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LONG-TIME BEHAVIOR OF SMALL SOLUTIONS TO QUASILINEAR DISSIPATIVE HYPERBOLIC EQUATIONS

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Abstract. We give sufficient conditions for the existence of global small solutions to the quasilinear dissipative hyperbolic equation

$$u_{tt} + 2u_t - a_{ij}(u_t, \nabla u)\partial_i\partial_j u = f$$

corresponding to initial values and source terms of sufficiently small size, as well as of small solutions to the corresponding stationary version, i.e. the quasilinear elliptic equation

$$-a_{ij}(0, \nabla v)\partial_i\partial_j v = h.$$

We then give conditions for the convergence, as $t \to \infty$, of the solution of the evolution equation to its stationary state.

Keywords: quasilinear evolution equation, quasilinear elliptic equation, a priori estimates, global existence, asymptotic behavior, stationary solutions

MSC 2010: 35J15, 35J60, 35L15, 35L70

1. Introduction

In this paper we consider the Cauchy problem for the quasilinear dissipative hyperbolic evolution equation

$$u_{tt} + 2u_t - a_{ij}(u_t, \nabla u)\partial_i\partial_j u = f \tag{1.1}$$

with $f = f(t,x)$, $t \geqslant 0$, $x \in \mathbb{R}^N$, $N \geqslant 3$, and u subject to the initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{1.2}$$

In (1.1), as well as in the sequel, summation for i, j from 1 to N is understood. Our goal is to prove a global existence result for small strong solutions of (1.1) (as defined in the beginning of Section 4 below), corresponding to small data u_0, u_1, and f (if the data are large, blow up of solutions of nonlinear hyperbolic problems in finite time is in general expected), and then to study the behavior of such solutions as $t \to \infty$. In particular, we look for sufficient conditions that imply the boundedness of the solutions in t, as well their convergence to the solutions of the corresponding stationary equation

$$-a_{ij}(0, \nabla v)\partial_i\partial_j v = h. \tag{1.3}$$

When $f \equiv 0$, global existence results for strong solutions of (1.1) corresponding to small initial values (1.2) were given by Matsumura [9], by means of direct energy estimates which yield an *a priori* bound on any local solution. In the case of a bounded domain, this method shows that solutions decay exponentially to 0. In the whole space case, polynomial decay to 0 can still be established by considering (1.1) as a "small" perturbation of the linear dissipative wave equation

$$u_{tt} + 2u_t - \Delta u = 0, \tag{1.4}$$

whose solutions are known to decay, via the variation of parameters formula (see Section 3, where we also report the explicit decay rates for solutions of (1.4) provided by Matsumura [8]). This procedure is reminiscent of the so-called "L^p-L^q estimates" technique, used e.g. by Klainerman [6] and Ponce [14], to prove a global existence result for small solutions of nonlinear non-dissipative wave equations, and Racke [15], for nonlinear dissipative wave equations of the general form

$$u_{tt} + u_t - \partial_j[a_{jk}(x)\partial_k u] = f(x, t, u, Du, \nabla u_t, \partial_x^2 u) \tag{1.5}$$

(which, however, does not include (1.1) when $f \equiv 0$). In fact, we will follow this same perturbation method to prove an a priori global bound on solutions of (1.1), which ensures their extendibility to a global bounded solution. In contrast, we will solve the stationary equation (1.3) directly, by means of a variational technique and a fixed point argument, and resort again to energy estimates to show the convergence of the solutions of (1.1) to those of (1.3). The nonhomogeneous case (non-zero f) is not explicitly treated by Matsumura and Racke; while it is not difficult to extend their methods in the bounded domain case, with solutions that remain bounded as $t \to \infty$, we have not been able to find analogous results in the whole space case. Our results in this direction seem to require an additional integrability condition at infinity on f, either with respect to t (namely, $f \in L^1(0, \infty; H^s)$), or with respect to x (i.e., $f(t,\cdot) \in L^q$ for some $q \in [1,2[$). We do not know if the latter conditions are necessary, although our results on the linear equation

$$u_{tt} + 2u_t - \Delta u = f \tag{1.6}$$

seem to indicate that they almost are. On the other hand, the integrability conditions in x also allow us to examine the convergence of solutions of (1.1) to those of the stationary equation (1.3): in fact, if the integrability of f on $[0,\infty[$ were a consequence of sufficiently fast decay of f, this would prevent its convergence to a non-zero source h of (1.3). In turn, this motivates us to consider the stationary equation in some detail, because while it is relatively straightforward to establish existence of small strong solutions in the bounded domain case, we have not been able to locate analogous results in the case of the whole space.

We refer to [11] for a more detailed discussion of our main motivations and possible applications of our results, which include, among others, models of heat equations with delay (see e.g. Li [7] and Cattaneo [2]), Maxwell's equations in ferro-magnetic materials ([10]), traffic flow models (Schochet [17]), as well as simple models of laser optic equations (Haus [4]) and random walk systems (Hadeler [3]). In these models, the reciprocal of the coefficient of the dissipation term u_t is a measure, respectively, of the delay shift-time; of the displacement of the currents, usually much stronger than the eddy ones; of the drivers' response time to sudden disturbances; of the low frequencies of the electromagnetic field, and of the turning rates of the moving particles. For example, the complete system of Maxwell's equations

$$D_t + J - \operatorname{curl} H = F, \quad B_t + \operatorname{curl} E = 0, \tag{1.7}$$

supplemented by the constituent relations $D = \varepsilon E$, $J = \sigma E$ (linear) and $H = \zeta(B)$ (nonlinear monotone), can be transformed into the quasilinear system

$$\varepsilon A_{tt} + \sigma A_t + \operatorname{curl} \zeta(\operatorname{curl} A) - \nabla \operatorname{div} A = -F, \tag{1.8}$$

which is of type (1.1), by means of the introduction of the electromagnetic potentials A, φ, defined by the relations $B = \operatorname{curl} A$, $E = -A_t + \nabla\varphi$, $\varepsilon\varphi_t + \sigma\varphi = \operatorname{div} A$ (the latter being one of various possible gauge relations). Typically, ferro-magnetic materials are characterized by very small frequency values, hence the interest in studying the long-time behavior of dissipative equations like (1.7) and, more generally, (1.1).

We also refer to [18] for another aspect of the asymptotic behavior of solutions of (1.1) (at least when (1.1) is in divergence form and $f \equiv 0$), consisting in showing that such behavior is the same as that of the solution of the corresponding parabolic equation

$$w_t - \partial_j(a_{ij}(0, \nabla w)\partial_i w) = 0, \tag{1.9}$$

in the sense that the difference $u - w$ decays to 0 faster than either u or w.

This paper is organized as follows. In Section 2 we introduce the notation and function spaces we use in the sequel. In Section 3 we consider the non-homogeneous linear equation (1.6), and give necessary and sufficient conditions for the existence of bounded solutions. In Section 4 we state a global existence and boundedness result on the quasilinear equation (1.1), an existence and uniqueness result for the stationary equation (1.3), and a convergence result of solutions of (1.1) to solutions of (1.3) as $t \to \infty$. We prove these results in Sections 5, 6, and 7.

2. Notation and function spaces

We adopt the following notation throughout this paper. If $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes its integer part. Bounded intervals of $\mathbb{R}$ are denoted by $[a, b]$ if closed, $]a, b[$ if open, $[a, b[$ or $]a, b]$ otherwise. If $u = u(t, x)$ is a smooth function, we denote its partial derivatives with respect to t by u_t, u_{tt}, etc., and with respect to the space variables by $\partial_j u$, $\partial_i\partial_j u$, etc. We also set $\nabla u := (\partial_1 u, \ldots, \partial_N u)$ and $Du := \{u_t, \nabla u\}$; more generally, given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}$, we denote by $|\alpha| := \alpha_1 + \ldots + \alpha_N$ its length, and set $\partial^\alpha u := \partial_1^{\alpha_1} \ldots \partial_N^{\alpha_N} u$. Given a positive integer k, we denote by $\partial_x^k u$ and $\partial_t^k u$ the set of all derivatives of u of order k with respect to the space or the time variables.

For $1 \leqslant p \leqslant \infty$, $|\cdot|_p$ denotes the norm in the Lebesgue space $L^p := L^p(\mathbb{R}^N)$. For $m \in \mathbb{N}$, H^m is the usual Sobolev space $W^{m,2}(\mathbb{R}^N)$ of those functions in L^2 whose distributional derivatives of order up to m are again in L^2. We identify $H^0 = L^2$, and denote by $\|\cdot\|_m$, $\langle\cdot,\cdot\rangle_m$, and $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, respectively, the norms and scalar products in H^m and L^2 (thus, $\|\cdot\| = \|\cdot\|_0 = |\cdot|_2$). Unless there is a risk of misunderstanding, with abuse of notation we also denote by L^p, H^m, etc., the product spaces $(L^p)^N$ and $(H^m)^N, \ldots$; that is, for instance, the formula $\nabla u \in H^m$ means that each component of ∇u is in H^m. Likewise, in many of the estimates we establish, we often omit one or both of the variables t, x, writing e.g. $u(t)$ instead of $u(t, \cdot)$, or $\|u\|_m$ instead of $\|u(t)\|_m$. Finally, we denote by $\mathcal{F}(f) = \hat{f}$ the Fourier transform of a function f whenever defined (e.g., in L^2), and by $\mathcal{F}^{-1}(g) = \check{g}$ its inverse transform whenever defined.

We denote by C a generic, universal constant, which may change from formula to formula, or even within the same formula. Some of these constants depend only on N or s, while other may depend on the coefficients a_{ij} of (1.1); however, they never depend on the data f, u_0, u_1, nor on t, nor on any of the functions involved in any of the formulas where such constants appear. When the specific value of a constant has to be fixed (for example, to define another one, or a parameter such as μ_0 in Theorem 4.1 below), we number that constant, denoting it C_1, C_2, etc. Unless otherwise specified, we assume that $C, C_1, \ldots \in [1, \infty[$.

We recall some results on the Sobolev spaces H^m which we need in the sequel; for a proof, see e.g. Adams-Fournier [1] for the first three, and Racke [16, Lemma 4.7] for the fourth.

1) The continuity of the Sobolev imbedding $H^1 \hookrightarrow L^p$, $2 \leqslant p \leqslant \bar{p} := 2N/(N-2)$, $N \geqslant 3$, in the "limit case" $p = \bar{p}$; that is,

$$|u|_{\bar{p}} \leqslant C_1 |\nabla u|_2, \quad u \in H^1. \tag{2.1}$$

2) The continuity of the imbeddings $H^{m+k} \hookrightarrow C^k \cap W^{k,\infty} \hookrightarrow W^{k,\infty}$, $m > \frac{1}{2}N$ and $k \geqslant 0$; in particular, for $k = 0$,

$$|u|_\infty \leqslant C_2 \|u\|_m, \quad u \in H^m. \tag{2.2}$$

When $N \geqslant 3$, (2.2) can be improved to

$$|u|_\infty \leqslant C_3 \|\nabla u\|_{m-1}, \tag{2.3}$$

as we easily see by direct use of the Fourier transform.

3) The continuity of the imbedding $H^p \cdot H^q \hookrightarrow H^r$, in the sense described by

Proposition 2.1. *Let $p, q, r \in \mathbb{N}$ be such that $p \geqslant r$, $q \geqslant r$, $p + q > r + \frac{1}{2}N$. If $u \in H^p$ and $v \in H^q$, the pointwise product uv is defined a.e. in H^r, and*

$$\|uv\|_r \leqslant C_4 \|u\|_p \|v\|_q, \tag{2.4}$$

with C_4 independent of u and v. In particular, for $p = q = r =: m > \frac{1}{2}N$,

$$\|uv\|_m \leqslant C_5 \|u\|_m \|v\|_m; \tag{2.5}$$

in fact, using (2.3) *and* (2.4), (2.5) *can be improved to*

$$\|uv\|_m \leqslant C_5 \|u\|_m \|\nabla v\|_{m-1}. \tag{2.6}$$

4) The Chain Rule in Sobolev spaces:

Proposition 2.2. *Let $m \in \mathbb{N}_{\geqslant 1}$, $u = (u^1, \dots, u^N) \in H^m$, and $\varphi \in C^m(\mathbb{R}^N; \mathbb{R})$. Then the function $x \mapsto \varphi(u(x)) - \varphi(0)$ is in H^m, and for all $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant m$,*

$$\|\partial^\alpha[\varphi(u) - \varphi(0)]\| \leqslant C_6 \beta_{m,\varphi}(|u|_\infty)(1 + |u|_\infty^{m-1})\|\partial^{|\alpha|}u\|, \tag{2.7}$$

where $\beta_{m,\varphi}(R) := \|\varphi\|_{C^m(\overline{B(0,R)})}$.

Corollary 2.1. *Let $m > \frac{1}{2}N$, $u, v \in H^m$, and φ as in Proposition* 2.2. *Then,*

$$\|\varphi(u)v\|_m \leqslant (|\varphi(0)| + C_5 C_6 \beta_{m,\varphi}(|u|_\infty)(1 + |u|_\infty^{m-1})\|u\|_m)\|v\|_m. \tag{2.8}$$

Proof. Writing $\varphi(u)v = (\varphi(u) - \varphi(0))v + \varphi(0)v$, (2.8) follows from inequalities (2.5) and (2.7). □

For integer $m \geqslant 1$, we define V^m as the completion of H^m with respect to the norm $\|\nabla u\|_{m-1}$. The space $C_0^\infty(\mathbb{R}^N)$ is dense in V^m, because it is dense in H^m, and the operator $\nabla\colon H^m \to H^{m-1}$ extends continuously to an operator in V^m, which we still denote by ∇. Likewise, the Laplacian $-\Delta$ is a positive-definite, self-adjoint operator from H^{m+1} into H^{m-1}, with the square root $(-\Delta)^{1/2}\colon (H^m \hookrightarrow H^{m-1}) \to H^{m-1}$. In fact, $(-\Delta)^{1/2}$ is an isometry from V^m into H^{m-1}, with

$$\|(-\Delta)^{1/2}u\|_{m-1} = \|\nabla u\|_{m-1} \tag{2.9}$$

for all $u \in V^m$. Also note that V^m is continuously imbedded into $C^0 \cap L^\infty$, see 2) above. Thus, if $m > \frac{1}{2}N$ and $N \geqslant 3$, V^m can be identified with a linear subspace of $C^0 \cap L^\infty$.

Proposition 2.3. *Let $m \geqslant 1$ and $h \in H^{m-1}$. A function h is in the range space $(-\Delta)^{1/2}(H^m)$ if and only if*

$$\kappa^2 := \int_{\mathbb{R}^N} \frac{|\hat{h}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi < \infty. \tag{2.10}$$

In particular, (2.10) *holds if $h \in H^{m-1} \cap L^p$, with $1 \leqslant p < 2N/(N+2)$ (which requires $N \geqslant 3$). If $h \in H^{m-1} \cap L^p$ and $h = (-\Delta)^{1/2}h_0$ for some $h_0 \in H^m$, then there is $C > 0$, depending only on N and m, such that*

$$\|h\|_{m-1} = \|\nabla h_0\|_{m-1} \leqslant \|h_0\|_m \leqslant C(\|h\|_{m-1} + |h|_p). \tag{2.11}$$

Proof. If $h \in H^{m-1}$ and $h = (-\Delta)^{1/2}h_0$ for some $h_0 \in H^m$, then $\kappa = \|\hat{h}_0\| = \|h_0\|$ is finite. Conversely, assume $h \in H^{m-1}$ satisfies (2.10). Then $h_0 := \mathcal{F}^{-1}(|\cdot|^{-1}\hat{h}) \in L^2$, and an induction procedure based on the easily proved identity

$$\|h_0\|_k^2 = \|h_0\|_{k-1}^2 + \|h\|_{k-1}^2, \tag{2.12}$$

$k \geqslant 1$, shows that, in fact, $h_0 \in H^m$. Assume now that $N \geqslant 3$. Since $h \in L^p$ and $1 \leqslant p < 2$, the Hausdorff-Young theorem implies that $\hat{h} \in L^q$, $1/p + 1/q = 1$. Let r be the conjugate index of $\frac{1}{2}q$; we compute that $r < \frac{1}{2}N$, so that the function $\xi\colon \mapsto 1/|\xi|$ is in $L^{2r}_{\mathrm{loc}}(\mathbb{R}^N)$. Thus, the function $\xi\colon \mapsto \hat{h}(\xi)/|\xi|$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, which implies (2.10). The equality in (2.11) follows from (2.9) with $u = h_0$; next, we sum the inequalities (2.12) from $k = 1$ to $k = m$, which yields

$$\|h_0\|_m^2 = \|h_0\|_0^2 + \sum_{k=1}^{m} \|h\|_{k-1}^2 \leqslant \|h_0\|^2 + C\|h\|_{m-1}^2. \tag{2.13}$$

We now recall that $h \in L^p$, $\hat{h} \in L^q$, and $|\cdot|^{-1} \in L^{2r}_{\mathrm{loc}}(\mathbb{R}^N)$; thus, letting $R := \left(\int_{|\xi|\leqslant 1} |\xi|^{-2r}\,\mathrm{d}\xi\right)^{1/r}$, by (2.10) we obtain

$$\|h_0\|^2 = \kappa^2 \leqslant |\hat{h}|_2^2 + R|\hat{h}|_q^2 \leqslant |h|_2^2 + CR|h|_p^2. \tag{2.14}$$

Inserting this into (2.13) yields the last inequality of (2.11). □

3. The linear equation

In this section we consider the linear equation (1.6) and its homogeneous version (1.4). Given a function $g = g(x)$, we denote by $t \mapsto S(t; g)$ the solution to (1.4) with initial data $u_0 = 0$ and $u_1 = g$. By the variation of parameters formula, we can then write the solution of (1.6) as

$$u(t) = S(t; 2u_0 + u_1) + \partial_t(S(t; u_0)) + \int_0^t S(t-\theta; f(\theta))\,\mathrm{d}\theta. \tag{3.1}$$

More generally, for $t \geqslant T \geqslant 0$ the function $t \mapsto S(t-T; g)$ denotes the solution to (1.4) satisfying $u(T) = 0$, $u_t(T) = g$.

We recall the following decay estimates for solutions of (1.4):

Proposition 3.1. *Let k, $m \in \mathbb{N}$, and set $r := \max\{k+m-1, 0\}$. For $q \in [1,2]$, set $\nu := \frac{1}{4}N(2/q - 1) + k + \frac{1}{2}m$. Let $w \in H^r \cap L^q$. Then for any multiindex α with $|\alpha| = m$,*

$$\|\partial_t^k \partial_x^\alpha (S(t; w))\| \leqslant C(1+t)^{-\nu}(\|w\|_r + |w|_q). \tag{3.2}$$

The same result holds for the equation

$$w_{tt} + 2w_t - c_{ij}\partial_i\partial_j w = 0, \tag{3.3}$$

where $C = [c_{ij}]$ is a positive definite matrix with constant entries.

P r o o f. The cases $q=1$ and $q=2$ are proved in Matsumura [8]; in this proof, the only place where the L^1 norm of w enters the argument is in the estimate of the term

$$J^2 := \int_{|\xi|\leqslant\delta} |\xi|^{2(2k+m)} \mathrm{e}^{-2t|\xi|^2} |\hat{w}(\xi)|^2 \,\mathrm{d}\xi, \tag{3.4}$$

where $\delta \in \,]0,\frac{1}{2}[$. Let $q \in \,]1,2[$, and let $p = q/(q-1)$ be its conjugate index. Then $p>2$ so that, with $c_p := p/(p-2)$,

$$J^2 \leqslant \left(\int_{|\xi|\leqslant\delta} |\hat{w}(\xi)|^p \,\mathrm{d}\xi\right)^{2/p} \left(\int_{|\xi|\leqslant\delta} |\xi|^{2c_p(2k+m)} \mathrm{e}^{-2c_p t|\xi|^2} \,\mathrm{d}\xi\right)^{1-2/p}. \tag{3.5}$$

By the Hausdorff-Young inequality,

$$\left(\int_{|\xi|\leqslant\delta} |\hat{w}(\xi)|^p \,\mathrm{d}\xi\right)^{2/p} = |\hat{w}|_p^2 \leqslant C|w|_q^2; \tag{3.6}$$

using inequality (10) of [8], we easily see that

$$\int_{|\xi|\leqslant\delta} |\xi|^{2c_p(2k+m)} \mathrm{e}^{-2c_p t|\xi|^2} \,\mathrm{d}\xi \leqslant C(1+t)^{-(N+2c_p(2k+m))/2}, \tag{3.7}$$

so that we deduce from (3.5) that

$$J \leqslant C|w|_q (1+t)^{-(N+2c_p(2k+m))/(4c_p)}, \tag{3.8}$$

from which (3.2) follows. Finally, the result also holds for (3.3) since this equation can be transformed into (1.4) by the change of variables $u(t,x) = w(t,\Gamma x)$, where $\Gamma^{-1} = C^{1/2}$. □

In particular, from (3.2) with $q=2$ we immediately deduce from (3.1) that the solution of (1.4), with initial data $u_0 \in H^{m+1}$ and $u_1 \in H^m$, $m \geqslant 0$, satisfies the estimate

$$\sup_{t\geqslant 0} \|u(t)\|_{m+1} \leqslant C(\|u_0\|_{m+1} + \|u_1\|_m). \tag{3.9}$$

Assume now that $u_0, u_1 \equiv 0$. Then the solution of (1.6) is formally given by $u(t,x) = (\mathcal{F}^{-1}v(t,\cdot))(x)$, where

$$v(t,\xi) = \int_0^t \hat{f}(t-\theta,\xi) h(\theta,\xi) \,\mathrm{d}\theta, \tag{3.10}$$

$$h(t,\xi) := \mathrm{e}^{-t} \frac{\sinh\big(\sqrt{1-|\xi|^2}\,t\big)}{\sqrt{1-|\xi|^2}}. \tag{3.11}$$

In particular, (3.10) is well defined if $f \in L^\infty(0,\infty;H^m)$ for some $m \in \mathbb{N}$, since then $\hat{f}(\cdot,\xi)$ is locally integrable for almost all $\xi \in \mathbb{R}^N$. Moreover, $u(t,\cdot) \in H^{m+1}$ for all $t \geqslant 0$. The following properties of h are easily deduced from (3.11):

Proposition 3.2. *For all $t \geqslant 0$ and $\xi \in \mathbb{R}^N$,*

$$|h(t,\xi)| \leqslant 2, \quad |\xi||h(t,\xi)| \leqslant 2. \tag{3.12}$$

More precisely: If $|\xi| \leqslant 1$, then

$$0 \leqslant h(t,\xi) \leqslant 2\mathrm{e}^{-t|\xi|^2/2}, \tag{3.13}$$

while if $|\xi| \geqslant 1$, then

$$|h(t,\xi)| \leqslant t\mathrm{e}^{-t}, \quad |\xi||h(t,\xi)| \leqslant 2(1+t)\mathrm{e}^{-t}. \tag{3.14}$$

Proposition 3.2 allows us to deduce that (3.9) can be improved to

$$\lim_{t\to\infty}(\|u(t)\|_{m+1} + \|u_t(t)\|_m) = 0. \tag{3.15}$$

In fact, considering only the case $u_0 = 0$ for simplicity, we note that, by (3.12), there is a constant C such that

$$\begin{aligned} \|u(t)\|_{m+1}^2 &\leqslant C\int_{\mathbb{R}^N}(1+|\xi|^2)^{m+1}|\hat{u}_1(\xi)|^2|h(t,\xi)|^2\,\mathrm{d}\xi \\ &\leqslant C\int_{\mathbb{R}^N}(1+|\xi|^2)^m|\hat{u}_1(\xi)|^2\,\mathrm{d}\xi. \end{aligned} \tag{3.16}$$

Since $h(t,\xi) \to 0$ as $t \to \infty$ for all $\xi \neq 0$, by Lebesgue's dominated convergence theorem we conclude that $\|u(t)\|_{m+1} \to 0$ as $t \to \infty$. The convergence $u_t(t) \to 0$ in H^m is proved similarly.

We now give some sufficient conditions for the boundedness of solutions to (1.6) (again, when $u_0, u_1 \equiv 0$) as $t \to \infty$.

Theorem 3.1. *Let $f \in L^\infty(0,\infty;H^m)$ for some $m \in \mathbb{N}$, and suppose there are positive constants λ, Λ such that for all $t \geqslant 0$,*

$$\int_{\mathbb{R}^N}\frac{|\hat{f}(t,\xi)|^2}{|\xi|^{4+\lambda}}\,\mathrm{d}\xi \leqslant \Lambda. \tag{3.17}$$

Then the function $t \mapsto \|u(t)\|_{m+1}$ is bounded on $[0,\infty[$. In particular, (3.17) holds for all $\lambda \in \,]0,\lambda_p[$, $\lambda_p := N(2p^{-1}-1)-4$, if $f \in L^\infty(0,\infty;H^m \cap L^p)$ with $1 \leqslant p < 2N/(N+4)$ (which requires $N \geqslant 5$ and implies $\lambda_p > 0$).

P r o o f. By Schwarz' inequality, we deduce from (3.10) that, for all $t \geqslant 0$,

$$|v(t,\xi)|^2 \leqslant \int_0^t |\hat{f}(t-\theta,\xi)|^2 |h(\theta,\xi)|\,\mathrm{d}\theta \int_0^t |h(\theta,\xi)|\,\mathrm{d}\theta. \tag{3.18}$$

If $|\xi| \geqslant 1$, by (3.14) we obtain from (3.18) that

$$\begin{aligned}(1+|\xi|^2)|v(t,\xi)|^2 &\leqslant \int_0^t |\hat{f}(t-\theta,\xi)|^2 \theta \mathrm{e}^{-\theta}\,\mathrm{d}\theta + 2\int_0^t |\hat{f}(t-\theta,\xi)|^2(1+\theta)\mathrm{e}^{-\theta}\,\mathrm{d}\theta \\ &\leqslant 3\int_0^t |\hat{f}(t-\theta,\xi)|^2(1+\theta)\mathrm{e}^{-\theta}\,\mathrm{d}\theta.\end{aligned} \tag{3.19}$$

By Fubini's theorem, (3.19) implies that

$$\begin{aligned}&\int_{|\xi|\geqslant 1} (1+|\xi|^2)^{m+1}|v(t,\xi)|^2\,\mathrm{d}\xi \\ &\qquad \leqslant 3\int_0^t (1+\theta)\mathrm{e}^{-\theta}\int_{|\xi|\geqslant 1}(1+|\xi|^2)^m |\hat{f}(t-\theta,\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}\theta \\ &\qquad \leqslant C\sup_{t\geqslant 0}\|f(t)\|_m^2.\end{aligned} \tag{3.20}$$

If $|\xi| \leqslant 1$, by Proposition 3.2 we obtain from (3.18) that

$$\begin{aligned}|v(t,\xi)|^2 &\leqslant 4|\xi|^{-2}\int_0^t |\hat{f}(t-\theta,\xi)|^2 \mathrm{e}^{-\theta|\xi|^2/2}\,\mathrm{d}\theta \\ &\leqslant 4M|\xi|^{-4-\lambda}\int_0^t |\hat{f}(t-\theta,\xi)|^2 \min(1,\theta^{-1-\lambda/2})\,\mathrm{d}\theta,\end{aligned} \tag{3.21}$$

where $M := \max\bigl\{1, \max_{r\geqslant 0}(r^{1+\lambda/2}\mathrm{e}^{-r/2})\bigr\}$. Again by Fubini's theorem,

$$\begin{aligned}&\int_{|\xi|\leqslant 1}(1+|\xi|^2)^{m+1}|v(t,\xi)|^2\,\mathrm{d}\xi \\ &\qquad \leqslant 2^{m+3}M\int_0^t\int_{|\xi|\leqslant 1}\frac{|\hat{f}(t-\theta,\xi)|^2}{|\xi|^{4+\lambda}}\min(1,\theta^{-1-\lambda/2})\,\mathrm{d}\xi\,\mathrm{d}\theta \\ &\qquad \leqslant 2^{m+3}M\Lambda\int_0^t \min(1,\theta^{-1-\lambda/2})\,\mathrm{d}\theta \leqslant 2^{m+3}M\Lambda(1+2\lambda^{-1}).\end{aligned} \tag{3.22}$$

Together with (3.20), (3.22) clearly implies the asserted boundedness of $\|u(t)\|_{m+1}$ as $t \to \infty$. The second part of the theorem is proved as in the second part of Proposition 2.3, noting that, now $r < \frac{1}{4}N$, and the smallness condition on λ guarantees that the function $|\cdot|^{-4-\lambda}$ is in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$. □

In an analogous way, we can also prove

Theorem 3.2. *Let $f \in L^\infty(0,\infty; H^m)$ for some $m \in \mathbb{N}$, and suppose there are positive constants λ, Λ such that*

$$\int_{\mathbb{R}^N} \frac{|\hat{f}(t,\xi)|^2}{|\xi|^{2+\lambda}} \, \mathrm{d}\xi \leqslant \Lambda \tag{3.23}$$

for all $t \geqslant 0$. Then the function $t \mapsto \|\nabla u(t,\cdot)\|_m$ is bounded on $[0,\infty[$. In particular, (3.23) holds for all $\lambda \in \,]0,\lambda_p[$, $\lambda_p := N(2p^{-1}-1)-2$, if $f \in L^\infty(0,\infty; H^m \cap L^p)$ with $1 \leqslant p < 2N/(N+2)$ (which requires $N \geqslant 3$ and implies $\lambda_p > 0$).

If f is independent of t, Theorems 3.1 and 3.2 can be refined as follows.

Theorem 3.3. *Let $f \in H^m$ for some $m \in \mathbb{N}$. The function $t \mapsto \|u(t,\cdot)\|_{m+1}$ is bounded on $[0,\infty[$ if and only if (3.17) holds with $\lambda = 0$; that is, if*

$$\Lambda := \int_{\mathbb{R}^N} \frac{|\hat{f}(\xi)|^2}{|\xi|^4} \, \mathrm{d}\xi < \infty. \tag{3.24}$$

In turn, (3.24) holds if and only if there is a (unique) $F \in H^{m+2}$ such that $-\Delta F = f$; in this case,

$$\|u(t,\cdot) - F\|_{m+1} \to 0 \quad \text{as } t \to \infty. \tag{3.25}$$

In particular, (3.24) holds if $f \in H^m \cap L^p$ with $1 \leqslant p < 2N/(N+4)$ (which requires $N \geqslant 5$). If $N \leqslant 4$, there are functions $f \in H^m \cap L^1$ such that (3.24) fails, and $\|u(t,\cdot)\|_m$ is unbounded.

P r o o f. If f does not depend on t, (3.10) reads

$$v(t,\xi) = \hat{f}(\xi) \int_0^t h(\theta,\xi) \, \mathrm{d}\theta =: \hat{f}(\xi) H(t,\xi). \tag{3.26}$$

Now,

$$H(t,\xi) = \frac{1}{|\xi|^2} \left(1 - \mathrm{e}^{-t} \left(\cosh(t\sqrt{1-|\xi|^2}) + \frac{\sinh\bigl(t\sqrt{1-|\xi|^2}\bigr)}{\sqrt{1-|\xi|^2}}\right)\right), \tag{3.27}$$

so that we easily see that for all $t \geqslant 0$ and $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$|H(t,\xi)| \leqslant 2|\xi|^{-2}, \quad \lim_{t\to\infty} H(t,\xi) = |\xi|^{-2}. \tag{3.28}$$

Together with (3.26), the first of (3.28) implies that for all $t \geqslant 0$,

$$\|u(t)\|_{m+1}^2 \leqslant \int_{\mathbb{R}^N} (1+|\xi|^2)^{m+1} |\hat{f}(\xi)|^2 4 |\xi|^{-4} \, \mathrm{d}\xi. \tag{3.29}$$

Splitting the integral over the regions $|\xi| \leqslant 1$ and $|\xi| \geqslant 1$, we deduce from (3.29) that, if (3.24) holds, then

$$\|u(t)\|_{m+1}^2 \leqslant 8(2^m \Lambda + \|f\|_m^2); \tag{3.30}$$

that is, $\|u(t)\|_{m+1}$ is bounded in t. Conversely, if (3.24) did not hold, Fatou's lemma would yield

$$\begin{aligned} \lim_{n\to\infty} \|u(n)\|_{m+1}^2 &\geqslant \int_{\mathbb{R}^N} (1+|\xi|^2)^{m+1} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,\mathrm{d}\xi \\ &\geqslant \int_{\mathbb{R}^N} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,\mathrm{d}\xi = \infty, \end{aligned} \tag{3.31}$$

contradicting the assumption that $\|u(t)\|_{m+1}$ is bounded. The claim concerning F is clear, since $-\Delta w = \mathcal{F}^{-1}(|\cdot|^2 \hat{w})$ for all $w \in H^2$. Then (3.25) follows from (3.28) and

$$\|u(t) - F\|_{m+1}^2 = \int_{\mathbb{R}^N} (1+|\xi|^2)^m |\hat{f}(\xi)|^2 (H(t,\xi) - |\xi|^{-2})^2\,\mathrm{d}\xi \tag{3.32}$$

via Lebesgue's dominated convergence theorem. Finally, the function $x \mapsto \mathrm{e}^{-|x|^2}$ is in $H^m \cap L^1$ for every $m \in \mathbb{N}$, but fails to satisfy (3.24) if $N \leqslant 4$. □

In fact, if f is independent of t, the solution u takes values in H^{m+2} and not just in H^{m+1} (this, of course, because the initial values, being zero, are smooth). It is then clear from the proof of Theorem 3.3 that, if (3.24) holds, also the function $t \mapsto \|u(t)\|_{m+2}$ is bounded on $[0,\infty[$. More specifically, proceeding as in Theorem 3.3 we can also prove

Theorem 3.4. *Let $f \in H^m$ for some $m \in \mathbb{N}$. The function $t \mapsto \|\nabla u(t,\cdot)\|_m$ is bounded on $[0,\infty[$ if and only if*

$$\int_{\mathbb{R}^N} \frac{|\hat{f}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi < \infty. \tag{3.33}$$

If (3.33) holds, there is $v \in V^{m+2}$ such that

$$\|\nabla(u(t) - v)\|_m \to 0 \quad \text{as } t \to \infty. \tag{3.34}$$

In particular, (3.33) holds if $f \in H^m \cap L^p$ with $1 \leqslant p < 2N/(N+2)$ (which requires $N \geqslant 3$). If $N \leqslant 2$, there are functions $f \in H^m \cap L^1$ such that (3.33) fails and $\|\nabla u(t,\cdot)\|_m$ is unbounded.

R e m a r k s. Comparing Theorems 3.3 with 3.1, or 3.4 with 3.2, the natural question arises whether we can allow $\lambda = 0$ in (3.17) or (3.23). In this case, our proof would allow for logarithmic growth of $\|u(t)\|_m$ or $\|\nabla u(t)\|_m$ as $t \to \infty$, but we do not know if this can truly happen. On the other hand, the last claim of Theorem 3.3 shows that if $N \leqslant 4$, Matsumura's estimate (3.2) with $k = m = 0$ is sharp, since even the L^2-boundedness of solutions of (1.6) may fail.

4. The quasi-linear equation

For $N \geqslant 3$ we consider integers $s > \frac{1}{2}N + 1$ so that, by (2.2), $H^{s-1} \hookrightarrow L^\infty$. We assume that the coefficients $a_{ij} \in C^s(\mathbb{R}^{1+N};\mathbb{R})$, the matrix $A(p) := [a_{ij}(p)]$ is symmetric for all $p \in \mathbb{R}^{1+N}$ and satisfies the uniformly strong ellipticity condition

$$\exists\, \nu > 0\colon\ \forall\, p \in \mathbb{R}^{1+N},\ q \in \mathbb{R}^N,\ a_{ij}(p)q^iq^j \geqslant \nu\,|q|^2; \tag{4.1}$$

without loss of generality, we take $\nu = 1$.

We assume that $u_0 \in H^{s+1}$, $u_1 \in H^s$ and, with $C_{\mathrm{b}} := C \cap L^\infty$,

$$f \in C_{\mathrm{b}}([0,\infty[; H^s \cap L^q), \quad 1 \leqslant q < \bar{q} := \frac{2N}{N+2}; \tag{4.2}$$

correspondingly, we set

$$\delta^2 := \|u_0\|_{s+1}^2 + \|u_1\|_s^2, \quad \sigma^2 := \sup_{t \geqslant 0}(\|f(t)\|_s^2 + |f(t)|_q^2). \tag{4.3}$$

Note that, since $q < \bar{q} < 2$, the interpolation inequality

$$|f|_{\bar{q}} \leqslant C|f|_q^\theta |f|_2^{1-\theta}, \quad \theta = \frac{1}{N}\Big(\frac{1}{q} - \frac{1}{2}\Big)^{-1} \in\,]0,1[, \tag{4.4}$$

implies that for all $t \geqslant 0$,

$$|f(t)|_{\bar{q}} \leqslant C\sigma. \tag{4.5}$$

For $0 \leqslant m \leqslant s$ and $T > 0$, we define

$$\mathcal{X}_T^m := \bigcap_{j=0}^m C^j([0,T]; H^{s+1-j}) \tag{4.6}$$

and, analogously,

$$\mathcal{X}_\infty^m := \bigcap_{j=0}^m C^j([0,\infty[; H^{s+1-j}); \tag{4.7}$$

correspondingly, we look for solutions of (1.1)–(1.2) in the space

$$\mathcal{Y} := \{u \in \mathcal{X}^2_\infty : \ Du \in C_{\rm b}([0,\infty;[H^s) \cap C^1_{\rm b}([0,\infty[; H^{s-1})\}. \tag{4.8}$$

As we shall see in Subsection 5.1, if (1.1) has a solution $u \in \mathcal{X}^1_T$ for some $T > 0$, then automatically $u \in \mathcal{X}^2_T$.

Under the above stated assumptions on the data and coefficients, in Section 5 we prove

Theorem 4.1. *Let $N \geqslant 3$. There exist $\mu_0 \in \,]0,1[$, $K_1, K_2 \geqslant 1$ such that, if $\delta + \sigma \leqslant \mu_0^2$, problem* (1.1)–(1.2) *has a unique solution $u \in \mathcal{Y}$, which satisfies the estimates*

$$\sup_{t\geqslant 0} \|Du(t)\|_s \leqslant \mu_0, \quad \sup_{t\geqslant 0} \|Du_t(t)\|_{s-1} \leqslant K_1\mu_0. \tag{4.9}$$

If $N \geqslant 5$, then $u \in C_{\rm b}([0,\infty[; L^2)$ and satisfies the estimate

$$\sup_{t\geqslant 0} \|u(t)\|_0 \leqslant K_2\mu_0. \tag{4.10}$$

The constants μ_0, K_1, and K_2 depend on the a_{ij}, N, s, and q, but not on u.

We recall that, as we have seen in Theorem 3.3 for the linear Cauchy problem, if $N = 3$ or $N = 4$ then there are data u_0, u_1, f such that the corresponding solution satisfies (4.9), but

$$\limsup_{t\to\infty} \|u(t)\|_0 = \infty. \tag{4.11}$$

We next consider the stationary equation (1.3). Given $m \in \mathbb{N}$ and $R > 0$, we denote by $B_m(R)$ the ball of V^m with center 0 and radius R. In Section 6 we prove

Theorem 4.2. *Assume $a_{ij} \in C^{s+1}(\mathbb{R}^{1+N};\mathbb{R})$, $N \geqslant 3$, and $h \in H^{s-1} \cap L^q$, $q \in [1,\bar{q}[$. There exists $R_0 \in \,]0,\frac{1}{2}[$ such that if*

$$\|h\|_{s-1} + |h|_q \leqslant R_0^2, \tag{4.12}$$

then (1.3) *has a unique solution $v \in B_{s+1}(R_0)$, which depends continuously on h.*

Finally, we consider the convergence, as $t \to \infty$, of the solutions to (1.1) to those of (1.3). In Section 7 we prove

Theorem 4.3. *Let $N \geqslant 3$, assume that $a_{ij} \in C^{s+2}(\mathbb{R}^{1+N};\mathbb{R})$, and let $h \in H^s \cap L^q$, $q \in [1, \bar{q}[$, satisfy* (4.12) *with $s-1$ replaced by s. Assume further that, as $t \to \infty$, $f(t,\cdot) \to h$ in the sense that*

$$\int_0^\infty (\|f(t)-h\|_s + |f(t)-h|_q)\,\mathrm{d}t =: \Lambda_0 < \infty. \tag{4.13}$$

The values of μ_0 and R_0 in Theorems 4.1 *and* 4.2 *can be chosen so small that, if $u \in \mathcal{Y}$ and $v \in B_{s+2}(R_0)$ are the corresponding solutions to problem* (1.1)–(1.2) *and equation* (1.3), *then $Du(t) \to Dv$ in H^s as $t \to \infty$; that is,*

$$\lim_{t\to\infty} \|D(u(t)-v)\|_s = 0. \tag{4.14}$$

5. Global existence and boundedness

In this section we prove Theorem 4.1. Our procedure is standard in that we show, by means of suitable *a priori* estimates, that local solutions of (1.1)–(1.2) can be extended to all of $[0,\infty[$. The value of μ_0 is determined in (5.48) below; in particular, we choose $\mu_0 \leqslant {C_2}^{-1}$, where C_2 is the norm of the Sobolev embedding in (2.2) for $m = s-1$. It follows that, if u satisfies (4.9), then $|Du(t)|_\infty \leqslant 1$ for all t.

5.1. Fundamental estimates. We first note that it is sufficient to prove the first inequality of (4.9), since this implies the other. Indeed, assume that (1.1) does have a solution $u \in \mathcal{Y}$ satisfying (4.9) with $\mu_0 \leqslant {C_2}^{-1}$. Then by Corollary 2.1 we deduce the estimate

$$\begin{aligned}\|a_{ij}(Du)\partial_i\partial_j u\|_{s-1} \\ \leqslant (|a_{ij}(0)| + C_5C_6\beta_{s-1,a_{ij}}(|Du|_\infty)(1+|Du|_\infty^{s-2}))\|\partial_i\partial_j u\|_{s-1}.\end{aligned} \tag{5.1}$$

Since $|a_{ij}(0)| \leqslant \beta_{s-1,a_{ij}}(1)$, estimating

$$|Du(t)|_\infty \leqslant C_2\,\|Du(t)\|_{s-1} \leqslant C_2\|Du(t)\|_s \leqslant C_2\mu_0 \leqslant 1 \leqslant C_5C_6 \tag{5.2}$$

we conclude from (5.1) that

$$\|a_{ij}(Du)\partial_i\partial_j u\|_{s-1} \leqslant 3C_5C_6\beta_{s,a_{ij}}(1)\|\nabla u\|_s =: K\|\nabla u\|_s. \tag{5.3}$$

Consequently, since $\sigma \leqslant \mu_0$,

$$\begin{aligned}\|u_{tt}\|_{s-1} &\leqslant \|f\|_{s-1} + 2\|u_t\|_{s-1} + \|a_{ij}(Du)\partial_i\partial_j u\|_{s-1} \\ &\leqslant \sigma + 2\mu_0 + K\mu_0 \leqslant (3+K)\mu_0 =: K_1\mu_0,\end{aligned} \tag{5.4}$$

from which the second inequality of (4.9) follows by virtue of $Du_t = \{u_{tt}, \nabla u_t\}$.

5.2. Further estimates. We now turn to the proof of the first inequality of (4.9). We define maps $Q\colon\ H^{s+1}\times H^s\to[0,\infty[$ and $F\colon\ H^{s+1}\times H^s\to\mathbb{R}$ by

$$Q(u,v):=\sum_{|\alpha|\leqslant s}\langle a_{ij}(v,\nabla u)\partial_i\partial^\alpha u,\partial_j\partial^\alpha u\rangle, \tag{5.5}$$

$$F(u,v):=\|v\|_s^2+2\langle u,v\rangle_s+Q(u,v); \tag{5.6}$$

then, for $u\in\mathcal{X}_T^1$ and $t\in[0,T]$, we abbreviate

$$F(t):=F(u(t),u_t(t)),\quad Q(t):=Q(u(t),u_t(t)); \tag{5.7}$$

note that (4.1) implies that

$$Q(u,v)\geqslant\|\nabla u\|_s^2 \tag{5.8}$$

for all $(u,v)\in H^{s+1}\times H^s$. Since $f\in L^2_{\mathrm{loc}}(0,\infty;H^s)$, Kato's results of [5] imply the existence of a unique local solution of problem (1.1)–(1.2); more precisely, we have

Theorem 5.1. *Under the above stated assumptions on the coefficients and the data, there exists $\tau\geqslant 0$ such that the problem has a unique solution $u\in\mathcal{X}_\tau^2$.*

The proof of uniqueness in Theorem 5.1 is independent of the size of τ; this yields the uniqueness claim of Theorem 4.1. By a standard continuation argument, we can extend the local solution u to a maximal interval $[0,T_c[$ with

$$T_c:=\sup\{T\geqslant 0\colon\ (1.1)\text{ has a solution }u\in\mathcal{X}_T^2\}. \tag{5.9}$$

Global existence corresponds to $T_c=\infty$, while if $T_c<\infty$, then

$$\limsup_{t\to T_c^-}\|Du(t)\|_s=\infty. \tag{5.10}$$

Indeed, if the function $t\mapsto\|Du(t)\|_s$ were bounded, then, as we have shown in Subsection 5.1, the function $t\mapsto\|Du_t(t)\|_{s-1}$ would also be bounded, and the function $t\mapsto\|u(t)\|_0$ could not blow up, since it would grow at most linearly; hence, $u\in\mathcal{Y}$.

To prove Theorem 4.1, it is then sufficient to establish an *a priori* estimate on the maximal solution of (1.1)–(1.2). To this end, we claim:

Proposition 5.1. *Let δ and σ be as in* (4.3)*, $T \in [0,T_c[$, and let $u \in \mathcal{X}_T^2$ be a solution of* (1.1)–(1.2)*. There exists $\mu_0 \in]0,1[$, independent of T, such that, if $\delta+\sigma \leqslant \mu_0^2$, u satisfies the estimates*

$$\|Du(t)\|_s \leqslant \mu_0, \quad \|Du_t(t)\|_{s-1} \leqslant (4+K_1)\mu_0 \tag{5.11}$$

for all $t \in [0,T]$ (compare to (4.9)*). Consequently, $T_c = \infty$.*

P r o o f. As we have shown in Subsection 5.1 above, it is sufficient to prove the first estimate in (5.11); before doing so, we prove the last claim of the proposition. If $T_c < \infty$, by (5.10) there would be $T_1 \in]0,T_c[$ such that $\mu_1 := \|Du(T_1)\|_s > \mu_0$; but then, since $\|Du(0)\|_s \leqslant \delta < \mu_0^2 < \mu_0$, there would also be $T \in]0,T_1[$ such that for all $t \in [0,T]$,

$$\|Du(t)\|_s \leqslant \frac{1}{2}(\mu_0+\mu_1) = \|Du(T)\|_s. \tag{5.12}$$

Since $\frac{1}{2}(\mu_0+\mu_1) > \mu_0$, (5.12) contradicts the first inequality of (5.11) for $t=T$.

Given $u \in \mathcal{X}_T^2$, we set $b_{ij} := a_{ij}(Du)$, $\mu := \max\limits_{0\leqslant t\leqslant T} \|Du(t)\|_s$, and

$$R_1 := 2\langle f, u_t+u\rangle_s, \quad R_2 := -2\langle(\partial_j b_{ij})\partial_i u, u\rangle. \tag{5.13}$$

We also denote by γ_μ a generic positive constant depending on μ in a continuous and non-decreasing way but independent of u and t; typically, such constants appear from applications of Corollary 2.1, as in the proof of (5.3) from (5.1) and (5.2), via the estimate $|Du|_\infty \leqslant C_2\mu$.

We first show that, with F as in (5.7), u satisfies for all $t \in [0,T]$ the estimate

$$\mathrm{e}^{2t}F(t) \leqslant F(0) + \int_0^t \mathrm{e}^{2\theta}(R_1+R_2+\gamma_\mu(\mu+\sigma)\|Du\|_s^2)\,\mathrm{d}\theta. \tag{5.14}$$

To this end, we note that the standard energy estimates on solutions of (1.1) involve multiplication of (1.1) by $2(u_t+u)$ in H^s; however, this is not directly possible, since some of the terms in the equation only take values in H^{s-1}. We thus resort to regularization, by means of Friedrichs' mollifiers $(\varrho^\varepsilon)_{\varepsilon>0}$ (see e.g. [1]). Setting $u^\varepsilon := \varrho^\varepsilon * u$, where the convolution refers to the space variables only, we deduce from (1.1) that u^ε satisfies the equation

$$u_{tt}^\varepsilon + 2u_t^\varepsilon - b_{ij}\partial_i\partial_j u^\varepsilon = f^\varepsilon + g^\varepsilon, \tag{5.15}$$

where

$$f^\varepsilon := \varrho^\varepsilon * f, \quad g^\varepsilon := \varrho^\varepsilon * (b_{ij}\partial_i\partial_j u) - b_{ij}\partial_i\partial_j u^\varepsilon. \tag{5.16}$$

Now, all terms in (5.15) take values in H^s; hence, we can multiply (5.15) in H^s by $2(u_t^\varepsilon + u^\varepsilon)$. In so doing, we note that the term

$$G^\varepsilon := -2\langle b_{ij}\partial_i\partial_j u^\varepsilon, u_t^\varepsilon + u^\varepsilon\rangle_s \tag{5.17}$$

can be written as

$$G^\varepsilon = \frac{\mathrm{d}}{\mathrm{dt}}Q^\varepsilon + 2Q^\varepsilon - R_2^\varepsilon - R_3^\varepsilon, \tag{5.18}$$

where

$$Q^\varepsilon := \sum_{|\alpha|\leqslant s}\langle b_{ij}\partial^\alpha\partial_i u^\varepsilon, \partial^\alpha\partial_j u^\varepsilon\rangle, \quad R_2^\varepsilon := -2\langle\partial_j b_{ij}\partial_i u^\varepsilon, u^\varepsilon\rangle, \tag{5.19}$$

and R_3^ε is the sum of terms containing only derivatives of u^ε, but not u^ε itself. Thus, defining

$$F^\varepsilon := \|u_t^\varepsilon\|_s^2 + 2\langle u^\varepsilon, u_t^\varepsilon\rangle_s + Q^\varepsilon, \quad R_1^\varepsilon := 2\langle f^\varepsilon + g^\varepsilon, u^\varepsilon + u_t^\varepsilon\rangle_s, \tag{5.20}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}F^\varepsilon + 2F^\varepsilon = R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon. \tag{5.21}$$

By means of Corollary 2.1, it is straightforward to see that, since $|Du|_\infty \leqslant C_2\|Du\|_{s-1} \leqslant C_2\mu$,

$$|R_3^\varepsilon(t)| \leqslant \gamma_\mu(\mu+\sigma)\|Du^\varepsilon\|_s^2, \tag{5.22}$$

where γ_μ is independent of ε. Thus, we deduce from (5.21) that

$$\frac{\mathrm{d}}{\mathrm{dt}}F^\varepsilon(t) + 2F^\varepsilon(t) \leqslant R_1^\varepsilon(t) + R_2^\varepsilon(t) + \gamma_\mu(\mu+\sigma)\|Du^\varepsilon\|_s^2, \tag{5.23}$$

from which

$$\mathrm{e}^{2t}F^\varepsilon(t) - F^\varepsilon(0) \leqslant \int_0^t \mathrm{e}^{2\theta}(R_1^\varepsilon + R_2^\varepsilon + \gamma_\mu(\mu+\sigma)\|Du^\varepsilon\|_s^2)\,\mathrm{d}\theta. \tag{5.24}$$

Following e.g. Mizohata [12], we can show that $\|g^\varepsilon(t)\|_s$ remains bounded for $t \in [0,T]$, $\varepsilon > 0$, and $\|g^\varepsilon(t)\|_s \to 0$ as $\varepsilon \to 0$ for fixed t. We also know that $f^\varepsilon(t) \to f(t)$ in H^s, $\|f^\varepsilon(t)\|_s \leqslant \|f(t)\|_s$, and analogously for $Du^\varepsilon(t)$ in H^{s-1}. Therefore, by Lebesque's dominated convergence theorem, we can let $\varepsilon \to 0$ in (5.24) and obtain (5.14).

We next show that for all $t \in [0, T]$

$$F(t) \leqslant \mathrm{e}^{-2t} F(0) + \gamma_\mu \mu(\sigma + \mu + \mu^2), \tag{5.25}$$

as follows from estimating the terms $R_1(t)$ and $R_2(t)$ under the integral sign in (5.14). Clearly,

$$\langle f, u_t \rangle_s + \langle \nabla f, \nabla u \rangle_{s-1} \leqslant \|f\|_s \|Du\|_s \leqslant \sigma\mu \tag{5.26}$$

and, by Hölder's inequality, (4.5), and (2.1),

$$\langle f, u \rangle \leqslant |f|_{\bar{q}} |u|_{\bar{p}} \leqslant C_1 |f|_{\bar{q}} |\nabla u|_2 \leqslant C C_1 \sigma\mu; \tag{5.27}$$

together with (5.26), this implies that

$$R_1(t) \leqslant C\sigma\mu. \tag{5.28}$$

Next, since $|w|_r \leqslant |w|_1 + |w|_2$ for $1 \leqslant r \leqslant 2$, we have

$$\begin{aligned} R_2(t) &\leqslant 2|\partial_j b_{ij} \partial_i u|_{\bar{q}} |u|_{\bar{p}} \\ &\leqslant 2C_1(|\partial_j b_{ij} \partial_i u|_1 + |\partial_j b_{ij} \partial_i u|_2)|\nabla u|_2 \\ &\leqslant 2C_1 |\nabla b_{ij}|_2 (|\nabla u|_2 + |\nabla u|_\infty)|\nabla u|_2 \end{aligned} \tag{5.29}$$

so that, by Proposition 2.2,

$$R_2(t) \leqslant \gamma_\mu |\nabla Du|_2 (|\nabla u|_2 (|\nabla u|_2 + C_2 \|\nabla u\|_{s-1}) |\nabla u|_2 \leqslant \gamma_\mu \mu^3. \tag{5.30}$$

Together with (5.28), this shows that

$$R_1(t) + R_2(t) + \gamma_\mu(\mu + \sigma)\|Du(t)\|_s^2 \leqslant \gamma_\mu \mu(\sigma + \sigma\mu + \mu^2); \tag{5.31}$$

inserting this into (5.14), we obtain (5.25).

Finally, we show that, if $\mu \leqslant 1$, then for all $t \in [0, T]$,

$$\|Du(t)\|_s^2 \leqslant \gamma_\mu (\delta^2 + \delta\mu + \sigma^2 + \sigma\mu + \mu^3). \tag{5.32}$$

Because of (5.25), to show (5.32) it is sufficient to estimate the term $\langle u, u_t \rangle_s$ of $F(t)$, which we can do by means of Proposition 3.1. Indeed, setting $c_{ij} := a_{ij}(0)$ and $\tilde{a}_{ij}(p) = a_{ij}(p) - c_{ij}$, we can rewrite equation (1.1) in the linearized form

$$u_{tt} + 2u_t - c_{ij} \partial_i \partial_j u = f + g, \tag{5.33}$$

where $g := \tilde{a}_{ij}(Du)\partial_i\partial_j u \in L^q$ for all $q \in [1,2]$, as follows from Proposition 2.2. In fact, noting that $\tilde{a}_{ij}(0) = 0$, (2.7) yields

$$|g|_q \leqslant |g|_1 + |g|_2 \leqslant |\tilde{a}_{ij}(Du)|_2(|\partial_i\partial_j u|_2 + |\partial_i\partial_j u|_\infty) \leqslant 2C_6\beta_1(C_2\mu)|Du|_2\|\nabla u\|_s \leqslant \gamma_\mu\mu^2. \tag{5.34}$$

We multiply equation (5.33) in L^2 by u, obtaining

$$\frac{\mathrm{d}}{\mathrm{dt}}\langle u, u_t\rangle - |u_t|_2^2 + 2\langle u, u_t\rangle + \langle c_{ij}\partial_j u, \partial_i u\rangle = \langle f + g, u\rangle; \tag{5.35}$$

since $\bar{q} \in [1,2]$, by (5.34) and (2.1) we have

$$|\langle f + g, u\rangle| \leqslant |f + g|_{\bar{q}}|u|_{\bar{p}} \leqslant \gamma_\mu(\sigma + \mu^2)|\nabla u|_2 \leqslant \gamma_\mu(\sigma + \mu^2)\mu. \tag{5.36}$$

We now use the variation of parameters formula (3.1), by which

$$u(t) = S(t; 2u_0 + u_1) + \partial_t(S(t; u_0)) + \int_0^t S(t - \theta; f(\theta) + g(\theta))\,\mathrm{d}\theta. \tag{5.37}$$

By (3.9), we easily see that

$$|\nabla S(t; 2u_0 + u_1) + \nabla\partial_t(S(t; u_0))|_2 \leqslant C\delta. \tag{5.38}$$

Next, we use Proposition 3.1, with q specified in assumption (4.2), to estimate

$$|\nabla S(t - \theta; f(\theta) + g(\theta))|_2 \leqslant C(1 + t - \theta)^{-\nu}(|f + g|_2 + |f + g|_q) \leqslant \gamma_\mu(1 + t - \theta)^{-\nu}(\sigma + \mu^2), \tag{5.39}$$

where

$$\nu = \frac{N}{4}\Big(\frac{2}{q} - 1\Big) + \frac{1}{2} > \frac{N}{4}\Big(\frac{2}{\bar{q}} - 1\Big) + \frac{1}{2} = 1 \tag{5.40}$$

(it is at this point that we need $q < \bar{q}$); therefore, from (5.37), (5.38), and (5.39) we obtain

$$|\nabla u|_2 \leqslant \gamma_\mu(\delta + \sigma + \mu^2). \tag{5.41}$$

From (5.35), (5.36), and (5.41) we deduce then that

$$-\frac{\mathrm{d}}{\mathrm{dt}}\langle u, u_t\rangle - 2\langle u, u_t\rangle \leqslant |\langle f + g, u\rangle| + |c_{ij}||\nabla u|_2^2 \leqslant \gamma_\mu(\delta^2 + \sigma^2 + \sigma\mu + \mu^3), \tag{5.42}$$

from which, finally, since $|\langle u_0, u_1\rangle| \leqslant \delta^2$,

$$-\langle u, u_t\rangle \leqslant \gamma_\mu(\delta^2 + \sigma^2 + \sigma\mu + \mu^3). \tag{5.43}$$

We are now ready to estimate $\langle u, u_t\rangle_s$. Clearly,

$$-\langle u, u_t\rangle_s = -\langle u, u_t\rangle - \langle \nabla u, \nabla u_t\rangle_{s-1} \leqslant -\langle u, u_t\rangle + \mu\|\nabla u\|_{s-1}; \tag{5.44}$$

we estimate $\|\nabla u\|_{s-1}$ by the same method we used to show (5.41), applying Proposition 3.1 with $k = 0$, $0 \leqslant m \leqslant s$, $0 \leqslant r \leqslant s-1$ (note that we cannot estimate $\|\nabla u\|_s$ in this way since this would require an estimate of $\|g(t)\|_s$; now, we do not know whether $g(t) \in H^s$, while we do have the estimate $\|g(t)\|_{s-1} \leqslant \gamma_\mu\mu^2$). We obtain

$$\|\nabla u\|_{s-1} \leqslant \gamma_\mu(\delta + \sigma + \mu^2); \tag{5.45}$$

therefore, from (5.54) and (5.43),

$$-\langle u, u_t\rangle_s \leqslant \gamma_\mu(\delta^2 + \delta\mu + \sigma^2 + \sigma\mu + \mu^3). \tag{5.46}$$

We now use (4.1) (with $\nu = 1$), (5.25), and (5.46) to conclude that

$$\begin{aligned} \|Du\|_s^2 = \|u_t\|_s^2 + \|\nabla u\|_s^2 &\leqslant \|u_t\|_s^2 + Q(t) = F(t) - 2\langle u, u_t\rangle_s \\ &\leqslant \gamma_\mu(\delta^2 + \delta\mu + \sigma^2 + \sigma\mu + \mu^3), \end{aligned} \tag{5.47}$$

which yields the desired estimate (5.32).

We can now conclude the proof of Proposition 5.1. Indeed, denote by $\Gamma(\mu)$ the specific constant γ_μ appearing in estimate (5.32), and choose $\mu_0 \in \,]0, 1/C_2[$ so small that

$$20\,\Gamma(1)\mu_0 \leqslant 1. \tag{5.48}$$

We claim that the first inequality of (5.11) holds with this choice of μ_0. Indeed, there would otherwise be $t_1 \in \,]0, T]$ such that $\|Du(t_1)\|_s > \mu_0$; but since

$$\|Du(0)\|_s = \delta \leqslant \mu_0^2 < \mu_0, \tag{5.49}$$

there would also be $t_2 \in [0, t_1[$ such that for all $t \in [0, t_2]$,

$$\|Du(t)\|_s \leqslant \mu_0 = \|Du(t_2)\|_s. \tag{5.50}$$

On the other hand, since $C_2\mu_0 \leqslant 1$, (5.32) with $\mu = \mu_0$ and $\delta + \sigma \leqslant \mu_0^2$ yields that for all $t \in [0, T]$,

$$\|Du(t)\|_s^2 \leqslant \Gamma(1)(2\mu_0^4 + 3\mu_0^3) \leqslant 5\,\Gamma(1)\mu_0^3 \leqslant \frac{1}{4}\mu_0^2. \tag{5.51}$$

Since for $t = t_2$ (5.50) contradicts (5.51), we conclude that (5.11) holds. As a consequence, the local solution cannot blow up; that is, as we have already seen, $T_c = \infty$. This concludes the proof of Proposition 5.1. □

Finally, the proof of (4.10) follows from (5.37) in the same way as (5.38) and (5.39); the only change is that, in the estimate for $|S(t-\theta; f(\theta)+g(\theta))|_2$, the rate of decay given by Proposition 3.1 is $r = \frac{1}{4}N$, and $r > 1$ if $N \geqslant 5$. This concludes the proof of Theorem 4.1. $\square$

6. The stationary equation

In this section we prove Theorem 4.2. For simplicity, we only consider the case $q = 1$; the case for $1 < q < \bar{q}$ follows in a similar way by interpolation, via the inequality $|w|_q \leqslant |w|_1 + |w|_2$. We find v as the fixed point of the map $\Phi\colon V^{s+1} \mapsto V^{s+1}$, which formally defines $u = \Phi(w)$ as the solution of the linear elliptic equation

$$A_w(u) := -\partial_j(a_{ij}(Dw)\partial_i u) = h - \underbrace{a'_{ij}(Dw)\cdot\nabla Dw\partial_i w}_{=:h_w} \tag{6.1}$$

where, with slight abuse of notation, we have set $Dw := (0, \nabla w)$.

We first show that for each fixed $w \in V^{s+1}$ with sufficiently small norm, we can solve the equation

$$A_w(u) = g, \quad g \in H^{s-1} \cap L^1, \tag{6.2}$$

for $u \in V^{s+1}$, by means of the Lax-Milgram theorem. Thus, let a_w be the bilinear form on $V^{s+1} \times V^{s+1}$ associated to A_w, defined by

$$a_w(u,v) := \langle a_{ij}(Dw)\partial_i u, \partial_j v\rangle_s, \quad u,v \in V^{s+1}. \tag{6.3}$$

Since $g \in H^{s-1} \cap L^1$ and $N \geqslant 3$, by Proposition 2.3 there is $G \in H^s$ such that $g = (-\Delta)^{1/2}G$. We claim that (6.2) is equivalent to finding $u \in V^{s+1}$ such that

$$a_w(u,v) = \langle G, (-\Delta)^{1/2}v\rangle_s \quad \forall\, v \in V^{s+1}. \tag{6.4}$$

In fact, assume $u \in V^{s+1}$ solves (6.4). The set $(-\Delta)^{1/2}(H^{s+1})$ is dense in H^s; therefore, (6.4) implies that

$$\langle a_{ij}(Dw)\partial_i u, \partial_j(-\Delta)^{-1/2}z\rangle_s = \langle G, z\rangle_s \tag{6.5}$$

for all $z \in H^s$. Now, $D_j := (-\Delta)^{-1/2}\partial_j$ is a bounded linear operator on H^s (indeed, $\mathcal{F}(D_j u)(\xi) = -\mathrm{i}(\xi_j/|\xi|)\hat{u}(\xi)$), with adjoint $-D_j$. Hence, (6.5) reads

$$\langle -(-\Delta)^{-1/2}\partial_j(a_{ij}(Dw)\partial_i u), z\rangle_s = \langle(-\Delta)^{-1/2}A_w(u), z\rangle_s = \langle G, z\rangle_s. \tag{6.6}$$

Since z in (6.6) is arbitrary, it follows that

$$(6.7) \qquad (-\Delta)^{-1/2} A_w(u) = G$$

in H^s; in turn, this implies that

$$(6.8) \qquad A_w(u) = (-\Delta)^{1/2} G = g$$

in H^{s-1}; that is, (6.4) implies (6.2). Reversing this argument proves the converse.

We solve (6.2) (for fixed small w) by means of the following results, which also yield a (small) solution to (1.3). For $r \in \mathbb{R}_{\geqslant 0}$ we set

$$(6.9) \qquad \psi(r) := \max_{0 \leqslant k \leqslant s+1} |\partial_r^k a_{ij}(0)| + C_*^3 \beta_{s+1, a_{ij}}(C_2 r)(1 + (C_2 r)^{s-1}) r,$$

where C_* is the largest of the constants $C_1, \dots, C_6$, and C_7, the last one being the algebraic constant appearing in (6.16) below.

Proposition 6.1. *Let A_w and a_w be as in* (6.1) *and* (6.3). *Then, for all $w \in V^{s+1}$:*

1) *A_w is a continuous, injective operator from V^{s+1} into H^{s-1}, and for all $u \in V^{s+1}$ we have*

$$(6.10) \qquad \|A_w(u)\|_{s-1} \leqslant \psi(\|\nabla w\|_s) \|\nabla u\|_s.$$

2) *For all $u, v \in V^{s+1}$ we have*

$$(6.11) \qquad |a_w(u, v)| \leqslant \psi(\|\nabla w\|_s) \|\nabla u\|_s \|\nabla v\|_s,$$

$$(6.12) \qquad a_w(u, u) \geqslant (1 - \psi(\|\nabla w\|_s) \|\nabla w\|_s) \|\nabla u\|_s^2.$$

Proof. 1) Let $u \in V^{s+1}$. Then $a_{ij}(Dw)\partial_i u \in H^s$, which is an algebra. Thus, (6.10) follows by (2.8) of Corollary 2.1, with $\varphi = a_{ij}$ and $m = s$; that is,

$$(6.13) \qquad \|A_w(u)\|_{s-1} \leqslant \|a_{ij}(Dw)\partial_i u\|_s \leqslant \psi(\|\nabla w\|_s) \|\nabla u\|_s.$$

Next, assume that $A_w(u) = 0$ for some $u \in V^{s+1}$, and choose a sequence $(u^n)_{n \geqslant 0} \subset H^{s+1}$ such that $\nabla u^n \to \nabla u$ in L^2. Then

$$(6.14) \qquad 0 = \langle A_w(u), u^n \rangle = \langle a_{ij}(Dw)\partial_i u, \partial_j u^n \rangle \to \langle a_{ij}(Dw)\partial_i u, \partial_j u \rangle;$$

consequently, by (4.1), $\nabla u = 0$, which implies $u = 0$ in V^{s+1}. That is, A_w is injective.

2) (6.11) is proved as in (6.13); when $v = u$, we compute

$$a_w(u,u) = \sum_{|\alpha|\leqslant s} \langle a_{ij}(Dw)\partial_i\partial^\alpha u, \partial_i\partial^\alpha u\rangle + \sum_{1\leqslant|\alpha|\leqslant s}\sum_{0<\beta\leqslant\alpha} \binom{\alpha}{\beta} \langle \partial^\beta[a_{ij}(Dw)]\partial^{\alpha-\beta}\partial_i u, \partial_i\partial^\alpha u\rangle. \tag{6.15}$$

We denote by S_2 the second sum of (6.15); since $|\beta| \geqslant 1$, we can estimate

$$\begin{aligned} S_2 &\leqslant C_7C_4\|\nabla[a_{ij}(Dw)]\|_{s-1}\|\nabla u\|_s^2 \\ &\leqslant C_7C_4C_6\beta_{s-1}(C_1\|\nabla w\|_{s-1})\|\nabla w\|_s\|\nabla u\|_s^2 \\ &\leqslant \psi(\|\nabla w\|_s)\|\nabla w\|_s\|\nabla u\|_s^2. \end{aligned} \tag{6.16}$$

Inserting this into (6.15) and recalling (4.1), (6.12) follows. □

Proposition 6.2. *Let $R_0 \in]0,1[$ be so small that $\psi(R_0)R_0 \leqslant \frac{1}{2}$, and assume $w \in B_{s+1}(R_0)$. Then for every $g \in H^{s-1} \cap L^1$ there is a unique $u \in V^{s+1}$, a solution of* (6.4) *(and, therefore, of* (6.2)*). Moreover, u satisfies the estimate*

$$\|\nabla u\|_s \leqslant C(\|g\|_{s-1} + |g|_1), \tag{6.17}$$

with C independent of w and u.

Proof. If $w \in B_{s+1}(R_0)$, from (6.12) we obtain that, for all $u \in V^{s+1}$,

$$a_w(u,u) \geqslant \frac{1}{2}\|\nabla u\|_s^2; \tag{6.18}$$

that is, a_w is coercive on V^{s+1}. In addition, by (6.11), for all $u, v \in V^{s+1}$ we have

$$|a_w(u,v)| \leqslant \psi(R_0)\|\nabla u\|_s\|\nabla v\|_s; \tag{6.19}$$

that is, a_w is bilinear continuous on $V^{s+1} \times V^{s+1}$. Recalling that $(-\Delta)^{1/2}$ is an isometry between V^{s+1} and H^s (see (2.9)), so that the right-side of (6.4) is a bounded linear functional on V^{s+1}, we can apply the Lax-Milgram theorem and deduce that (6.4) admits a unique solution $u \in V^{s+1}$. From (6.18), (6.4), and (2.11) with $m = s$ we deduce that

$$\frac{1}{2}\|\nabla u\|_s^2 \leqslant \|G\|_s\|\nabla u\|_s \leqslant C(\|g\|_{s-1} + |g|_1)\|\nabla u\|_s, \tag{6.20}$$

from which (6.17) follows. □

Proposition 6.3. *Let C be as in (6.17), and assume $R_0 \in]0,1[$ is so small that*

$$12\, C R_0 (1 + \psi(2R_0)) \leqslant 1. \tag{6.21}$$

Assume h satisfies (4.12). Then:

1) *For all $w \in B_{s+1}(R_0)$, (6.1) has a unique solution $u \in B_{s+1}(R_0)$; this defines a map $w \mapsto u = \Phi(w)$ from $B_{s+1}(R_0)$ into itself.*
2) *Φ is a strict contraction on $B_{s+1}(R_0)$ with*

$$\|\nabla(\Phi(w) - \Phi(\tilde{w}))\|_s \leqslant \frac{1}{2}\|\nabla(w - \tilde{w})\|_s \tag{6.22}$$

for all $w, \tilde{w} \in V^{s+1}$.

P r o o f. 1) We first show that for all $w \in V^{s+1}$ we have $h_w \in H^{s-1} \cap L^1$ with

$$\|h_w\|_{s-1} + |h_w|_1 \leqslant 2\psi(\|\nabla w\|_s)\|\nabla w\|_s^2. \tag{6.23}$$

This follows from the fact that, by Corollary 2.1,

$$\|h_w\|_{s-1} \leqslant \|a'_{ij}(Dw)\partial_j Dw\|_{s-1}\|\partial_i w\|_{s-1} \leqslant \psi(\|\nabla w\|_s)\|\nabla w\|_s^2 \tag{6.24}$$

and

$$|h_w|_1 \leqslant \beta_1(|Dw|_\infty)|\partial^2 w|_2 |\nabla w|_2 \leqslant \psi(\|\nabla w\|_s)\|\nabla w\|_s^2. \tag{6.25}$$

Assume now that $w \in B_{s+1}(R_0)$. By Proposition 6.2, (6.1) has a unique solution $u \in V^{s+1}$ which, by (6.17), satisfies the estimate

$$\|\nabla u\|_s \leqslant C(\|h - h_w\|_{s-1} + |h - h_w|_1). \tag{6.26}$$

By (4.12), (6.23), and (6.21) it then follows that

$$\|\nabla u\|_s \leqslant C(R_0^2 + R_0^2 \psi(R_0)) \leqslant R_0, \tag{6.27}$$

that is, $\Phi(w) \in B_{s+1}(R_0)$, as claimed.

2) Given $w, \tilde{w} \in B_{s+1}(R_0)$, set $u = \Phi(w)$, $\tilde{u} = \Phi(\tilde{w})$, $v = w - \tilde{w}$, $z = u - \tilde{u}$. Then $z \in V^{s+1}$, and it satisfies the equation

$$A_w(z) = \underbrace{\partial_j((a_{ij}(Dw) - a_{ij}(D\tilde{w}))\partial_i \tilde{u})}_{=:\chi} + h_{\tilde{w}} - h_w. \tag{6.28}$$

It is now straightforward to see that $\chi \in H^{s-1} \cap L^1$; in fact, writing

$$a_{ij}(Dw) - a_{ij}(D\tilde{w}) = \int_0^1 a'_{ij}(\lambda Dw + (1-\lambda)D\tilde{w}) \cdot (Dw - D\tilde{w})\,\mathrm{d}\lambda, \tag{6.29}$$

we obtain that

$$\|\chi\|_{s-1} + |\chi|_1 \leqslant 2\psi(2R_0)R_0\|\nabla v\|_s. \tag{6.30}$$

As proved in the first part of Proposition 6.3, $h_w - h_{\tilde{w}} \in H^{s-1} \cap L^1$; hence, by (6.17) of Proposition 6.2,

$$\|\nabla z\|_s \leqslant C(2R_0\psi(R_0)\|\nabla v\|_s + \|h_w - h_{\tilde{w}}\|_{s-1} + |h_w - h_{\tilde{w}}|_1). \tag{6.31}$$

To estimate $h_w - h_{\tilde{w}}$, we decompose

$$h_w - h_{\tilde{w}} = U_3 + U_4 + U_5, \tag{6.32}$$

where

$$U_3 := (a'_{ij}(Dw) \cdot D\partial_j w)\partial_i v, \tag{6.33}$$
$$U_4 := (a'_{ij}(Dw) \cdot D\partial_j v)\partial_i \tilde{w}, \tag{6.34}$$
$$U_5 := \left(\int_0^1 a''_{ij}(\lambda Dw + (1-\lambda)D\tilde{w})(Dv, D\partial_j\tilde{w})\,\mathrm{d}\lambda\right)\partial_i\tilde{w}. \tag{6.35}$$

The terms U_3 and U_4 can be estimated exactly as the term h_w in (6.24) and (6.25), yielding

$$\|U_3\|_{s-1} + \|U_4\|_{s-1} \leqslant 2\psi(R_0)R_0\|\nabla v\|_s; \tag{6.36}$$

likewise, the term U_5 can be estimated as χ in (6.30) (it is at this point that we need the additional regularity of the coefficients a_{ij}), yielding

$$\|U_5\|_{s-1} \leqslant 2\psi(2R_0)R_0\|\nabla v\|_s. \tag{6.37}$$

In conclusion, we obtain that

$$\|h_w - h_{\tilde{w}}\|_{s-1} + |h_w - h_{\tilde{w}}|_1 \leqslant 4R_0\psi(R_0)\|\nabla v\|_s, \tag{6.38}$$

and, putting this into (6.31),

$$\|\nabla z\|_s \leqslant 6CR_0\psi(2R_0)\|\nabla v\|_s \leqslant \frac{1}{2}\|\nabla v\|_s. \tag{6.39}$$

By (6.21), (6.22) follows. □

As a consequence of Proposition 6.3, Φ admits a unique fixed point $u \in B_{s+1}(R_0)$, which is clearly the desired solution to the nonlinear elliptic equation (1.3). We now proceed to show that this solution depends continuously on h. Let $u, \tilde{u} \in B_{s+1}(R_0)$ be the solutions to (1.3) corresponding, respectively, to source terms $h, \tilde{h} \in H^{s-1} \cap L^1$. Then, as in (6.28), the difference $z = u - \tilde{u}$ satisfies the equation

$$A_w(z) = \underbrace{\partial_j((a_{ij}(Du) - a_{ij}(D\tilde{u}))\partial_i \tilde{u})}_{=:U_6} + \tilde{h} - h. \tag{6.40}$$

The term U_6 can again be estimated as χ in (6.30), with

$$\|U_6\|_{s-1} + |U_6|_1 \leqslant 2R_0\psi(2R_0)\|\nabla z\|_s; \tag{6.41}$$

thus,

$$\|\nabla z\|_s \leqslant 2CR_0\psi(2R_0)\|\nabla z\|_s + C(\|h - \tilde{h}\|_{s-1} + |h - \tilde{h}|_1). \tag{6.42}$$

By (6.21), $2CR_0\psi(2R_0) \leqslant \frac{1}{6}$, so that we deduce from (6.42)

$$\|\nabla u - \nabla \tilde{u}\|_s \leqslant C(\|h - \tilde{h}\|_{s-1} + |h - \tilde{h}|_1), \tag{6.43}$$

which shows the asserted continuous dependence of u on h. This ends the proof of Theorem 4.2. □

7. Convergence as $t \to \infty$

In this section we prove Theorem 4.3; again, we only consider the case $q = 1$ and, with abuse of notation, we write $Dv := (0, \nabla v)$. We denote by γ_j, $j \in \mathbb{N}$, a generic positive constant, depending on μ_0 and R_0 in a continuous and non-decreasing way.

The difference $z(t) := u(t) - v$ satisfies the equation

$$z_{tt} + 2z_t - b_{ij}\partial_i\partial_j z = f - h + g, \tag{7.1}$$

where, as before, $b_{ij} = a_{ij}(Du)$, and

$$g := (a_{ij}(Du) - a_{ij}(Dv))\partial_i\partial_j v = (A_{ij} \cdot Dz)\partial_i\partial_j v \tag{7.2}$$

with

$$A_{ij} := \int_0^1 a'_{ij}(\lambda Du + (1-\lambda)(Dv))\,\mathrm{d}\lambda; \tag{7.3}$$

note that $g \in L^\infty(0,\infty; H^s \cap L^1)$, since $h \in V^{s+2}$ (the L^1 part of this assertion follows by noting that $a_{ij}(Du) - a_{ij}(Dv) = (a_{ij}(Du) - a_{ij}(0)) + (a_{ij}(0) - a_{ij}(Dv)) \in L^2$).

We proceed to establish energy estimates on ∇z, similar to those on u established in the proof of Theorem 4.1. We multiply (7.1) in V^s by $2(z_t + z)$ (this procedure is formal, but can be justified by regularization, as in the proof of (5.14)). In doing so, we realize that the term $B := -2\langle \nabla(b_{ij}\partial_i\partial_j z), \nabla(z_t + z)\rangle_{s-1}$ can be written as

$$B = \frac{\mathrm{d}}{\mathrm{dt}}\tilde{Q} + 2\tilde{Q} + \tilde{R}_3, \tag{7.4}$$

where

$$\tilde{Q} := \sum_{|\beta|\leqslant s-1}\sum_{k=1}^{N}\langle b_{ij}\partial_i\partial_x^\beta\partial_k z, \partial_j\partial_x^\beta\partial_k z\rangle, \tag{7.5}$$

and $\tilde{R}_3$ is the sum of terms containing only derivatives of z, but not z itself. The terms with derivatives of z of order higher than 1 can be easily estimated, as we have done for R_3^ε (cf. (5.22)), in terms of $(\mu_0 + R_0)\|\nabla Dz(t)\|_{s-1}$; the terms with first order derivatives of z have the form

$$\tilde{R}_{3,1} = \langle a_{ij}'(Du)\cdot D\partial_k u\partial_j\partial_l z, \partial_r z\rangle, \tag{7.6}$$

and these can be estimated by

$$\begin{aligned}\tilde{R}_{3,1} &\leqslant |a_{ij}'(Du)|_\infty|\nabla Du|_N|\partial^2 z|_2|\nabla z|_{\bar{p}}\\ &\leqslant \gamma_0\|Du\|_{s-1}\|\partial^2 z\|^2 \leqslant \gamma_0\mu_0\|\nabla Dz\|^2.\end{aligned} \tag{7.7}$$

Thus, defining

$$G := \|\nabla z_t\|_{s-1}^2 + 2\langle\nabla z, \nabla z_t\rangle_{s-1} + 2\|\nabla z\|_{s-1}^2 + \tilde{Q}, \tag{7.8}$$

we obtain an estimate of the form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}G(t) + 2\|\nabla Dz(t)\|_{s-1}^2\\ &\qquad\leqslant \gamma_1(\mu_0 + R_0)\|\nabla Dz\|_{s-1}^2 + 2\langle\nabla(f - h + g), \nabla(z_t + z)\rangle_{s-1}.\end{aligned} \tag{7.9}$$

We can choose μ_0 and R_0 so small as to absorb the first term on the right side of (7.9) into the positive term $\|\nabla Dz(t)\|_{s-1}^2$ on its left, thus leading to an inequality of the form

$$G'(t) + \frac{3}{2}\|\nabla Dz(t)\|_{s-1}^2 \leqslant 2\langle\nabla(f - h + g), \nabla(z_t + z)\rangle_{s-1}. \tag{7.10}$$

Clearly,

$$\langle \nabla(f-h+g), \nabla z_t \rangle_{s-1} \leqslant \|f-h+g\|_s \|\nabla z_t\|_{s-1}, \tag{7.11}$$

and, integrating by parts,

$$\langle \nabla(f-h+g), \nabla z \rangle_{s-1} \leqslant \|f-h+g\|_{s-1} \|\Delta z\|_{s-1}; \tag{7.12}$$

thus, we deduce from (7.10) that

$$G'(t) + \frac{3}{2}\|\nabla Dz(t)\|_{s-1}^2 \leqslant C\|f-h+g\|_s \|\nabla Dz(t)\|_{s-1}. \tag{7.13}$$

By (2.6),

$$\|g\|_s \leqslant C\|A_{ij}\partial_i\partial_j v\|_s \|\nabla Dz\|_{s-1} \leqslant \gamma_2 R_0 \|\nabla Dz\|_{s-1}; \tag{7.14}$$

inserting this into (7.13), by the smallness of R_0 we obtain

$$G'(t) + \|\nabla Dz(t)\|_{s-1}^2 \leqslant C\|f-h\|_s \|\nabla Dz\|_{s-1}. \tag{7.15}$$

Now, for all $t \geqslant 0$,

$$\|\nabla Dz(t)\|_{s-1} \leqslant \|Dz(t)\|_s \leqslant \|Du(t)\|_s + \|\nabla v\|_s \leqslant \mu_0 + R_0 \leqslant 2, \tag{7.16}$$

so that, by virtue of (7.15),

$$G'(t) + \|\nabla Dz(t)\|_s^2 \leqslant 2C\|f-h\|_s. \tag{7.17}$$

Integrating this and recalling that, by Schwarz' inequality, $G(t) \geqslant 0$ for all $t \geqslant 0$, we deduce that

$$\int_0^t \|\nabla Dz(\theta)\|_{s-1}^2 \, \mathrm{d}\theta \leqslant G(0) + 2C\Lambda_0. \tag{7.18}$$

Thus, the function $t \mapsto \int_0^t \|\nabla Dz(\theta)\|_{s-1}^2 \, \mathrm{d}\theta$ is bounded. Finally, applying the standard energy estimate for (7.1) (i.e., multiplying the gradient of (7.1) only by $2\nabla z_t$ in H^{s-1}) and using the bounds so far obtained, we can deduce that the function $t \mapsto (\mathrm{d}/\mathrm{d}t)\|\nabla Dz(t)\|_{s-1}^2$ is also bounded from above. In conclusion, the positive function $t \mapsto \|\nabla Dz(t)\|_{s-1}^2$ is absolutely continuous on $[0, \infty[$, has a bounded integral and a derivative bounded above. These conditions imply that $\|\nabla Dz(t)\|_{s-1}^2 \to 0$ as $t \to \infty$.

To prove (4.14), we still need to show that also

$$\|\nabla z(t)\| + \|z_t(t)\| \to 0. \tag{7.19}$$

To this end, we rewrite (7.1) as in (5.33) in the form

$$z_{tt} + 2z_t - c_{ij}\partial_i\partial_j z = f - h + g + \tilde{g}, \tag{7.20}$$

where $c_{ij} = a_{ij}(0)$ and

$$\tilde{g}(\cdot) := (a_{ij}(Du) - a_{ij}(0))\partial_i\partial_j z. \tag{7.21}$$

From the first part of this proof we know that for every $\varepsilon > 0$ there is $T_1 \geqslant 0$ such that for all $t \geqslant T_1$,

$$\|\nabla Dz(t)\|_{s-1} \leqslant \varepsilon. \tag{7.22}$$

Assumption (4.13) on the data f and h also implies that there is $T_2 \geqslant 0$ such that for all $t \geqslant T_2$,

$$\int_t^\infty \underbrace{(\|f(\theta) - h\| + |f(\theta) - h|_1)}_{=\zeta(\theta)} \,\mathrm{d}\theta \leqslant \varepsilon. \tag{7.23}$$

Fix $\varepsilon \in \,]0,1[$ and let $T_\varepsilon := \max\{T_1, T_2\}$, $\varphi_\varepsilon := z(T_\varepsilon)$, $\psi_\varepsilon := z_t(T_\varepsilon)$; also, set $\Phi := f - h + \tilde{g}$. Then, for all $t \geqslant T_\varepsilon$, by the variation of parameters formula we obtain as in (5.37)

$$\begin{aligned}\nabla z(t) = {} & S(t - T_\varepsilon; \nabla(2\varphi_\varepsilon + \psi_\varepsilon)) + \partial_t S(t - T_\varepsilon; \nabla\varphi_\varepsilon) \\ & + \int_{T_\varepsilon}^t \nabla S(t-\theta; \Phi(\theta))\,\mathrm{d}\theta + \int_{T_\varepsilon}^t \nabla S(t-\theta; g(\theta))\,\mathrm{d}\theta.\end{aligned} \tag{7.24}$$

As in (3.15), the fact that $\nabla(\varphi_\varepsilon + \psi_\varepsilon) \in L^2$ implies that

$$\|S(t - T_\varepsilon; \nabla(2\varphi_\varepsilon + \psi_\varepsilon))\| \to 0 \quad \text{as } t \to \infty; \tag{7.25}$$

by Matsumura's estimate (3.2) with $q = 2$, $m = 0$ and $\nu = k = 1$,

$$\|\partial_t S(t - T_\varepsilon; \nabla\varphi_\varepsilon)\| \leqslant C(1 + t - T_\varepsilon)^{-1}\|\nabla\varphi_\varepsilon\|. \tag{7.26}$$

We estimate the third term of (7.24) again by (3.2), now with $q = 1$, $k = 0$ and $m = 1$; then $\nu_1 := \frac{1}{4}N + \frac{1}{2} \geqslant \frac{5}{4} > 1$ and

$$\begin{aligned} W_1(t) &:= \int_{T_\varepsilon}^{t} \|\nabla S(t-\theta; \Phi(\theta))\| \, \mathrm{d}\theta \\ &\leqslant C \int_{T_\varepsilon}^{t} (1+t-\theta)^{-\nu_1} (|\Phi(\theta)|_2 + |\Phi(\theta)|_1) \, \mathrm{d}\theta. \end{aligned} \tag{7.27}$$

Recalling (7.21), we conclude that

$$\begin{aligned} |\Phi(\theta)|_2 &\leqslant \|f(\theta) - h\| + |a_{ij}(Du) - a_{ij}(0)|_\infty |\partial_i \partial_j z|_2 \\ &\leqslant \|f(\theta) - h\| + \gamma_3 \|\nabla D z\|. \end{aligned} \tag{7.28}$$

Likewise, by (2.7),

$$\begin{aligned} |\Phi(\theta)|_1 &\leqslant |f(\theta) - h|_1 + |a_{ij}(Du) - a_{ij}(0)|_2 |\partial_i \partial_j z|_2 \\ &\leqslant |f(\theta) - h|_1 + \gamma_4 \|\nabla D z\|. \end{aligned} \tag{7.29}$$

Inserting (7.28) and (7.29) into (7.27) and recalling (7.23) and (7.22), we obtain that for all $t \geqslant T_\varepsilon$,

$$\begin{aligned} W_1(t) &\leqslant C \int_{T_\varepsilon}^{t} (1+t-\theta)^{-\nu_1} (\zeta(\theta) + \|\nabla D z(\theta)\|_1) \, \mathrm{d}\theta \\ &\leqslant C \int_{T_\varepsilon}^{\infty} \zeta(\theta) \, \mathrm{d}\theta + C\gamma_5 \varepsilon \int_0^t (1+\tau)^{-\nu_1} \, \mathrm{d}\tau. \end{aligned} \tag{7.30}$$

To estimate the last term of (7.24), we need to proceed in a slightly different manner because, even if, as we know, $g(\theta) \in L^1$, we do not know how to estimate $|g(\theta)|_1$ in terms of $\|\nabla D z(\theta)\|_{s-1}$. Thus, we first define positive numbers λ, μ by $\lambda^{-1} = 1 - \frac{1}{4}N^{-1}$, $\mu^{-1} = \frac{1}{2} - \frac{1}{4}N^{-1}$; it is immediate to verify that $\lambda \in \,]1, \bar{q}[$ and $\mu \in \,]2, \bar{p}[$. We apply (3.2) again, with $q = \lambda$, $k = 0$ and $m = 1$, noting that, since $\lambda < \bar{q}$, it still follows that $\nu_\lambda := \frac{1}{4}N(2\lambda^{-1} - 1) + \frac{1}{2} > 1$. Then

$$\begin{aligned} W_2(t) &:= \int_{T_\varepsilon}^{t} \|\nabla S(t-\theta; g(\theta))\| \, \mathrm{d}\theta \\ &\leqslant C \int_{T_\varepsilon}^{t} (1+t-\theta)^{-\nu_\lambda} (|g(\theta)|_2 + |g(\theta)|_\lambda) \, \mathrm{d}\theta. \end{aligned} \tag{7.31}$$

Since $\lambda^{-1} = \mu^{-1} + \frac{1}{2}$, using interpolation we estimate

$$|g|_\lambda \leqslant C |A_{ij}|_\infty |Dz|_\mu |\partial_i \partial_j v|_2 \leqslant \gamma_6 |Dz|_\mu \leqslant C\gamma_6 |Dz|_2^{3/4} |Dz|_{\bar{p}}^{1/4}. \tag{7.32}$$

Arguing as in (7.16) and recalling (7.14), we conclude from (7.32) and (7.22) that

$$|g|_2 + |g|_\lambda \leqslant \gamma_2 R_0 \|\nabla Dz\|_{s-1} + \gamma_7 |\nabla Dz|_2^{1/4} \leqslant \gamma_8 \varepsilon^{1/4}; \tag{7.33}$$

and, therefore, from (7.31) we obtain

$$W_2(t) \leqslant \gamma_8 \varepsilon^{1/4} \int_0^t (1+\tau)^{-\nu_\lambda} \,\mathrm{d}\tau \leqslant C\varepsilon^{1/4}. \tag{7.34}$$

Together with (7.25), (7.26), and (7.30), (7.34) allows us to assert, via (7.24), that $\|\nabla z(t)\| \to 0$ as $t \to \infty$. The proof that $\|z_t(t)\| \to 0$ as $t \to \infty$ is similar; we can therefore conclude the proof of Theorem 4.3. □

Remark. To prove (4.14), it would be natural to multiply (7.1) in H^s by $2(z_t+z)$ as in the proof of Theorem 4.1; however, we cannot do so because, in contrast to that situation, we do not know that $u(t) - v \in L^2$, because we do not know that $v \in L^2$. Note also that Theorem 4.1 guarantees that $u(t) \in L^2$, boundedly in t, only when $N \geqslant 5$, and, as we have noted already in the linear case (Theorem 3.3), the map $t \mapsto u(t)$ may fail to be bounded in L^2 if $N = 3$ or $N = 4$.

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References

[1] *R. Adams, J. Fournier*: Sobolev Spaces, 2nd ed. Academic Press, New York, 2003.
[2] *C. Cattaneo*: Sulla Conduzione del Calore. Atti Semin. Mat. Fis., Univ. Modena *3* (1949), 83–101. (In Italian.)
[3] *KP. Hadeler*: Random walk systems and reaction telegraph equations. In: Dynamical Systems and their Applications (S. van Strien, S. V. Lunel, eds.). Royal Academy of the Netherlands, 1995.
[4] *H. A. Haus*: Waves and Fields in Optoelectronics. Prentice Hall, 1984.
[5] *T. Kato*: Abstract Differential Equations and Nonlinear Mixed Problems. Fermian Lectures. Academie Nazionale dei Licei, Pisa, 1985.
[6] *S. Klainerman*: Global existence for nonlinear wave equations. Commun. Pure Appl. Math. *33* (1980), 43–101.
[7] *Ta-Tsien Li*: Nonlinear Heat Conduction with Finite Speed of Popagation. Proceedings of the China-Japan Symposium on Reaction Diffusion Equations and their Applcations to Computational Aspects. World Scientific, Singapore, 1997.
[8] *A. Matsumura*: On the asymptotic behavior of solutions to semi-linear wave equations. Publ. Res. Inst. Mat. Sci., Kyoto Univ. *12* (1976), 169–189.

[9] *A. Matsumura*: Global existence and asymptotics of the solutions of second-order quasilinear hyperbolic equations with first-order dissipation. Publ. Res. Inst. Mat. Sci., Kyoto Univ. *13* (1977), 349–379.
[10] *A. Milani*: The quasi-stationary Maxwell equations as singular limit of the complete equations: The quasi-linear case. J. Math. Anal. Appl. *102* (1984), 251–274.
[11] *A. Milani*: Global existence via singular perturbations for quasilinear evolution equations. Adv. Math. Sci. Appl. *6* (1996), 419–444.
[12] *S. Mizohata*: The Theory of Partial Differential Equations. Cambridge University Press, London, 1973.
[13] *J. Moser*: A rapidly convergent iteration method and non-linear differential equations. Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. *20* (1966), 265–315.
[14] *G. Ponce*: Global existence of small solutions to a class of nonlinear evolution equations. Nonlin. Anal., Theory Methods Appl. *9* (1985), 399–418.
[15] *R. Racke*: Non-homogeneous nonlinear damped wave equations in unbounded domains. Math. Methods Appl. Sci. *13* (1990), 481–491.
[16] *R. Racke*: Lectures on Nonlinear Evolution Equations. Initial Value Problems. Vieweg, Braunschweig, 1992.
[17] *S. Schochet*: The instant-response limit in Whitham's nonlinear traffic flow model: Uniform well-posedness and global existence. Asymptotic Anal. *1* (1988), 263–282.
[18] *H. Yang, A. Milani*: On the diffusion phenomenon of quasilinear hyperbolic waves. Bull. Sci. Math. *124* (2000), 415–433.

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