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ON THE MAXIMAL OPERATOR OF WALSH-KACZMARZ-FEJÉR MEANS

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Abstract. In this paper we prove that the maximal operator

$$\tilde{\sigma}^{\kappa,*} f := \sup_{n \in \mathbb{P}} \frac{|\sigma_n^\kappa f|}{\log^2(n+1)},$$

where $\sigma_n^\kappa f$ is the n-th Fejér mean of the Walsh-Kaczmarz-Fourier series, is bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

Keywords: Walsh-Kaczmarz system, Fejér means, maximal operator

MSC 2010: 42C10

1. Introduction

The a.e. convergence of Walsh-Fejér means $\sigma_n f$ was proved by Fine [2]. In 1975 Schipp [11] showed that the maximal operator σ^* is of weak type $(1,1)$ and of type (p,p) for $1 < p \leqslant \infty$. The boundedness fails to hold for $p = 1$. But, Fujii [3] proved that σ^* is bounded from the dyadic Hardy space H_1 to the space L_1. The theorem of Fujii was extended by Weisz [21], he showed that the maximal operator σ^* is bounded from the martingale Hardy space H_p to the space L_p for $p > 1/2$. Simon gave a counterexample [13], which shows that the boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ is due to Goginava [9]. In the endpoint case $p = 1/2$ two positive results were showed. Weisz [22] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. In 2008 Goginava [8] proved that the maximal operator $\tilde{\sigma}^*$ defined by

$$\tilde{\sigma}^* f := \sup_{n \in \mathbb{P}} \frac{|\sigma_n f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. He also proved that for any nondecreasing function $\varphi\colon\ \mathbb{P}\to[1,\infty)$ satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{\log^2(n+1)}{\varphi(n)}=+\infty \tag{1.1}$$

the maximal operator $\sup\limits_{n\in\mathbb{P}}|\sigma_n f|/\varphi(n)$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

In 1948 Šneider [16] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty}\frac{D_n^{\kappa}(x)}{\log n}\geqslant C>0$$

holds a.e. In 1974 Schipp [12] and Young [18] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [15] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous function f. Gát [4] proved, for any integrable function, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. He showed that the maximal operator of Walsh-Kaczmarz-Fejér means $\sigma^{\kappa,*}$ is weak type $(1,1)$ and of type (p,p) for all $1<p\leqslant\infty$. Gát's result was generalized by Simon [14], who showed that the maximal operator $\sigma^{\kappa,*}$ is of type (H_p,L_p) for $p>1/2$.

In the endpoint case $p=1/2$ the first author [6] proved that maximal operator is not of type $(H_{1/2},L_{1/2})$ and Weisz [22] showed that the maximal operator is of weak type $(H_{1/2},L_{1/2})$.

In the present paper we prove that the maximal operator $\tilde\sigma^{\kappa,*}$ defined by

$$\tilde\sigma^{\kappa,*}:=\sup_{n\in\mathbb{P}}\frac{|\sigma_n^{\kappa}f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. We also prove that for any nondecreasing function $\varphi\colon\ \mathbb{P}\to[1,\infty)$ satisfying the condition (1.1) the maximal operator $\sup\limits_{n\in\mathbb{P}}|\sigma_n^{\kappa}f|/\varphi(n)$ is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

2. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis [1], [10]. Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}:=\mathbb{P}\cup\{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of countably many copies of the compact group $\mathbb{Z}_2$. The elements of G are of the form $x=(x_0,x_1,\dots,x_k,\dots)$ with

$x_k \in \{0,1\}$ $(k \in \mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G,\ I_n(x) := I_n(x_0,\dots,x_{n-1}) \\ &:= \{y \in G\colon\ y = (x_0,\dots,x_{n-1},y_n,y_{n+1},\dots)\}, \end{aligned}$$

$(x \in G,\ n \in \mathbb{N})$. These sets are called dyadic intervals. Let $0 = (0\colon\ i \in \mathbb{N}) \in G$ denote the null element of G, and $I_n := I_n(0)$ $(n \in \mathbb{N})$. Set $e_n := (0,\dots,0,1,0,\dots) \in G$, the n-th coordinate of which is 1 and the rest are zeros $(n \in \mathbb{N})$.

For $k \in \mathbb{N}$ and $x \in G$ denote

$$r_k(x) := (-1)^{x_k}$$

the k-th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum\limits_{i=0}^{\infty} n_i 2^i$ can be written, where $n_i \in \{0,1\}$ $(i \in \mathbb{N})$, i.e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbb{N}\colon\ n_j \neq 0\}$, that is $2^{|n|} \leqslant n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_k} \quad (x \in G,\ n \in \mathbb{P}).$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geqslant 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum\limits_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions is the same in dyadic blocks. Namely,

$$\{\kappa_n\colon\ 2^k \leqslant n < 2^{k+1}\} = \{w_n\colon\ 2^k \leqslant n < 2^{k+1}\}$$

for all $k \in \mathbb{P}$ and $\kappa_0 = w_0$.

V. A. Skvortsov (see [15]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions with the help of the transformation $\tau_A\colon G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots)$$

for $A \in \mathbb{N}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).$$

The Dirichlet kernels are defined by

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

where $\alpha_n = w_n$ or κ_n ($n \in \mathbb{P}$), $D_0^\alpha := 0$. The 2^n-th Dirichlet kernels have a closed form (see e.g. [10])

$$D_{2^n}^w(x) = D_{2^n}^\kappa(x) = D_{2^n}(x) = \begin{cases} 0 & \text{if } x \notin I_n, \\ 2^n & \text{if } x \in I_n. \end{cases} \tag{2.1}$$

The σ-algebra generated by the dyadic intervals of measure 2^{-k} will be denoted by F_k ($k \in \mathbb{N}$). Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to $(F_n, n \in \mathbb{N})$ (for details see, e. g. [19], [20]). The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

In case $f \in L_1(G)$, the maximal function can also be given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) \, \mathrm{d}\mu(u) \right|, \quad x \in G.$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G)$ consists of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G)$, then it is easy to show that the sequence $(S_{2^n} f \colon n \in \mathbb{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \dots)$ then the Walsh-(Kaczmarz)-Fourier coefficients must be defined in a little bit different way:

$$\hat{f}(i) = \lim_{k \to \infty} \int_G f^{(k)}(x) \alpha_i(x) \, \mathrm{d}\mu(x) \quad (\alpha_i = w_i \text{ or } \kappa_i).$$

The Walsh-(Kaczmarz)-Fourier coefficients of $f \in L_1(G)$ are the same as the ones of the martingale $(S_{2^n} f \colon n \in \mathbb{N})$ obtained from f.

The partial sums of the Walsh-(Kaczmarz)-Fourier series are defined as follows:

$$S_M^\alpha(f; x) := \sum_{i=0}^{M-1} \hat{f}(i) \alpha_i(x) \quad (\alpha = w \text{ or } \kappa).$$

For $n = 1, 2, \dots$ and a martingale f the Fejér mean of order n of the Walsh-(Kaczmarz)-Fourier series of the function f is given by

$$\sigma_n^\alpha(f;x) = \frac{1}{n}\sum_{j=0}^{n-1} S_j^\alpha(f;x).$$

The Fejér kernel of order n of the Walsh-(Kaczmarz)-Fourier series defined by

$$K_n^\alpha(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k^\alpha(x).$$

For a martingale f we consider the maximal operators

$$\sigma^{\kappa,*} f = \sup_{n\in\mathbb{P}} |\sigma_n^\kappa(f;x)|,$$
$$\widetilde{\sigma}^{\kappa,*} f = \sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa(f;x)|}{\log^2(n+1)}.$$

3. Auxiliary propositions and main results

First, we formulate our main theorems.

Theorem 3.1. *The maximal operator $\tilde{\sigma}^{\kappa,*}$ is bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.*

Theorem 3.2. *Let $\varphi\colon \mathbb{P} \to [1,\infty)$ be a nondecreasing function satisfying the condition* (1.1). *Then the maximal operator*

$$\sup_{n\in\mathbb{P}} \frac{|\sigma_n^\kappa f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

To prove our Theorem 3.1 we need the following Lemmas:

Lemma 3.3 (Skvortsov [15]). *For $n \in \mathbb{P}$, $x \in G$*

$$nK_n^\kappa(x) = 1 + \sum_{i=0}^{|n|-1} 2^i D_{2^i}(x) + \sum_{i=0}^{|n|-1} 2^i r_i(x) K_{2^i}^w(\tau_i(x))$$
$$+ (n - 2^{|n|})(D_{2^{|n|}}(x) + r_{|n|}(x) K_{n-2^{|n|}}^w(\tau_{|n|}(x))).$$

Lemma 3.4 (Gát [4]). *Let A, $t \in \mathbb{N}$, $A > t$. Suppose that $x \in I_t \setminus I_{t+1}$. Then*

$$K_{2^A}^w(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_A, \\ 2^{t-1} & \text{if } x - x_t e_t \in I_A. \end{cases}$$

If $x \in I_A$, then $K_{2^A}^w(x) = \frac{1}{2}(2^A - 1)$.

A bounded measurable function a is a p-atom if there exists a dyadic interval I, such that

a) $\int_I a \, \mathrm{d}\mu = 0$,
b) $\|a\|_\infty \leqslant \mu(I)^{-1/p}$,
c) $\operatorname{supp} a \subset I$.

Lemma 3.5 (Weisz [20]). *Suppose that the operator T is sublinear and p-quasilocal for any $0 < p \leqslant 1$. If T is bounded from L_∞ to L_∞, then*

$$\|Tf\|_p \leqslant c_p \|f\|_{H_p} \quad \text{for all } f \in H_p.$$

Lemma 3.6 (Gát, Goginava, Nagy [5]). *Let $n < 2^{A+1}$, $A > N$ and $x \in I_N(x_0, \ldots, x_{m-1}, x_m = 1, 0, \ldots, 0, x_l = 1, 0, \ldots, 0) =: J_N^{m,l}$, $l = 0, \ldots, N-1$, $m = -1, 0, \ldots, l$. Then*

$$\int_{I_N} n |K_n^w(\tau_A(x+t))| \, \mathrm{d}t \leqslant c \frac{2^A}{2^{m+l}},$$

where

$$I_N(x_0, \ldots, x_m = 1, 0, \ldots, 0, x_l = 1, 0, \ldots, 0) := I_N(0, \ldots, 0, x_l = 1, 0, \ldots, 0)$$

for $m = -1$.

Lemma 3.7 (Goginava [7]). *Let $2 < A \in \mathbb{P}$ and $q_A := 2^{2A} + 2^{2A-2} + \ldots + 2^2 + 2^0$. Then*

$$q_{A-1} |K_{q_{A-1}}(x)| \geqslant 2^{2m+2s-3}$$

for $x \in I_{2A}(0, \ldots, 0, x_{2m} = 1, 0, \ldots, 0, x_{2s} = 1, x_{2s+1}, \ldots, x_{2A-1})$, $m = 0, 1, \ldots, A-3$, $s = m+2, m+3, \ldots, A-1$.

First, we prove Theorem 3.1.

Proof of Theorem 3.1. Lemma 3.3 yields that

$$\begin{aligned}\tilde{\sigma}_n^{\kappa} f &= \frac{|f * K_n^{\kappa}|}{\log^2(n+1)} \\ &\leqslant \left| f * \frac{1}{n|n|^2}\Bigl(1 + \sum_{i=0}^{|n|-1} 2^i D_{2^i}\Bigr)\right| + \left| f * \frac{1}{n|n|^2} \sum_{i=0}^{|n|-1} 2^i r_i K_{2^i}^w \circ \tau_i \right| \\ &\quad + \left| f * \frac{n - 2^{|n|}}{n|n|^2}(D_{2^{|n|}} + r_{|n|} K^w_{n-2^{|n|}} \circ \tau_{|n|})\right| =: \sum_{i=1}^{3} |f * L_n^i|\end{aligned}$$

and

$$\tilde{\sigma}^{\kappa,*} f \leqslant \sup_{n\in\mathbb{P}} |f * L_n^1| + \sup_{n\in\mathbb{P}} |f * L_n^2| + \sup_{n\in\mathbb{P}} |f * L_n^3| =: R^1 f + R^2 f + R^3 f.$$

With the help of Lemma 3.5 we show that the operators R^i $(i = 1, 2, 3)$ are of type $(H_{1/2}, L_{1/2})$. The boundedness from the space L_∞ to the space L_∞ follows from (2.1) and

$$\|K_n^w \circ \tau_i\|_1 = \|K_n^w\|_1 \leqslant 2$$

for $i \leqslant |n|$, $n \in \mathbb{P}$ (see Yano [17]). By Lemma 3.5, the proof will be complete if we show that the maximal operators R^i $(i = 1, 2, 3)$ are 1/2-quasilocal. That is, there exists a constant c such that

$$\int_{\overline{I}} |R^i a|^{1/2} \, \mathrm{d}\mu \leqslant c < \infty$$

for every 1/2-atom a, where the dyadic interval I is the support of the 1/2-atom a.

Let a be an arbitrary 1/2-atom with support I, and $\mu(I) = 2^{-N}$. Without loss of generality, we may assume that $I := I_N$.

It is evident that $\tilde{\sigma}_n^{\kappa}(a) = 0$ and $a * L_n^i = 0$ $(i = 1, 2, 3)$ if $n \leqslant 2^N$. Therefore, we set $n > 2^N$.

By $\|a\|_\infty \leqslant c2^{2N}$ we have that

$$|a * L_n^i| \leqslant \int_{I_N} |a(s)||L_n^i(x+s)| \, \mathrm{d}\mu(s) \leqslant c2^{2N} \int_{I_N} |L_n^i(x+s)| \, \mathrm{d}\mu(s)$$

and

$$|R^i a| \leqslant c2^{2N} \sup_{n > 2^N} \int_{I_N} |L_n^i(x+s)| \, \mathrm{d}\mu(s). \tag{4.1}$$

Now, we write

$$\overline{I_N} = \bigcup_{j=0}^{N-1} (I_j \setminus I_{j+1}).$$

Set $x \in I_j \setminus I_{j+1}$ and $s \in I_N$, then $x + s \in I_j \setminus I_{j+1}$ for $j = 0, \dots, N-1$. Thus, we have

$$\begin{aligned}\sup_{n>2^N} \int_{I_N} |L_n^1(x+s)| \,\mathrm{d}\mu(s) &\leqslant \sup_{n>2^N} \int_{I_N} \frac{1}{n|n|^2}\left(1 + \sum_{i=0}^{j} 2^i D_{2^i}(x+s)\right) \mathrm{d}\mu(s)\\ &\leqslant \frac{c}{2^N N^2} 2^{2j} 2^{-N} \leqslant \frac{c 2^{2j}}{N^2 2^{2N}}\end{aligned}$$

and

$$\begin{aligned}\int_{\overline{I_N}} |R^1 a(x)|^{1/2} \,\mathrm{d}\mu(x) &= \sum_{j=0}^{N-1} \int_{I_j \setminus I_{j+1}} |R^1 a(x)|^{1/2} \,\mathrm{d}\mu(x)\\ &\leqslant c 2^N \sum_{j=0}^{N-1} \int_{I_j \setminus I_{j+1}} \left(\frac{2^{2j}}{N^2 2^{2N}}\right)^{1/2} \mathrm{d}\mu(x) \leqslant \frac{cN}{N} \leqslant c.\end{aligned}$$

Now, we discuss $\int_{\overline{I_N}} |R^2 a|^{1/2} \,\mathrm{d}\mu$. We use the disjoint decomposition of $\overline{I_N}$ above. That is, $\overline{I_N} = \bigcup_{t=0}^{N-1} (I_t \setminus I_{t+1})$ and we decompose the sets $I_t \setminus I_{t+1}$ as the following disjoint union:

$$I_t \setminus I_{t+1} = \bigcup_{l=t+1}^{N} J_t^l,$$

where $J_t^l := I_N(0, \dots, 0, x_t = 1, 0, \dots, 0, x_l = 1, x_{l+1}, \dots, x_{N-1})$ for $t < l < N$ and $J_t^l := I_N(e_t)$ for $l = N$.

$$\begin{aligned}\int_{\overline{I_N}} |R^2 a(x)|^{1/2} \,\mathrm{d}\mu(x) &= \sum_{t=0}^{N-1} \sum_{l=t+1}^{N} \int_{J_t^l} |R^2 a(x)|^{1/2} \,\mathrm{d}\mu(x)\\ &= \sum_{t=0}^{N-1} \sum_{l=t+1}^{N-1} \int_{J_t^l} |R^2 a(x)|^{1/2} \,\mathrm{d}\mu(x)\\ &\quad + \sum_{t=0}^{N-1} \int_{J_t^N} |R^2 a(x)|^{1/2} \,\mathrm{d}\mu(x) =: \Sigma_1 + \Sigma_2.\end{aligned}$$

Let $x \in J_t^l$ and $s \in I_N$, then $x + s \in J_t^l$ $(0 \leqslant t < N,\ t < l \leqslant N)$.

For $0 \leqslant t < l < N$, Lemma 3.4 and $K^w_{2^i}(\tau_i(x+s)) \neq 0$ imply that $i \leqslant l$, $K^w_{2^i}(\tau_i(x+s)) = 2^{i-t-2}$ for $l > i > t$ and $K^w_{2^i}(\tau_i(x+s)) = \frac{1}{2}(2^i - 1)$ for $i \leqslant t$. Thus,

$$\begin{aligned}
\sup_{n>2^N} \int_{I_N} |L^2_n(x+s)| \,\mathrm{d}\mu(s) &\leqslant \sup_{n>2^N} \frac{1}{n|n|^2} \int_{I_N} \sum_{i=0}^{l} 2^i |K^w_{2^i}(\tau_i(x+s))| \,\mathrm{d}\mu(s) \\
&\leqslant \sup_{n>2^N} \frac{c}{n|n|^2} \int_{I_N} \left(\sum_{i=0}^{t} 2^{2i} + \sum_{i=t+1}^{l} 2^i 2^{i-t} \right) \mathrm{d}\mu(s) \\
&\leqslant \frac{c(2^{2t} + 2^{2l-t})}{2^{2N} N^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Sigma_1 &\leqslant c \sum_{t=0}^{N-1} \sum_{l=t+1}^{N-1} \int_{J^l_t} 2^N \left(\frac{(2^{2t} + 2^{2l-t})}{2^{2N} N^2} \right)^{1/2} \mathrm{d}\mu(x) \\
&\leqslant c \sum_{t=0}^{N-1} \sum_{l=t+1}^{[3t/2]} \int_{J^l_t} \frac{2^t}{N} \,\mathrm{d}\mu(x) + c \sum_{t=0}^{N-1} \sum_{l=[3t/2]+1}^{N-1} \int_{J^l_t} \frac{2^{l-t/2}}{N} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{t=0}^{N-1} \sum_{l=t+1}^{[3t/2]} \sum_{\substack{y_i=0 \\ i\in\{l+1,\ldots,N-1\}}}^{1} \int_{I_N(y+e_t+e_l)} \frac{2^t}{N} \,\mathrm{d}\mu(x) \\
&\quad + c \sum_{t=0}^{N-1} \sum_{l=[3t/2]+1}^{N-1} \sum_{\substack{y_i=0 \\ i\in\{l+1,\ldots,N-1\}}}^{1} \int_{I_N(y+e_t+e_l)} \frac{2^{l-t/2}}{N} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{t=0}^{N-1} \sum_{l=t+1}^{[3t/2]} \frac{2^t}{N} 2^{-l} + c \sum_{t=0}^{N-1} \sum_{l=[3t/2]+1}^{N-1} \frac{2^{l-t/2}}{N} 2^{-l} \leqslant c.
\end{aligned}$$

For $l = N$, let $x \in J^N_t$. Lemma 3.4 yields

$$\begin{aligned}
&\sup_{n>2^N} \int_{I_N} |L^2_n(x+s)| \,\mathrm{d}\mu(s) \leqslant \sup_{n>2^N} \frac{1}{n|n|^2} \int_{I_N} \sum_{i=0}^{|n|-1} 2^i |K^w_{2^i}(\tau_i(x+s))| \,\mathrm{d}\mu(s) \\
&\leqslant c \sup_{n>2^N} \frac{1}{n|n|^2} \left(\int_{I_N} \left(\sum_{i=0}^{t} 2^{2i} + \sum_{i=t+1}^{N} 2^i 2^{i-t} \right) \mathrm{d}\mu(s) + \sum_{i=N+1}^{|n|-1} \int_{I_i(x_{N,i-1})} 2^i 2^{i-t} \,\mathrm{d}\mu(s) \right) \\
&\leqslant c \sup_{n>2^N} \frac{2^{2t-N} + 2^{N-t} + 2^{|n|-t}}{n|n|^2} \leqslant \frac{c 2^{2t}}{2^{2N} N^2} + \frac{c}{2^t N^2},
\end{aligned}$$

where $x_{N,i-1} := \sum\limits_{j=N}^{i-1} x_j e_j$;

$$\begin{aligned}
\Sigma_2 &\leqslant c \sum_{t=0}^{N-1} \int_{J_t^N} 2^N \Big(\frac{2^{2t}}{2^{2N} N^2} + \frac{1}{2^t N^2} \Big)^{1/2} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{t=0}^{[2N/3]} \int_{J_t^N} \frac{2^N}{2^{t/2} N} \,\mathrm{d}\mu(x) + c \sum_{t=[2N/3]+1}^{N-1} \int_{J_t^N} 2^N \frac{2^t}{2^N N} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{t=0}^{[2N/3]} \frac{1}{2^{t/2} N} + c \sum_{t=[2N/3]+1}^{N-1} \frac{2^t}{2^N N} \leqslant c.
\end{aligned}$$

To discuss $\int_{\overline{I_N}} |R^3 a|^{1/2} \,\mathrm{d}\mu$ we use Lemma 3.6 and the following disjoint decomposition of $\overline{I_N}$:

$$\overline{I_N} = \bigcup_{l=0}^{N-1} \bigcup_{m=-1}^{l} J_N^{l,m},$$

where the set $J_N^{l,m}$ is defined in Lemma 3.6.

If $x \in \overline{I_N}$ and $s \in I_N$, then $x+s \in \overline{I_N}$ and $D_{2^{|n|}}(x+s) = 0$. Moreover, if $x \in J_N^{l,m}$, then $x + s \in J_N^{l,m}$ and by Lemma 4 we have

$$\begin{aligned}
\sup_{n>2^N} \int_{I_N} |L_n^3(x+s)| \,\mathrm{d}\mu(s) &\leqslant \sup_{n>2^N} \int_{I_N} \frac{n - 2^{|n|}}{n|n|^2} |K^w_{n-2^{|n|}}(\tau_{|n|}(x+s))| \,\mathrm{d}\mu(s) \\
&\leqslant c \sup_{n>2^N} \frac{1}{n|n|^2} \frac{2^{|n|}}{2^{l+m}} \leqslant \frac{c}{2^{l+m} N^2}.
\end{aligned}$$

By the above

$$\begin{aligned}
\int_{\overline{I_N}} |R^3 a(x)|^{1/2} \,\mathrm{d}\mu(x) &= \sum_{l=0}^{N-1} \sum_{m=-1}^{l} \int_{J_N^{l,m}} |R^3 a(x)|^{1/2} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{l=0}^{N-1} \sum_{m=-1}^{l} \int_{J_N^{l,m}} \frac{2^N}{2^{(l+m)/2} N} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{l=0}^{N-1} \sum_{m=-1}^{l} \sum_{\substack{y_i=0 \\ i\in\{0,\ldots,m-1\}}}^{1} \int_{I_N(y+e_m+e_l)} \frac{2^N}{2^{(l+m)/2} N} \,\mathrm{d}\mu(x) \\
&\leqslant c \sum_{l=0}^{N-1} \sum_{m=-1}^{l} \frac{2^N}{2^{(l+m)/2} N} 2^{-N+m} \leqslant c.
\end{aligned}$$

This completes the proof of Theorem 3.1. $\square$

Next, we prove Theorem 3.2.

P r o o f of Theorem 3.2. Let $\{n_k\colon\ k\in\mathbb{P}\}$ be an increasing sequence of positive integers such that

$$\lim_{k\to\infty}\frac{\log^2 n_k}{\varphi(n_k)}=+\infty.$$

It is evident that for every n_k there exists a positive integer m_k' such that

$$q_{m_k'}\leqslant n_k<q_{m_k'+1}<5q_{m_k'}.$$

Since $\varphi(n)$ is nondecreasing function we have

$$\overline{\lim_{k\to\infty}}\frac{(m_k')^2}{\varphi(q_{m_k'})}\geqslant c\lim_{k\to\infty}\frac{\log^2 n_k}{\varphi(n_k)}=+\infty.$$

Let $\{m_k\colon\ k\in\mathbb{P}\}\subset\{m_k'\colon\ k\in\mathbb{P}\}$ such that

$$\lim_{k\to\infty}\frac{(m_k)^2}{\varphi(q_{m_k})}=+\infty.$$

Let

$$f_{m_k}(x):=D_{2^{2m_k+1}}(x)-D_{2^{2m_k}}(x).$$

It is evident that

$$\hat f^{\kappa}_{m_k}(i)=\begin{cases}1, & \text{if } i=2^{2m_k},\dots,2^{2m_k+1}-1,\\ 0, & \text{otherwise.}\end{cases}$$

Then we can write

$$S_i^{\kappa}(f_{m_k};x)=\begin{cases}D_i^{\kappa}(x)-D_{2^{2m_k}}(x), & i=2^{2m_k}+1,\dots,2^{2m_k+1}-1,\\ f_{m_k}(x), & i\geqslant 2^{2m_k+1},\\ 0, & \text{otherwise.}\end{cases}\tag{4.2}$$

Since,

$$f^*_{m_k}(x)=\sup_{n\in\mathbb{N}}|S_{2^n}(f_{m_k};x)|=|f_{m_k}(x)|,$$

from (2.1) we get

$$\|f_{m_k}\|_{H_p}=\|f^*_{m_k}\|_p=\|D_{2^{2m_k}}\|_p=2^{2m_k(1-1/p)}.\tag{4.3}$$

Since we have

$$D_n^{\kappa}(x)=D_{2^{|n|}}(x)+r_{|n|}(x)D^w_{n-2^{|n|}}(\tau_{|n|}(x)),$$

from (4.2) we can write

$$\begin{aligned}
\sup_{n\in\mathbb{P}}\frac{|\sigma_n^\kappa(f_{m_k};x)|}{\varphi(n)} &\geqslant \frac{|\sigma_{q_{m_k}}^\kappa(f_{m_k};x)|}{\varphi(q_{m_k})}\\
&= \frac{1}{\varphi(q_{m_k})}\frac{1}{q_{m_k}}\Bigg|\sum_{j=0}^{q_{m_k}-1} S_j^\kappa(f_{m_k};x)\Bigg|\\
&= \frac{1}{\varphi(q_{m_k})}\frac{1}{q_{m_k}}\Bigg|\sum_{j=2^{2m_k}}^{q_{m_k}-1} S_j^\kappa(f_{m_k};x)\Bigg|\\
&= \frac{1}{\varphi(q_{m_k})}\frac{1}{q_{m_k}}\Bigg|\sum_{i=2^{2m_k}}^{q_{m_k}-1} (D_i^\kappa(x)-D_{2^{2m_k}}(x))\Bigg|\\
&= \frac{1}{\varphi(q_{m_k})}\frac{1}{q_{m_k}}\Bigg|\sum_{i=0}^{q_{m_k-1}-1} (D_{i+2^{2m_k}}^\kappa(x)-D_{2^{2m_k}}(x))\Bigg|\\
&= \frac{1}{\varphi(q_{m_k})}\frac{1}{q_{m_k}}\Bigg|\sum_{i=0}^{q_{m_k-1}-1} D_i^w(\tau_{2m_k}(x))\Bigg|\\
&= \frac{1}{\varphi(q_{m_k})}\frac{q_{m_k-1}}{q_{m_k}}|K_{q_{m_k-1}}^w(\tau_{2m_k}(x))|.
\end{aligned}$$

Let $x\in J_{2m_k}^{2A-2s-1,2A-2l-1}$ for some $l<s<m_k$. Then from Lemma 3.7 we have

$$\frac{\sigma_{q_{m_k}}^\kappa(f_{m_k};x)}{\varphi(q_{m_k})}\geqslant c\frac{2^{2s+2l-2m_k}}{\varphi(q_{m_k})}.$$

Hence, we can write

$$\begin{aligned}
&\int_G\left(\sup_{n\in\mathbb{P}}\frac{|\sigma_n^\kappa(f_{m_k};x)|}{\varphi(n)}\right)^{1/2}\mathrm{d}\mu(x)\\
&\geqslant c\sum_{l=0}^{m_k-3}\sum_{s=l+2}^{m_k-1}\int_{J_{2m_k}^{2A-2s-1,2A-2l-1}}\left(\sup_{n\in\mathbb{P}}\frac{|\sigma_n^\kappa(f_{m_k};x)|}{\varphi(n)}\right)^{1/2}\mathrm{d}\mu(x)\\
&\geqslant c\sum_{l=0}^{m_k-3}\sum_{s=l+2}^{m_k-1}\sum_{\substack{y_i=0\\ i\in\{0,\dots,2m_k-2s-2\}}}^{1}\int_{I_{2m_k}(y+e_{2A-2s-1}+e_{2A-2l-1})}\left(\frac{|\sigma_{q_{m_k}}^\kappa(f_{m_k};x)|}{\varphi(q_{m_k})}\right)^{1/2}\mathrm{d}\mu(x)\\
&\geqslant c\sum_{l=0}^{m_k-3}\sum_{s=l+2}^{m_k-1}\frac{2^{2m_k-2s}}{2^{2m_k}}\frac{2^{s+l-m_k}}{\sqrt{\varphi(q_{m_k})}}\\
&\geqslant\frac{c}{\sqrt{\varphi(q_{m_k})}}\frac{m_k}{2^{m_k}}.
\end{aligned}$$

Then from (4.3) we obtain

$$\left\{\int_G \left(\sup_{n\in\mathbb{P}} \frac{|\sigma_n^{\kappa}(f_{m_k};x)|}{\varphi(n)}\right)^{1/2} \mathrm{d}\mu(x)\right\}^2 \Big/ \|f_{m_k}\|_{H_{1/2}}$$
$$\geqslant \frac{cm_k^2}{\varphi(q_{m_k})2^{-2m_k}2^{2m_k}} = \frac{cm_k^2}{\varphi(q_{m_k})} \to \infty \quad \text{as } k \to \infty.$$

Theorem 3.2 is proved. □

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