Paolo Lipparini

Every weakly initially $\mathfrak{m}$-compact topological space is $\mathfrak{m}$pcap

Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 3, 781–784

Persistent URL: http://dml.cz/dmlcz/141638

Czechoslovak Mathematical Journal, 61 (136) (2011), 781–784

EVERY WEAKLY INITIALLY $\mathfrak{m}$-COMPACT TOPOLOGICAL SPACE IS $\mathfrak{m}$PCAP

Paolo Lipparini, Roma

(Received May 15, 2010)

Abstract. The statement in the title solves a problem raised by T. Retta. We also present a variation of the result in terms of $[\mu, \kappa]$-compactness.

Keywords: weak initial compactness, $\mathfrak{m}$pcap, $[\mu, \kappa]$-compactness, pseudo-(κ, λ)-compactness, covering number

MSC 2010: 54D20, 03E75

Let $\mathfrak{m}$ be an infinite cardinal. A topological space is *weakly initially* $\mathfrak{m}$*-compact* if and only if every open cover of cardinality $\leqslant \mathfrak{m}$ has a finite subset with a dense union.

A topological space X is said to be $\mathfrak{m}$*pcap* [6] if and only if every family of $\leqslant \mathfrak{m}$ open sets in X has a complete accumulation point, i.e., a point each neighborhood of which meets κ members of the family, where κ is the cardinality of the family. The acronym $\mathfrak{m}$pcap stands for $\mathfrak{m}$*-pseudocompact in the sense of complete accumulation points.*

The next theorem solves the last problem in [6]. Notice that the results from [6] have subsequently found deep applications (see, e.g., [1]).

Theorem 1. *For every infinite cardinal* $\mathfrak{m}$*, every weakly initially* $\mathfrak{m}$*-compact topological space is* $\mathfrak{m}$*pcap.*

Before proving the theorem, we recall some known facts about the notions involved in its statement.

The notion of weak initial $\mathfrak{m}$-compactness was introduced by Frolík [3] under the name *almost* $\mathfrak{m}$*-compactness*, and has been studied by various authors under various names, such as *weak*-$\mathfrak{m}$-$\aleph_0$*-compactness*, or $\mathcal{O}$-$[\omega, \mathfrak{m}]$*-compactness.* See [5] for references. By taking complements, it is trivial to see that a topological space X is weakly initially $\mathfrak{m}$-compact if and only if the following holds. For every sequence $(C_\alpha)_{\alpha\in\mathfrak{m}}$ of closed sets of X, if for every finite $F \subseteq \mathfrak{m}$ there exists a nonempty open set O_F of X such that $\bigcap_{\alpha\in F} C_\alpha \supseteq O_F$, then $\bigcap_{\alpha\in\mathfrak{m}} C_\alpha \neq \emptyset$.

Let $\kappa \leqslant \lambda$ be infinite cardinals. A topological space is said to be *pseudo-*(κ,λ)*-compact* [2] if and only if for every λ-indexed sequence $(O_\alpha)_{\alpha\in\lambda}$ of nonempty open subsets of X, there is $x \in X$ such that for every neighborhood U of x, $|\{\alpha \in \lambda\colon U \cap O_\alpha \neq \emptyset\}| \geqslant \kappa$. Actually, Comfort and Negrepontis used the notation in which the order of κ and λ is reversed. The present notation we are adopting is consistent with the others generally employed in the topological literature when dealing with similar notions.

T. Retta [6, Theorem 3(d)] proved that a space is mpcap if and only if it is pseudo-(κ,κ)-compact for each $\kappa \leqslant \mathfrak{m}$.

Proof of the theorem. If $\kappa \leqslant \mathfrak{m}$, then trivially every weakly initially $\mathfrak{m}$-compact topological space is weakly initially κ-compact. Thus if we prove that for every infinite cardinal κ, every weakly initially κ-compact topological space is pseudo-(κ,κ)-compact, then we have that every weakly initially $\mathfrak{m}$-compact topological space is pseudo-(κ,κ)-compact, for every $\kappa \leqslant \mathfrak{m}$, and we are done by the mentioned result from [6, Theorem 3(d)].

Hence let X be a weakly initially κ-compact topological space, and let $(O_\alpha)_{\alpha\in\kappa}$ be a sequence of nonempty open subsets of X. Let $S_\omega(\kappa)$ be the set of all finite subsets of κ. Since $|S_\omega(\kappa)| = \kappa$, we can reindex the sequence $(O_\alpha)_{\alpha\in\kappa}$ as $(O_F)_{F\in S_\omega(\kappa)}$. For every $\alpha \in \kappa$, let $C_\alpha = \overline{\bigcup\{O_F\colon F \in S_\omega(\kappa), \alpha \in F\}}$. For every finite subset F of κ, we have that $\bigcap_{\alpha\in F} C_\alpha$ contains the nonempty open set O_F. By weak initial κ-compactness, $\bigcap_{\alpha\in\kappa} C_\alpha \neq \emptyset$.

Let $x \in \bigcap_{\alpha\in\kappa} C_\alpha$. We are going to show that for every neighborhood U of x, we have that $|\{F \in S_\omega(\kappa)\colon U \cap O_F \neq \emptyset\}| = \kappa$, thus X is pseudo-(κ,κ)-compact, and the theorem is proved.

So, let U be a neighborhood of x, and suppose by contradiction that the cardinality of $H = \{F \in S_\omega(\kappa)\colon U \cap O_F \neq \emptyset\}$ is $< \kappa$. Then $|\bigcup H| < \kappa$. Choose $\alpha \in \kappa$ such that $\alpha \notin \bigcup H$. Thus if $F \in S_\omega(\kappa)$ and $\alpha \in F$, then $F \notin H$, hence $U \cap O_F = \emptyset$. Then we also get $U \cap \bigcup\{O_F\colon F \in S_\omega(\kappa), \alpha \in F\} = \emptyset$, hence $x \notin C_\alpha$ since $C_\alpha = \overline{\bigcup\{O_F\colon F \in S_\omega(\kappa), \alpha \in F\}}$, and U is a neighborhood of x. We have reached a contradiction, and the theorem is proved. □

In fact, our argument gives something more. Let us say that a topological space is *weakly* $[\lambda,\kappa]$*-compact* if and only if every open cover of cardinality $\leqslant \kappa$ has a subset of cardinality $< \lambda$ with a dense union. In this sense, weak initial κ-compactness is the same as weak $[\omega, \kappa]$-compactness. The notion of weak $[\lambda, \kappa]$-compactness has been studied in [4], [5], sometimes under the name $\mathcal{O}$-$[\lambda, \kappa]$-compactness.

For $\kappa \geqslant \lambda \geqslant \mu$, let $\mathrm{COV}(\kappa, \lambda, \mu)$ denote the minimal cardinality of a family of subsets of κ, each of cardinality $< \lambda$, such that every subset of κ of cardinality $< \mu$ is contained in at least one set of the family. Highly non trivial results about $\mathrm{COV}(\kappa, \lambda, \mu)$ are proved in [7] under the terminology $\mathrm{cov}(\kappa, \lambda, \mu, 2)$. See [7, II, Definition 5.1]. Notice that, trivially, $\mathrm{COV}(\kappa, \lambda, \mu) \leqslant |S_\mu(\kappa)| = \sup_{\mu' < \mu} \kappa^{\mu'}$. In particular, $\mathrm{COV}(\kappa, \lambda, \omega) = \kappa$, hence the next proposition is stronger than Theorem 1, via [6, Theorem 3(d)].

Proposition 2. *Suppose that $\kappa \geqslant \lambda \geqslant \mu$ are infinite cardinals, and either $\kappa > \lambda$, or κ is regular. Then every weakly $[\mu, \kappa]$-compact topological space is pseudo-$(\kappa, \mathrm{COV}(\kappa, \lambda, \mu))$-compact.*

Proof. The proof is essentially the same as the proof of Theorem 1. We shall only point out the differences. Let K be a subset of $S_\lambda(\kappa)$ witnessing $|K| = \mathrm{COV}(\kappa, \lambda, \mu)$. Suppose that X is a weakly $[\mu, \kappa]$-compact topological space and let $(O_Z)_{Z \in K}$ be a sequence of nonempty open sets of X. For $\alpha \in \kappa$, put $C_\alpha = \overline{\bigcup\{O_Z \colon Z \in K, \alpha \in Z\}}$. If $W \subseteq \kappa$, and $|W| < \mu$, then there is $Z \in K$ such that $Z \supseteq W$, so that $\bigcap_{\alpha \in W} C_\alpha \supseteq \bigcap_{\alpha \in Z} C_\alpha$ contains the nonempty open set O_Z, hence, by weak $[\mu, \kappa]$-compactness, $\bigcap_{\alpha \in \kappa} C_\alpha \neq \emptyset$.

Now notice that the union of $< \kappa$ sets, each of cardinality $< \lambda$, has cardinality $< \kappa$, and this is the only fact that is used in the final part of the proof of Theorem 1. □

For κ a regular cardinal, Proposition 2 implies that every weakly $[\kappa, \kappa]$-compact topological space is pseudo-(κ, κ)-compact. Indeed, for κ a regular cardinal, weak $[\kappa, \kappa]$-compactness and pseudo-(κ, κ)-compactness are equivalent, as proved in [4] under different terminology.

By replacing everywhere nonempty open sets by points in Proposition 2, we get the following result which, in the present generality, might be new.

Proposition 3. *Suppose that $\kappa \geqslant \lambda \geqslant \mu$ are infinite cardinals, and either $\kappa > \lambda$, or κ is regular, and let $\nu = \mathrm{COV}(\kappa, \lambda, \mu)$. If X is a $[\mu, \kappa]$-compact topological space, then, for every ν-indexed family $(x_\beta)_{\beta \in \nu}$ of elements of X, there is an element $x \in X$ such that for every neighborhood U of x, the set $\{\beta \in \nu \colon x_\beta \in U\}$ has cardinality $\geqslant \kappa$.*

A common generalization of both Propositions 2 and 3 can be given along the abstract framework presented in [4], [5]. If X is a topological space, and $\mathcal{F}$ is a family of subsets of X, we say that X is $\mathcal{F}$-$[\mu, \kappa]$-compact if and only if the following holds.

For every sequence $(C_\alpha)_{\alpha\in\kappa}$ of closed sets of X if, for every $Z \subseteq \kappa$ with $|Z| < \mu$, there exists a set $F_Z \in \mathcal{F}$ such that $\bigcap_{\alpha\in Z} C_\alpha \supseteq F_Z$, then $\bigcap_{\alpha\in\kappa} C_\alpha \neq \emptyset$.

Proposition 4. *Suppose that $\kappa \geqslant \lambda \geqslant \mu$ are infinite cardinals, and either $\kappa > \lambda$, or κ is regular, and let $\nu = \operatorname{cov}(\kappa, \lambda, \mu)$. Suppose that X is a topological space, and $\mathcal{F}$ is a family of subsets of X. If X is $\mathcal{F}$-$[\mu, \kappa]$-compact, then for every ν-indexed family $(F_\beta)_{\beta\in\nu}$ of elements of $\mathcal{F}$, there is an element $x \in X$ such that for every neighborhood U of x, the set $\{\beta \in \nu\colon F_\beta \cap U \neq \emptyset\}$ has cardinality $\geqslant \kappa$.*

Proposition 2 is the particular case of Proposition 4 when we take $\mathcal{F}$ to be the family of all nonempty open subsets of X. Proposition 3 is the particular case of Proposition 4 when we take $\mathcal{F}$ to be the family of all singletons of X.

References

[1] *A. V. Arhangel'skii*: Strongly τ-pseudocompact spaces. Topology Appl. *89* (1998), 285–298.

[2] *W. W. Comfort, S. Negrepontis*: Chain Conditions in Topology. Cambridge Tracts in Mathematics 79, Cambridge University Press, Cambridge-New York, 1982.

[3] *Z. Frolík*: Generalisations of compact and Lindelöf spaces. Czech. Math. J. *9* (1959), 172–217. (In Russian. English summary.)

[4] *P. Lipparini*: Some compactness properties related to pseudocompactness and ultrafilter convergence. Topol. Proc. *40* (2012), 29–51.

[5] *P. Lipparini*: More generalizations of pseudocompactness. Topology Appl. *158* (2011), 1655–1666.

[6] *T. Retta*: Some cardinal generalizations of pseudocompactness. Czech. Math. J. *43* (1993), 385–390.

[7] *S. Shelah*: Cardinal Arithmetic. Oxford Logic Guides, 29, The Clarendon Press, Oxford University Press, New York, 1994.

Author's address: Paolo Lipparini, Math. Dept., Vialis Ricercæ Scientificæ, II Università di Roma (Tor Vergata), I-00133 Rome, Italy, http://www.mat.uniroma2.it/~lipparin.