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ON TYPICALLY REAL FUNCTIONS WHICH ARE GENERATED BY A FIXED TYPICALLY REAL FUNCTION

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Abstract. Let T be the family of all typically real functions, i.e. functions that are analytic in the unit disk $\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and such that $\operatorname{Im} z \operatorname{Im} f(z) \geqslant 0$ for $z \in \Delta$.

In this paper we discuss the class T_g defined as

$$\mathrm{T}_g := \{\sqrt{f(z)g(z)}\colon f \in \mathrm{T}\}, \quad g \in \mathrm{T}.$$

We determine the sets $\bigcup\limits_{g\in\mathrm{T}} \mathrm{T}_g$ and $\bigcap\limits_{g\in\mathrm{T}} \mathrm{T}_g$. Moreover, for a fixed g, we determine the superdomain of local univalence of T_g, the radii of local univalence, of starlikeness and of univalence of T_g.

Keywords: typically real functions, superdomain of local univalence, radius of local univalence, radius of starlikeness, radius of univalence

MSC 2010: 30C45, 30C55

1. Some properties of the class T

Let A be the family of all functions which are analytic in the unit disk $\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let T denote the well-known class which consists of all typically real functions. Recall that a function $f \in \mathrm{A}$ belongs to T if and only if the condition $\operatorname{Im} z \operatorname{Im} f(z) \geqslant 0$ for $z \in \Delta$ is satisfied. Moreover, let us denote $\mathrm{T}^{(2)} := \{f \in \mathrm{T}\colon f(-z) = -f(z)\}$.

Rogosinski [4] gave the explicit relation between a function $f \in \mathrm{T}$ and a probability measure μ defined on $[-1, 1]$. Namely,

$$f \in \mathrm{T} \iff f(z) = \int_{-1}^{1} k_t(z)\,\mathrm{d}\mu(t), \qquad \text{where } k_t(z) = \frac{z}{1 - 2tz + z^2}.$$

In this paper we determine the radii of starlikeness r_{ST}, of local univalence r_{LU} and of univalence r_S, and the superdomains of local univalence in certain classes. Let us recall some definitions.

Definition 1. We say that $r_{ST}(A)$ is the radius of starlikeness in the class A, if it is the maximum of the numbers r such that the inequality $\mathrm{Re}(zf'(z)/f(z)) \geqslant 0$ holds in $|z| \leqslant r$ for each function $f \in \mathrm{A}$.

Definition 2. The set $G \subset \Delta$ is called the set of local univalence in the class A, if $\forall_{f\in \mathrm{A}}\ \forall_{z\in G}\ f'(z) \neq 0$ and $\forall_{z\in\Delta\setminus G}\ \exists_{f\in A}\ \ f'(z) = 0$.

We say that $r_{LU}(A)$ is the radius of local univalence in the class A, if it is the maximum of numbers r such that every function $f \in \mathrm{A}$ is locally univalent in $|z| \leqslant r$.

Remark 1. If for a class A the following condition

$$f \in \mathrm{A} \Rightarrow F(z) := \frac{f(tz)}{t} \in \mathrm{A}, \quad -1 \leqslant t \leqslant 1 \tag{1}$$

is satisfied, then the set of local univalence is a domain (starlike domain).

Definition 3. We say that $r_S(A)$ is the radius of univalence in the class A, if it is the maximum of the numbers r such that every function $f \in \mathrm{A}$ is univalent in $|z| \leqslant r$.

In the class A the following inequality is satisfied:

$$r_{ST}(A) \leqslant r_S(A) \leqslant r_{LU}(A). \tag{2}$$

P. Todorov in [5] gave an estimate for the operator $\mathrm{Re}(zf'(z)/f(z))$ for $f \in \mathrm{T}$.

Theorem 1 [P. G. Todorov]**.** *For each typically real function we have*

(i)

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geqslant \frac{1-6r^2+r^4}{1-r^4} \quad \textit{for } 2-\sqrt{3} \leqslant r = |z| < 1$$

with equality for the function $f(z) = z(1+z^2)/(1-z^2)^2$ *at the points* $z = \pm \mathrm{i}r$;

(ii)

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geqslant \frac{1-r}{1+r} \quad \textit{for } 0 \leqslant r = |z| \leqslant 2-\sqrt{3}$$

with equality for the functions $k_1(z)$ *and* $k_{-1}(z)$ *at the points* $\pm r$.

2. Some properties of the class T_g

For typically real functions $f, g \in \mathrm{T}$ and $\varepsilon \in [0,1]$ we have $f^{\varepsilon} g^{1-\varepsilon} \in \mathrm{T}$, $\varepsilon \in [0,1]$. In this paper, we investigate functions $f^{\varepsilon} g^{1-\varepsilon}$ for $\varepsilon = \frac{1}{2}$, i.e.

$$\sqrt{f(z)\cdot g(z)} := z \cdot \sqrt{\frac{f(z)}{z}\cdot\frac{g(z)}{z}}.$$

Denote

$$\mathrm{T}_g := \{\sqrt{f(z)g(z)}\colon\ f \in \mathrm{T}\}, \quad g \in \mathrm{T}. \tag{3}$$

Observe that the class T_g defined by (3) is not empty, because the function g belongs to T_g. In the next two theorems we introduce successively important properties of the class T_g.

Theorem 2. *The class* T_g *is convex.*

Proof. Recall that the class $\mathscr{A} \subset \mathrm{A}$ is convex if

$$\forall_{f_1, f_2 \in \mathscr{A}}\ \forall_{\varepsilon \in [0,1]}\ \varepsilon f_1 + (1-\varepsilon) f_2 \in \mathscr{A}.$$

Assume that $F_1, F_2 \in \mathrm{T}_g$. This means $F_1(z) = z \cdot \sqrt{(f_1(z)/z)\cdot(g(z)/z)}$, $F_2(z) = z \cdot \sqrt{(f_2(z)/z)\cdot(g(z)/z)}$, where $f_1, f_2 \in \mathrm{T}$. Notice that

$$\begin{aligned}&\varepsilon F_1(z) + (1-\varepsilon)F_2(z)\\ &\qquad = z \cdot \sqrt{\frac{\varepsilon^2 f_1(z) + 2\varepsilon(1-\varepsilon)\sqrt{f_1(z)}\sqrt{f_2(z)} + (1-\varepsilon)^2 f_2(z)}{z}} \cdot \sqrt{\frac{g(z)}{z}}.\end{aligned}$$

From the facts that the class T is convex and $f_1 \in \mathrm{T}$, $f_2 \in \mathrm{T}$, $\sqrt{f_1}\sqrt{f_2} \in \mathrm{T}$ and $\varepsilon^2 + 2\varepsilon(1-\varepsilon) + (1-\varepsilon)^2 = 1$, it follows that $\varepsilon^2 f_1 + 2\varepsilon(1-\varepsilon)\sqrt{f_1}\sqrt{f_2} + (1-\varepsilon)^2 f_2 \in \mathrm{T}$. Hence, $\varepsilon F_1(z) + (1-\varepsilon)F_2(z) \in \mathrm{T}_g$. $\square$

Theorem 3.
(i) $\bigcup_{g \in \mathrm{T}} \mathrm{T}_g = \mathrm{T}$.
(ii) $\bigcap_{g \in \mathrm{T}} \mathrm{T}_g = \left\{\dfrac{z}{1-z^2}\right\}$.

Proving Theorem 3 we use the following lemmas:

Lemma 1.

$$\forall_{g\in \mathrm{T}}\ \frac{z}{1-z^2}\in \mathrm{T}_g.$$

P r o o f. It is known ([2], [4]) that for $g\in \mathrm{T}$ we have:

$$g(z)=\frac{z\cdot p(z)}{1-z^2},\quad p\in P_R, \tag{4}$$

where P_R denotes the family of all functions p analytic in Δ such that they have real coefficients and satisfy the conditions $\operatorname{Re} p(z)>0$ and $p(0)=1$. From (4) and the fact that $p\in P_R \Leftrightarrow 1/p\in P_R$ we obtain

$$\frac{1}{p(z)}=\frac{z}{(1-z^2)\cdot g(z)}\in P_R.$$

Let $h(z)\in \mathrm{T}$. Then

$$h(z)=\frac{z}{1-z^2}\cdot\frac{1}{p(z)}=\Big(\frac{z}{1-z^2}\Big)^2\cdot\frac{1}{g(z)}$$

and

$$\sqrt{h(z)\cdot g(z)}=\frac{z}{1-z^2}\cdot\frac{1}{\sqrt{g(z)}}\cdot\sqrt{g(z)}=\frac{z}{1-z^2}.$$

Hence, $z/(1-z^2)\in \mathrm{T}_g$. □

Lemma 2.

(i) *If* $F(z)=z+A_2z^2+\ldots\in \mathrm{T}_{k_1}$, *then* $A_2\geqslant 0$.

(ii) *If* $F(z)=z+A_2z^2+\ldots\in \mathrm{T}_{k_{-1}}$, *then* $A_2\leqslant 0$.

P r o o f. Let $F(z)=z+A_2z^2+\ldots\in \mathrm{T}_{k_1}$. From the definition of the class T_{k_1} it follows that

$$F(z)=\frac{z}{1-z}\sqrt{\frac{f(z)}{z}}$$

for $f(z)=z+a_2z^2+\ldots\in \mathrm{T}$. We have $A_2=\frac12 a_2+1$. Since $a_2\in[-2,2]$, we conclude that $A_2\geqslant 0$.

Analogously, if $F(z)=z+A_2z^2+\ldots\in \mathrm{T}_{k_{-1}}$, then

$$F(z)=\frac{z}{1+z}\sqrt{\frac{f(z)}{z}}$$

and $A_2=\frac12 a_2-1$. Therefore, $A_2\leqslant 0$. □

P r o o f of Theorem 3. Observe that $\mathrm{T}_g \subset \mathrm{T}$. Hence, $\bigcup\limits_{g\in\mathrm{T}} \mathrm{T}_g \subset \mathrm{T}$. Furthermore, $g \in \mathrm{T}_g$, $\mathrm{T} = \{g\colon\ g \in \mathrm{T}\} \subset \{\mathrm{T}_g\colon\ g \in \mathrm{T}\}$. This means that $\mathrm{T} \subset \bigcup\limits_{g\in\mathrm{T}} \mathrm{T}_g$. From these facts we conclude $\bigcup\limits_{g\in\mathrm{T}} \mathrm{T}_g = \mathrm{T}$.

Now we prove the second part of Theorem 3.

From Lemma 1 it appears that $\{z/(1-z^2)\} \subset \bigcap\limits_{g\in\mathrm{T}} \mathrm{T}_g$. Notice that $\bigcap\limits_{g\in\mathrm{T}} \mathrm{T}_g \subset \mathrm{T}_{k_1} \cap \mathrm{T}_{k_{-1}}$. From Lemma 2 we obtain that if $F \in \mathrm{T}_{k_1} \cap \mathrm{T}_{k_{-1}}$, then $A_2 = 0$. Assume $F \in \mathrm{T}_{k_1}$ and $A_2 = 0$. The definition of the class T_{k_1} gives $F(z) = \sqrt{zf(z)}/(1-z)$, where $f(z) = z + a_2 z^2 + \ldots \in \mathrm{T}$. Since $A_2 = 0$, from Lemma 2 we have $a_2 = -2$. This fact is equivalent to $f(z) = z/(1+z)^2$. Hence, we obtain $F(z) = z/(1-z^2)$. $\square$

For some fixed functions $g \in \mathrm{T}$, let us determine the superdomain of local univalence of T_g, the radii of local univalence, of starlikeness, and of univalence of T_g.

3. The class T_{id}

Let us consider the class T_g, where $g(z) = z$. Denote this class as T_{id}. Then for $F \in \mathrm{T}_{\mathrm{id}}$ we have

$$\frac{zF'(z)}{F(z)} = \frac{1}{2}\Big(\frac{zf'(z)}{f(z)} + 1\Big), \quad f \in \mathrm{T}. \tag{5}$$

From (5) and from Todorov's inequalities given in Theorem 1 (i) and (ii) we get

$$2\operatorname{Re}\frac{zF'(z)}{F(z)} = \operatorname{Re}\Big(\frac{zf'(z)}{f(z)} + 1\Big) \geqslant \begin{cases} \dfrac{1-r}{1+r} + 1 & \text{for } 0 \leqslant r < 2-\sqrt{3}, \\[2mm] \dfrac{1-6r^2+r^4}{1-r^4} + 1 & \text{for } 2-\sqrt{3} \leqslant r < 1. \end{cases}$$

Therefore, $\operatorname{Re}(zF'(z)/F(z)) > 0$ for $0 \leqslant r < \frac{1}{3}\sqrt{3}$. Thus, $r_{ST}(\mathrm{T}_{\mathrm{id}}) \geqslant \frac{1}{3}\sqrt{3}$. Observe that $\min \operatorname{Re}(zF'(z)/F(z))$ is reached by the function $F_0(z) = z\sqrt{1+z^2}/(1-z^2)$ for $z = \mathrm{i}r$, $r \in [2-\sqrt{3}, 1)$. This follows from the fact that $\min \operatorname{Re}(zf'(z)/f(z))$ is reached by the function f given in Theorem 1 (i) for $z = \mathrm{i}r$, $r \in [2-\sqrt{3}, 1)$.

Moreover, we have $F_0'(\frac{1}{3}\sqrt{3}\mathrm{i}) = 0$. Hence, we obtain that $r_{LU}(\mathrm{T}_{\mathrm{id}}) \leqslant \frac{1}{3}\sqrt{3}$. But the inequality (2) and the above facts give $r_{ST}(\mathrm{T}_{\mathrm{id}}) = r_{LU}(\mathrm{T}_{\mathrm{id}}) = r_S(\mathrm{T}_{\mathrm{id}}) = \frac{1}{3}\sqrt{3}$.

From (5) we obtain $F'(z) \neq 0 \Leftrightarrow zf'(z) + f(z) \neq 0$. Putting into this inequality

$$f(z) = \varepsilon k_1(z) + (1-\varepsilon)k_{-1}(z), \quad \varepsilon \in [0,1]$$

we get $\varepsilon 2z/(1-z)^3 + (1-\varepsilon)2z/(1+z)^3 \neq 0$. Therefore,

$$\frac{1+z}{1-z} \neq \sqrt[3]{-1}\cdot\lambda, \quad \lambda = \sqrt[3]{\frac{1-\varepsilon}{\varepsilon}} \in [0,\infty).$$

Let us find the set which is represented by the equality $(1+z)/(1-z) = \sqrt[3]{-1}\,\lambda$ for $\lambda \in [0,\infty)$. For $\sqrt[3]{-1} = -1$ we have $z = (\lambda+1)/(\lambda-1) \notin \Delta$. For $\sqrt[3]{-1} = \mathrm{e}^{\mathrm{i}\pi/3}$ we get $z = (\lambda \mathrm{e}^{\mathrm{i}\pi/3} - 1)/(\lambda \mathrm{e}^{\mathrm{i}\pi/3} + 1)$, $\lambda \geqslant 0$. This is the circle $\{z\colon\ |z + \frac{1}{3}\sqrt{3}\mathrm{i}| = \frac{2}{3}\sqrt{3}\}$. If $\sqrt[3]{-1} = \mathrm{e}^{5\mathrm{i}\pi/3}$ we have $z = (\lambda \mathrm{e}^{5\mathrm{i}\pi/3} - 1)/(\lambda \mathrm{e}^{5\mathrm{i}\pi/3} + 1)$, $\lambda \geqslant 0$ and this is the circle $\{z\colon\ |z - \frac{1}{3}\sqrt{3}\mathrm{i}| = \frac{2}{3}\sqrt{3}\}$. Hence, the set which we have found is the following lens: $\{z\colon\ |z - \frac{1}{3}\sqrt{3}\mathrm{i}| \leqslant \frac{2}{3}\sqrt{3}\} \cap \{z\colon\ |z + \frac{1}{3}\sqrt{3}\mathrm{i}| \leqslant \frac{2}{3}\sqrt{3}\}$ (see Figure 1).

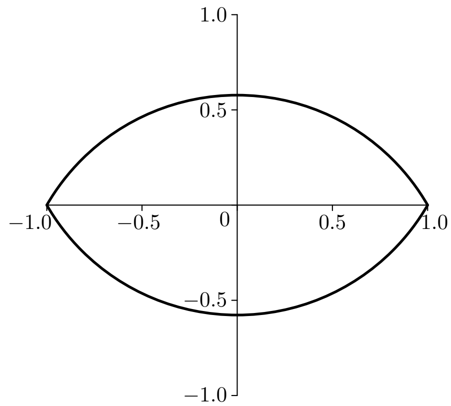

Figure 1. The boundary of the superdomain of local univalence of T_{id}.

Since the class T_{id} has the property (1), so the set of local univalence is a domain containing the origin. Hence, the domain of local univalence is included in this lens.

We have proved the following theorem:

Theorem 4.

(i) $r_{ST}(\mathrm{T}_{\mathrm{id}}) = r_{LU}(\mathrm{T}_{\mathrm{id}}) = r_S(\mathrm{T}_{\mathrm{id}}) = \frac{1}{3}\sqrt{3}$.

(ii) *The superdomain of local univalence of the class* T_{id} *is the set*

$$\Big\{z\colon\ \Big|z - \frac{\sqrt{3}}{3}\mathrm{i}\Big| \leqslant \frac{2\sqrt{3}}{3}\Big\} \cap \Big\{z\colon\ \Big|z + \frac{\sqrt{3}}{3}\mathrm{i}\Big| \leqslant \frac{2\sqrt{3}}{3}\Big\}.$$

4. The class T_g for $g(z) = z/(1-z^2)$

Let us study the class T_g, where $g(z) = z/(1-z^2)$. Then for $F \in \mathrm{T}_g$ we have:

$$\frac{zF'(z)}{F(z)} = \frac{1}{2}\Big(\frac{zf'(z)}{f(z)} + \frac{1+z^2}{1-z^2}\Big), \qquad f \in \mathrm{T}.$$

Taking into account Theorem 1 (i) and (ii) we obtain

$$
\begin{aligned}
2\operatorname{Re}\frac{zF'(z)}{F(z)} &= \operatorname{Re}\Big(\frac{zf'(z)}{f(z)}+\frac{1+z^2}{1-z^2}\Big)\\
&\geqslant \begin{cases} \dfrac{1-r}{1+r}+\dfrac{1-r^2}{1+r^2} & \text{for } 0\leqslant r<2-\sqrt{3},\\[2mm] \dfrac{1-6r^2+r^4}{1-r^4}+\dfrac{1-r^2}{1+r^2} & \text{for } 2-\sqrt{3}\leqslant r<1,\end{cases}\\
&= \begin{cases} \dfrac{(1-r)(2+r-r^3)}{(1+r)(1+r^2)} & \text{for } 0\leqslant r<2-\sqrt{3},\\[2mm] \dfrac{2(r^4-4r^2+1)}{1-r^4} & \text{for } 2-\sqrt{3}\leqslant r<1.\end{cases}
\end{aligned}
$$

Therefore, $\operatorname{Re}(zF'(z)/F(z))>0$ for $0\leqslant r<\sqrt{2-\sqrt{3}}$. Thus, $r_{ST}(\mathrm{T}_g)\geqslant\sqrt{2-\sqrt{3}}$. Observe that $\min\operatorname{Re}(zF'(z)/F(z))$ is reached by the function

$$
F_0(z)=\frac{z}{1-z^2}\sqrt{\frac{1+z^2}{1-z^2}}\quad \text{for } z=\mathrm{i}r,\ r\in[2-\sqrt{3},1).
$$

This follows from the fact that $\min\operatorname{Re}(zf'(z)/f(z))$ is reached by the function f given in Theorem 1 (i) for $z=\mathrm{i}r$, $r\in[2-\sqrt{3},1)$.

Furthermore, we have $F_0'\big(\sqrt{2-\sqrt{3}}\mathrm{i}\big)=0$. Hence we obtain $r_{LU}(\mathrm{T}_g)\leqslant\sqrt{2-\sqrt{3}}$, but the inequality (2) and the above facts give $r_{LU}(\mathrm{T}_g)=r_{ST}(\mathrm{T}_g)=r_S(\mathrm{T}_g)=\sqrt{2-\sqrt{3}}$.

We have proved the following theorem:

Theorem 5. *For $g(z)=z/(1-z^2)$ we have $r_{LU}(\mathrm{T}_g)=r_{ST}(\mathrm{T}_g)=r_S(\mathrm{T}_g)=\sqrt{2-\sqrt{3}}$.*

5. The class T_g for $g(z)=z(1+z^2)/(1-z^2)^2$

Let us study the class T_g, where $g(z)=\frac{1}{2}k_1(z)+\frac{1}{2}k_{-1}(z)=z(1+z^2)/(1-z^2)^2$. Then for $F\in\mathrm{T}_g$ we have

$$
\frac{zF'(z)}{F(z)}=\frac{1}{2}\Big(\frac{zf'(z)}{f(z)}+\frac{1+6z^2+z^4}{1-z^4}\Big),\quad f\in\mathrm{T}.
$$

Taking into consideration Theorem 1 (i) and (ii) we get

$$
\begin{aligned}
2\operatorname{Re}\frac{zF'(z)}{F(z)} &= \operatorname{Re}\Big(\frac{zf'(z)}{f(z)}+\frac{1+6z^2+z^4}{1-z^4}\Big)\\
&\geqslant \begin{cases} \dfrac{1-r}{1+r}+\dfrac{1-6r^2+r^4}{1-r^4} & \text{for } 0\leqslant r<2-\sqrt{3},\\[2mm] \dfrac{1-6r^2+r^4}{1-r^4}+\dfrac{1-6r^2+r^4}{1-r^4} & \text{for } 2-\sqrt{3}\leqslant r<1,\end{cases}\\
&= \begin{cases} \dfrac{r^4-r^3-2r^2-r+1}{1-r^4} & \text{for } 0\leqslant r<2-\sqrt{3},\\[2mm] \dfrac{2(r^4-6r^2+1)}{1-r^4} & \text{for } 2-\sqrt{3}\leqslant r<1.\end{cases}
\end{aligned}
$$

Therefore, $\operatorname{Re}(zF'(z)/F(z))>0$ for $0\leqslant r<\sqrt{2}-1$. Thus, $r_{ST}(\mathrm{T}_g)\geqslant\sqrt{2}-1$. Observe that $\min\operatorname{Re}(zF'(z)/F(z))$ is reached by the function

$$F_0(z)=\frac{z(1+z^2)}{(1-z^2)^2}\quad\text{for } z=\mathrm{i}r,\ r\in[2-\sqrt{3},1).$$

This follows from the fact that $\min\operatorname{Re}(zf'(z)/f(z))$ is reached by the function f given in Theorem 1 (i) for $z=\mathrm{i}r$, $r\in[2-\sqrt{3},1)$.

Moreover, we have $F_0'((\sqrt{2}-1)\mathrm{i})=0$. Hence, we obtain that $r_{LU}(\mathrm{T}_g)\leqslant\sqrt{2}-1$, but the inequality (2) and the above facts give $r_{ST}(\mathrm{T}_g)=r_{LU}(\mathrm{T}_g)=r_S(\mathrm{T}_g)=\sqrt{2}-1$.

We have proved the following theorem:

Theorem 6. *For* $g(z)=z(1+z^2)/(1-z^2)^2$ *we have* $r_{LU}(\mathrm{T}_g)=r_{ST}(\mathrm{T}_g)=r_S(\mathrm{T}_g)=\sqrt{2}-1$.

6. The class $\mathrm{T}_g^{(2)}$ for $g(z)=z/(1+z^2)$

We can investigate an analogous problem in the class $\mathrm{T}_g^{(2)}$, i.e. $\mathrm{T}_g^{(2)}=\{\sqrt{f(z)g(z)}\colon f\in\mathrm{T}^{(2)}\}$ for $g\in\mathrm{T}^{(2)}$.

Let us study the class $\mathrm{T}_g^{(2)}$ for $g(z)=z/(1+z^2)$. From [3] we know that $f\in\mathrm{T}^{(2)}$ is given by $f(z)=(1+z^2)h(z^2)/z$ for some $h\in\mathrm{T}$. So $\mathrm{T}_g^{(2)}=\{\sqrt{h(z^2)}\colon h\in\mathrm{T}\}$. For $F\in\mathrm{T}_g^{(2)}$ we have $F(z)=\sqrt{h(z^2)}$ and

$$\frac{zF'(z)}{F(z)}=\frac{z^2\cdot h'(z^2)}{h(z^2)},\qquad h\in\mathrm{T}.$$

Taking into consideration Theorem 1 (i) and (ii) we get

$$\operatorname{Re}\frac{zF'(z)}{F(z)}=\operatorname{Re}\frac{z^2\cdot h'(z^2)}{h(z^2)}\geqslant\begin{cases}\dfrac{1-r^2}{1+r^2} & \text{for } 0\leqslant r<\sqrt{2-\sqrt{3}},\\[2ex] \dfrac{1-6r^4+r^8}{1-r^8} & \text{for } \sqrt{2-\sqrt{3}}\leqslant r<1.\end{cases}$$

Therefore, $\operatorname{Re}(zF'(z)/F(z))>0$ for $0\leqslant r<\sqrt{\sqrt{2}-1}$. Thus, $r_{ST}(\mathrm{T}_g^{(2)})\geqslant\sqrt{\sqrt{2}-1}$. Observe that $\min\operatorname{Re}(zF'(z)/F(z))$ is reached by the function

$$F_0(z)=\frac{z\sqrt{1+z^4}}{1-z^4}\quad\text{for } z=r\sqrt{\mathrm{i}},\ r\in\bigl[\sqrt{2-\sqrt{3}},1\bigr).$$

This follows from the fact that $\min\operatorname{Re}(zf'(z)/f(z))$ is reached by the function f given in Theorem 1 (i) for $z=r\sqrt{\mathrm{i}}$, $r\in[\sqrt{2-\sqrt{3}},1)$.

Moreover, we have $F_0'(\sqrt{\sqrt{2}-1}\cdot\sqrt{\mathrm{i}})=0$. Hence, we obtain that $r_{ST}(\mathrm{T}_g^{(2)})=r_{LU}(\mathrm{T}_g^{(2)})=r_S(\mathrm{T}_g^{(2)})=\sqrt{\sqrt{2}-1}$.

Let us observe that for $F\in\mathrm{T}_g^{(2)}$ we have $F'(z)=z\cdot h'(z^2)/\sqrt{h(z^2)}$ and

$$F'(z)\neq 0\Leftrightarrow h'(z^2)\neq 0. \tag{6}$$

We know (see [1]) that the domain of local univalence for the class T is the set

$$\Bigl\{z\in\Delta\colon \Bigl|z+\frac{1}{z}\Bigr|>2\Bigr\}. \tag{7}$$

From (6) and (7) we get that the domain of local univalence for the class T is given by

$$\Bigl\{z\in\Delta\colon \Bigl|z^2+\frac{1}{z^2}\Bigr|>2\Bigr\}.$$

Putting $z=re^{\mathrm{i}\theta}$, $r<1$, $\theta\in[0,2\pi]$ into the equation $|z^4+1|=2|z|^2$, we get the boundary of the domain of local univalence (see Figure 2), which is given by the following parametrical equation:

$$\begin{cases}x=\sqrt[4]{2-\cos 4\theta-\sqrt{(3-\cos 4\theta)(1-\cos 4\theta)}}\cdot\cos\theta,\\ y=\sqrt[4]{2-\cos 4\theta-\sqrt{(3-\cos 4\theta)(1-\cos 4\theta)}}\cdot\sin\theta,\end{cases}\qquad \theta\in[0,2\pi].$$

Hence, we have proved the following theorem:

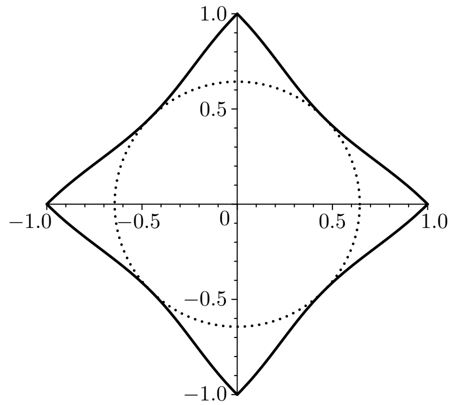

Figure 2. The boundary of the domain of local univalence of $\mathrm{T}_g^{(2)}$ (solid line) and $|z| = \sqrt{\sqrt{2}-1}$ (dot line).

Theorem 7. *For the class* $\mathrm{T}_g^{(2)}$ *where* $g(z) = z/(1+z^2)$ *we have*

(i) $r_{LU}(\mathrm{T}_g^{(2)}) = r_{ST}(\mathrm{T}_g^{(2)}) = r_S(\mathrm{T}_g^{(2)}) = \sqrt{\sqrt{2}-1}$;

(ii) *the domain of local univalence is the set which is bounded and symmetric with respect to both axes, whose boundary in the first quadrant of the complex plane is of the form*

$$\begin{cases} x = \sqrt[4]{2-\cos 4\theta - \sqrt{(3-\cos 4\theta)(1-\cos 4\theta)}} \cdot \cos\theta, \\ y = \sqrt[4]{2-\cos 4\theta - \sqrt{(3-\cos 4\theta)(1-\cos 4\theta)}} \cdot \sin\theta, \end{cases} \qquad \theta \in \left[0, \frac{\pi}{2}\right].$$

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