0$ where $B = \bigcup\limits_{i\in\mathbb{N}} B_i$ (hence $\mu(B_i) = 0$ for every $i \in \mathbb{N}$).

Theorem 3.2. $C^2_\mu = C^{*2}_\mu$ *iff μ satisfies the condition* (APO_2).

Proof. The proof is parallel to the proof of the corresponding theorems in [7] and is omitted. $\square$

4. New double sequence spaces

Recall that a mapping $g\colon X \to \mathbb{R}$ is called a paranorm on X if it satisfies the following conditions:

(i) $g(\theta) = 0$ where θ is the zero element of the space;

(ii) $g(x) = g(-x)$;

(iii) $g(x + y) \leqslant g(x) + g(y)$;

(iv) $\lambda_n \to \lambda$ $(n \to \infty)$ and $g(x^n - x) \to 0$ $(n \to \infty)$ imply $g(\lambda_n x^n - \lambda x) \to 0$ $(n \to \infty)$ for all $x, y \in X$ ([18], see also [25]). The ordered pair (X, g) is called a paranormed space with respect to the paranorm g.

Now we first define the following sequence space.

Definition 4.1. Let $p = \{p_{ij}\}_{i,j\in\mathbb{N}}$ be a sequence of non-negative real numbers. $l''(2-p) = \Big\{x \in S''(2-X)\colon \sum\limits_{s,t\in\mathbb{N}} \|x_{st}, z\|^{p_{st}} < \infty, \forall\, z \in X\Big\}$.

We now state an inequality which will be used throughout our study: If $\{p_{ij}\}_{i,j\in\mathbb{N}}$ is a bounded double sequence of non-negative real numbers and $\sup\limits_{i,j\in\mathbb{N}} p_{ij} = H$ and $D = \mathrm{Max}\{1, 2^{H-1}\}$, then

$$|a_{ij} + b_{ij}|^{p_{ij}} \leqslant D\{|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}}\}$$

for all i, j, and $a_{ij}, b_{ij} \in \mathbb{C}$, the set of all complex numbers. Also,

$$|a|^{p_{ij}} \leqslant \mathrm{Max}\{1, |a|^H\}$$

for all $a \in \mathbb{C}$.

Lemma 4.1. *The sequence space $l''(2-p)$ is a linear space.*

Proof. The proof is parallel to the proof of Lemma 3.1 in [25] and so is omitted. □

Theorem 4.1. *$l''(2-p)$ is a paranormed space with the paranorm defined by $g\colon l''(2-p) \to \mathbb{R}$, $g(x) = \Big(\sum\limits_{s,t\in\mathbb{N}} \|x_{st}, z\|^{p_{st}}\Big)^{1/M}$, where $\{p_{ij}\}_{i,j\in\mathbb{N}}$ is a bounded double sequence of non-negative real numbers and $\sup\limits_{i,j\in\mathbb{N}} p_{ij} = H$ and $M = \mathrm{Max}(1, H)$.*

Proof. The proof is modelled after the proof of Theorem 3.3 in [25] with necessary modifications.

(i) $g(\theta) = \Big(\sum\limits_{s,t\in\mathbb{N}} \|\theta_{st}, z\|^{p_{st}}\Big)^{1/M} = 0.$

(ii) $g(-x) = \Big(\sum\limits_{s,t\in\mathbb{N}} \|-x_{st}, z\|^{p_{st}}\Big)^{1/M} = \Big(\sum\limits_{s,t\in\mathbb{N}} |-1|\|x_{st}, z\|^{p_{st}}\Big)^{1/M} = g(x).$

(iii) Using the well-known inequalities

$$\begin{aligned} g(x+y) &= \Big(\sum_{s,t\in\mathbb{N}} \|x_{st} + y_{st}, z\|^{p_{st}}\Big)^{1/M} \\ &\leqslant \Big(\sum_{s,t\in\mathbb{N}} (\|x_{st}, z\|^{p_{st}/M})^M\Big)^{1/M} + \Big(\sum_{s,t\in\mathbb{N}} (\|y_{st}, z\|^{p_{st}/M})^M\Big)^{1/M} \\ &= g(x) + g(y). \end{aligned}$$

(iv) Let $\lambda^n \to \lambda$ as $n \to \infty$ and let $g(x^n - x) \to 0$ as $n \to \infty$, where $x^n = \{x_{ij}^n\}_{i,j\in\mathbb{N}}$ and $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$. Then using Minkowski's inequalities (see [29])

$$\begin{aligned} g(\lambda^n x^n - \lambda x) &= \left(\sum_{s,t\in\mathbb{N}} \|\lambda^n x_{st}^n - \lambda x_{st}, z\|^{p_{st}}\right)^{1/M} \\ &\leqslant |\lambda^n|^{H/M} \left(\sum_{s,t\in\mathbb{N}} \|x_{st}^n - x_{st}, z\|^{p_{st}}\right)^{1/M} \\ &\quad + \left(\sum_{s,t\in\mathbb{N}} |\lambda^n - \lambda| \|x_{st}, z\|^{p_{st}}\right)^{1/M}. \end{aligned}$$

In this inequality, the first term of the right-hand side tends to zero because $g(x^n - x) \to 0$ as $n \to \infty$. On the other hand, since $\lambda^n \to \lambda$ as $n \to \infty$, the second term also tends to zero by Lemma 5.1. $\square$

Let $\Lambda = \{\lambda_m\}_{m\in\mathbb{N}}$ and $\upsilon = \{\upsilon_n\}_{n\in\mathbb{N}}$ be non decreasing sequences of positive real numbers such that each tends to ∞ and

$$\lambda_{m+1} \leqslant \lambda_m + 1, \quad \lambda_1 = 0$$

and

$$\upsilon_{n+1} \leqslant \upsilon_n + 1, \quad \upsilon_1 = 0.$$

The generalized double de la Valée-Pousin mean is defined by

$$t_{mn}(x) = \frac{1}{\lambda_m \upsilon_n} \sum_{i\in J_m} \sum_{j\in K_n} x_{ij}$$

where $J_m = [m - \lambda_m + 1, m]$ and $K_n = [n - \upsilon_n + 1, n]$. Writing $I_{mn} = J_m \times K_n$ and $\lambda_{mn}^2 = \lambda_m \upsilon_n$ we can write t_{mn} as

$$t_{mn}(x) = \frac{1}{\lambda_{mn}^2} \sum_{(i,j)\in I_{mn}} x_{ij},$$

which will be used throughout the paper.

Definition 4.2. Suppose also that as before μ is a two valued measure on $\mathbb{N} \times \mathbb{N}$ and M is an Orlicz function and $(X, \|\cdot,\cdot\|)$ is a 2-normed space. Further, let $p = \{p_{ij}\}_{i,j\in\mathbb{N}}$ be a bounded sequence of positive real numbers. Now we introduce the following different types of sequence spaces, for all $\varepsilon > 0$:

$$W^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|) = \Big\{x \in S''(2-X)\colon\ \mu\Big((i,j) \in \mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st} - L}{\varrho}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \varepsilon\Big) = 0, \text{ for some } \varrho > 0 \text{ and } L \in X \text{ and each } z \in X\Big\},$$

$$W_0^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|) = \Big\{x \in S''(2-X)\colon\ \mu\Big((i,j) \in \mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \varepsilon\Big) = 0, \text{ for some } \varrho > 0 \text{ and each } z \in X\Big\},$$

$$W_{\infty}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|) = \Big\{x \in S''(2-X)\colon \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}}\frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho}, z\Big\|\Big)\Big]^{p_{st}} \leqslant k, \text{ for some } k > 0, \text{ for some } \varrho > 0 \text{ and each } z \in X\Big\},$$

$$W_{\infty}^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|) = \Big\{x \in S''(2-X)\colon\ \exists\, k > 0,\ \mu\Big(\Big\{(i,j) \in \mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho}, z\Big\|\Big)\Big]^{p_{st}} \geqslant k\Big\}\Big) = 0, \text{ for some } \varrho > 0 \text{ and each } z \in X\Big\},$$

where $I_{ij} = J_i \times K_j$, $\Lambda^2 = \{\lambda_m \upsilon_n\}_{m,n\in\mathbb{N}}$ and Δ^m denotes the generalized m-th order difference, i.e.

$$\Delta(x) = \{x_{j+1,k+1} + x_{jk} - x_{j,k+1} - x_{j+1,k}\}_{j,k\in\mathbb{N}}$$

and

$$\Delta^m(x) = \Delta(\Delta^{m-1}(x)) \quad \text{for } m > 1.$$

We now have

Theorem 4.2. *$W^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|)$, $W_0^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|)$ and $W_{\infty}^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|)$ are linear spaces. Here $(X, \|\cdot,\cdot\|)$ is a 2-normed space.*

P r o o f. We shall prove the theorem for $W_0^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|)$ while the others can be proved similarly. Let $\varepsilon > 0$ be given. Assume that $x, y \in W_0^{\mu}(\Lambda^2, M, \Delta^m, p, \|\cdot,\cdot\|)$ and $\alpha, \beta \in \mathbb{R}$, where $x = \{x_{ij}\}_{i,j\in\mathbb{N}}$ and $y \in \{y_{ij}\}_{i,j\in\mathbb{N}}$. Further, let $z \in X$. Then

$$\mu\Big(\Big\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho_1}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \varepsilon\Big\}\Big) = 0 \tag{4.1}$$

for some $\varrho_1 > 0$ and

$$\mu\Big(\Big\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m y_{st}}{\varrho_2}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \varepsilon\Big\}\Big) = 0 \tag{4.2}$$

for some $\varrho_2 > 0$.

Since $\|\cdot,\cdot\|$ is a 2-norm, Δ^m is linear, therefore the following inequality holds:

$$\begin{aligned}
&\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m(\alpha x_{st} + \beta y_{st})}{|\alpha|\varrho_1 + |\beta|\varrho_2}, z\Big\|\Big)\Big]^{p_{st}} \\
&\leqslant D\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[\frac{|\alpha|\varrho_1}{|\alpha|\varrho_1 + |\beta|\varrho_2} M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho_1}, z\Big\|\Big)\Big]^{p_{st}} \\
&\quad + D\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[\frac{|\beta|\varrho_2}{|\alpha|\varrho_1 + |\beta|\varrho_2} M\Big(\Big\|\frac{\Delta^m y_{st}}{\varrho_2}, z\Big\|\Big)\Big]^{p_{st}} \\
&\leqslant DF\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho_1}, z\Big\|\Big)\Big]^{p_{st}} + DF\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m y_{st}}{\varrho_2}, z\Big\|\Big)\Big]^{p_{st}},
\end{aligned}$$

where $F = \operatorname{Max}\{1, [|\alpha|\varrho_1/(|\alpha|\varrho_1 + |\beta|\varrho_2)]^H, [|\beta|\varrho_2/(|\alpha|\varrho_1 + |\beta|\varrho_2)]^H\}$, and $D = \operatorname{Max}\{1, 2^{H-1}\}$ as defined before.

From the above inequality we get

$$\begin{aligned}
&\Big\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m(\alpha x_{st} + \beta y_{st})}{|\alpha|\varrho_1 + |\beta|\varrho_2}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \varepsilon\Big\} \\
&\quad \subseteq \Big\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon DF\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho_1}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \frac{\varepsilon}{2}\Big\} \\
&\qquad \cup \Big\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon DF\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \Big[M\Big(\Big\|\frac{\Delta^m y_{st}}{\varrho_2}, z\Big\|\Big)\Big]^{p_{st}} \geqslant \frac{\varepsilon}{2}\Big\}.
\end{aligned}$$

Hence (4.1) and (4.2) yield the required result. $\square$

Theorem 4.3. *For any fixed $(i,j)\in\mathbb{N}\times\mathbb{N}$, $W_\infty^\mu(\Lambda^2,M,\Delta^m,p,\|\cdot,\cdot\|)$ is a paranormed space with respect to the paranorm $g_{ij}\colon X\to\mathbb{R}$, defined by*

$$g_{ij}(x)=\inf_{z\in X}\sum_{(s,t)\in I_{ij}}\|x_{st},z\| + \inf\Big\{\varrho^{p_{ij}/H}\colon \varrho>0 \text{ s.t. } \sup_{(s,t)\in\mathbb{N}\times\mathbb{N}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\leqslant 1,\ \forall\, z\in X\Big\}.$$

Proof. The identities $g_{ij}(\theta)=0$ and $g_{ij}(-x)=g_{ij}(x)$ are easy to prove. So we omit them.

(iii) Let us take $x=\{x_{ij}\}_{i,j\in\mathbb{N}\times\mathbb{N}}$ and $y=\{y_{ij}\}_{i,j\in\mathbb{N}\times\mathbb{N}}$ in $W_\infty^\mu(\Lambda^2,M,\Delta^m,p,\|\cdot,\cdot\|)$. Let us construct the following sets:

$$A(x)=\Big\{\varrho>0\colon \sup_{(s,t)\in\mathbb{N}\times\mathbb{N}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\leqslant 1,\ \forall\, z\in X\Big\}$$

and

$$A(y)=\Big\{\varrho>0\colon \sup_{(s,t)\in\mathbb{N}\times\mathbb{N}}\Big[M\Big(\Big\|\frac{\Delta^m y_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\leqslant 1,\ \forall\, z\in X\Big\}.$$

Let $\varrho_1\in A(x)$ and $\varrho_2\in A(y)$ and $\varrho_0=\varrho_1+\varrho_2$. Then

$$M\Big(\Big\|\frac{\Delta^m(x_{st}+y_{st})}{\varrho_0},z\Big\|\Big) \leqslant \frac{\varrho_1}{\varrho_1+\varrho_2}M\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho_1},z\Big\|\Big)+\frac{\varrho_2}{\varrho_1+\varrho_2}M\Big(\Big\|\frac{\Delta^m y_{st}}{\varrho_2},z\Big\|\Big).$$

Thus

$$\sup_{(s,t)\in\mathbb{N}\times\mathbb{N}} M\Big(\Big\|\frac{\Delta^m(x_{st}+y_{st})}{\varrho_0},z\Big\|\Big)\leqslant 1.$$

Therefore

$$\begin{aligned} g_{ij}(x+y)\leqslant{}& \inf_{z\in X}\sum_{(s,t)\in I_{ij}}\|x_{st}+y_{st},z\| \\ &+\inf\{(\varrho_1+\varrho_2)^{p_{ij}/H}\colon \varrho_1\in A(x),\ \varrho_2\in A(y)\} \\ \leqslant{}& \inf_{z\in X}\sum_{(s,t)\in I_{ij}}\|x_{st},z\|+\inf\{\varrho_1^{p_{ij}/H}\colon \varrho_1\in A(x)\} \\ &+\inf_{z\in X}\sum_{(s,t)\in I_{ij}}\|y_{st},z\|+\inf\{\varrho_2^{p_{ij}/H}\colon \varrho_2\in A(y)\} \\ ={}& g_{ij}(x)+g_{ij}(y). \end{aligned}$$

(iv) Let $\sigma^m \to \sigma$ as $m \to \infty$, where $\sigma, \sigma^m \in \mathbb{C}$ and let $g_{ij}(x^m - x) \to 0$ as $m \to \infty$, where $x^m = \{x^m_{pq}\}_{p,q\in\mathbb{N}}$ and $x = \{x_{pq}\}_{p,q\in\mathbb{N}}$. Let

$$A(x^m) = \Big\{\varrho_m > 0\colon \sup_{s,t\in\mathbb{N}} \Big[M\Big(\Big\|\frac{\Delta^m x^m_{st}}{\varrho_m}, z\Big\|\Big)\Big]^{p_{st}} \leqslant 1,\ \forall\, z \in X\Big\},$$
$$A(x^m - x) = \Big\{\varrho'_m > 0\colon \sup_{s,t\in\mathbb{N}} \Big[M\Big(\Big\|\frac{\Delta^m (x^m_{st} - x_{st})}{\varrho'_m}, z\Big\|\Big)\Big]^{p_{st}} \leqslant 1,\ \forall\, z \in X\Big\}.$$

If $\varrho_m \in A(x^m)$ and $\varrho'_m \in A(x^m - x)$ then we observe that

$$\begin{aligned}
&M\Big(\Big\|\frac{\Delta^m(\sigma^m x^m_{st} - \sigma x_{st})}{\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|}, z\Big\|\Big)\\
&\quad\leqslant M\Big(\Big\|\frac{\Delta^m(\sigma^m x^m_{st} - \sigma x^m_{st})}{\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|}, z\Big\| + \Big\|\frac{\Delta^m(\sigma x^m_{st} - \sigma x_{st})}{\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|}, z\Big\|\Big)\\
&\quad\leqslant \frac{|\sigma^m - \sigma|\varrho_m}{\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|} M\Big(\Big\|\frac{\Delta^m x^m_{st}}{\varrho_m}, z\Big\|\Big)\\
&\qquad + \frac{|\sigma|\varrho'_m}{\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|} M\Big(\Big\|\frac{\Delta^m(x^m_{st} - x_{st})}{\varrho'_m}, z\Big\|\Big).
\end{aligned}$$

From the above inequality it now readily follows that

$$\Big[M\Big(\Big\|\frac{\Delta^m(\sigma^m x^m_{st} - \sigma x_{st})}{\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|}, z\Big\|\Big)\Big]^{p_{st}} \leqslant 1$$

and consequently

$$\begin{aligned}
&g_{ij}(\sigma^m x^m - \sigma x)\\
&\quad\leqslant \inf_{z\in X} \sum_{(s,t)\in I_{ij}} \|\sigma^m x^m_{st} - \sigma x_{st}, z\|\\
&\qquad + \inf\{(\varrho_m|\sigma^m - \sigma| + \varrho'_m|\sigma|)^{p_{ij}/H}\colon\ \varrho_m \in A(x^m),\ \varrho'_m \in A(x^m - x)\}\\
&\quad\leqslant |\sigma^m - \sigma| \inf_{z\in X} \sum_{(s,t)\in I_{ij}} \|x^m_{st}, z\| + |\sigma| \inf_{z\in X} \sum_{(s,t)\in I_{ij}} \|x^m_{st} - x_{st}, z\|\\
&\qquad + (|\sigma^m - \sigma|)^{p_{ij}/H} \inf\{(\varrho_m)^{p_{ij}/H}\colon\ \varrho_m \in A(x^m)\}\\
&\qquad + (|\sigma|)^{p_{ij}/H} \inf\{(\varrho'_m)^{p_{ij}/H}\colon\ \varrho'_m \in A(x^m - x)\}\\
&\quad\leqslant \max\{|\sigma^m - \sigma|, (|\sigma^m - \sigma|)^{p_{ij}/H}\} g_{ij}(x^m) + \max\{|\sigma|, (|\sigma|)^{p_{ij}/H}\} g_{ij}(x^m - x).
\end{aligned}$$

Note that $g_{ij}(x^m) \leqslant g_{ij}(x) + g_{ij}(x^m - x)$ for all $m \in \mathbb{N}$. Hence by our assumption the right-hand side tends to 0 as $m \to \infty$ and the result follows. This completes the proof of the theorem. $\square$

Theorem 4.4. *Let M, M_1, M_2 be Orlicz functions. Then*

(i) $W_0^{\mu}(\Lambda^2, M_1, \Delta^m, p, \|\cdot,\cdot\|) \subseteq W_0^{\mu}(\Lambda^2, MoM_1, \Delta^m, p, \|\cdot,\cdot\|)$ *provided* $\{p_{ij}\}_{i,j\in\mathbb{N}\times\mathbb{N}}$ *is such that* $H_0 = \inf p_{ij} > 0$;

(ii) $W_0^{\mu}(\Lambda^2, M_1, \Delta^m, p, \|\cdot,\cdot\|) \cap W_0^{\mu}(\Lambda^2, M_2, \Delta^m, p, \|\cdot,\cdot\|) \subseteq W_0^{\mu}(\Lambda^2, M_1 + M_2, \Delta^m, p, \|\cdot,\cdot\|)$.

Proof. Let $\varepsilon > 0$ be given. Choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon$. Now using the continuity of M choose $0 < \delta < 1$ such that $0 < t < \delta$ implies that $M(t) < \varepsilon_0$. Let $\{x_{ij}\}_{i,j\in\mathbb{N}\times\mathbb{N}} \in W_0^{\mu}(\Lambda^2, M_1, \Delta^m, p, \|\cdot,\cdot\|)$. Now from the definition $\mu(A(\delta)) = 0$, where

$$A(\delta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \left[M_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]^{p_{st}} \geqslant \delta^H\right\}.$$

Thus if $(i,j) \notin A(\delta)$ then

$$\frac{1}{\lambda_{ij}^2} \sum_{(s,t)\in I_{ij}} \left[M_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]^{p_{st}} < \delta^H$$

i.e.

$$\sum_{(s,t)\in I_{ij}} \left[M_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]^{p_{st}} < \lambda_{ij}^2 \delta^H$$

i.e.

$$\left[M_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]^{p_{st}} < \delta^H$$

for all $(s,t) \in I_{ij}$. Hence

$$\left[M_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right] < \delta$$

for all $(s,t) \in I_{ij}$.

Hence from the above using the continuity of M we have

$$M\left(\left[M_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]\right) < \varepsilon_0$$

for all $(s,t) \in I_{ij}$. This implies that

$$\left[MoM_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]^{p_{st}} < \max\{\varepsilon_0^{H_0}, \varepsilon_0^H\}$$

for all $(s,t) \in I_{ij}$, i.e.

$$\sum_{(s,t)\in I_{ij}} \left[MoM_1\left(\left\|\frac{\Delta^m x_{st}}{\varrho}, z\right\|\right)\right]^{p_{st}} < \lambda_{ij}^2 \max\{\varepsilon_0^{H_0}, \varepsilon_0^H\} < \lambda_{ij}^2 \varepsilon,$$

which again implies that

$$\frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[MoM_1\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}<\varepsilon.$$

This shows that

$$\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[MoM_1\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\varepsilon\Big\}\subseteq A(\delta).$$

Therefore

$$\mu\Big(\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[MoM_1\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\varepsilon\Big\}\Big)=0.$$

Thus

$$\{x_{ij}\}_{i,j\in\mathbb{N}}\in W_0^{\mu}(\Lambda^2,M_1,\Delta^m,p,\|\cdot,\cdot\|).$$

(ii) Let $\{x_{ij}\}_{i,j\in\mathbb{N}}\in W_0^{\mu}(\Lambda^2,M_1,\Delta^m,p,\|\cdot,\cdot\|)\cap W_0^{\mu}(\Lambda^2,M_2,\Delta^m,p,\|\cdot,\cdot\|)$. Then the inequality

$$\begin{aligned}\frac{1}{\lambda_{ij}^2}&\Big[(M_1+M_2)\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\\&\leqslant\frac{D}{\lambda_{ij}^2}\Big[M_1\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}+\frac{D}{\lambda_{ij}^2}\Big[M_2\Big(\Big\|\frac{\Delta^m x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\end{aligned}$$

gives the result. This completes the proof of the theorem. □

Theorem 4.5. *Let $X(\Delta^{m-1})$, $m\geqslant 1$ stand for $W^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$ or $W_0^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$ or $W_\infty^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$. Then $X(\Delta^{m-1})\subsetneqq X(\Delta^m)$. In general $X(\Delta^i)\subsetneqq X(\Delta^m)$ for all $i=1,2,3,\dots,m-1$.*

Proof. We give the proof for $W_0^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$ only. It can be proved in a similar way for $W^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$ and $W_\infty^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$.

Let $x=\{x_{ij}\}_{i,j\in\mathbb{N}}\in W_0^{\mu}(\Lambda^2,M,\Delta^{m-1},p,\|\cdot,\cdot\|)$. Let also $\varepsilon>0$ be given. Then

$$\mu\Big(\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^{m-1} x_{st}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\varepsilon\Big\}\Big)=0 \tag{4.3}$$

for some $\varrho > 0$. Since M is non-decreasing and convex it follows that

$$
\begin{aligned}
&\frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{4\varrho},z\Big\|\Big)\Big]^{p_{st}}\\
&=\frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t+1}-\Delta^{m-1}x_{s+1,t}-\Delta^{m-1}x_{s,t+1}+\Delta^{m-1}x_{st}}{4\varrho},z\Big\|\Big)\Big]^{p_{st}}\\
&\leqslant\frac{D^2}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big(\Big[\frac{1}{4}M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t+1}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}+\Big[\frac{1}{4}M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\\
&\quad+\Big[\frac{1}{4}M\Big(\Big\|\frac{\Delta^{m-1}x_{s,t+1}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}+\Big[\frac{1}{4}M\Big(\Big\|\frac{\Delta^{m-1}x_{s,t}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\Big)\\
&\leqslant\frac{D^2G}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big(\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t+1}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}+\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\\
&\quad+\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s,t+1}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}+\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s,t}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\Big)
\end{aligned}
$$

where $G=\mathrm{Max}\{1,(\frac{1}{4})^H\}$. Hence we have

$$
\begin{aligned}
&\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{4\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\varepsilon\Big\}\\
&\quad\subseteq\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{D^2G}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t+1}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\frac{\varepsilon}{4}\Big\}\\
&\qquad\cup\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{D^2G}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s+1,t}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\frac{\varepsilon}{4}\Big\}\\
&\qquad\cup\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{D^2G}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s,t+1}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\frac{\varepsilon}{4}\Big\}\\
&\qquad\cup\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{D^2G}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^{m-1}x_{s,t}}{\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\frac{\varepsilon}{4}\Big\}.
\end{aligned}
$$

Using (4.3) we get

$$\mu\Big(\Big\{(i,j)\in\mathbb{N}\times\mathbb{N}\colon \frac{1}{\lambda_{ij}^2}\sum_{(s,t)\in I_{ij}}\Big[M\Big(\Big\|\frac{\Delta^m x_{st}}{4\varrho},z\Big\|\Big)\Big]^{p_{st}}\geqslant\varepsilon\Big\}\Big)=0.$$

Therefore $x=\{x_{ij}\}_{i,j\in\mathbb{N}}\in W_0^{\mu}(\Lambda^2,M,\Delta^m,p,\|\cdot,\cdot\|)$. This completes the proof. $\square$

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