Mathematica Bohemica

Volker Reitmann

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Mathematica Bohemica, Vol. 136 (2011), No. 2, 185–194

Persistent URL: `http://dml.cz/dmlcz/141581`

136 (2011) MATHEMATICA BOHEMICA No. 2, 185–194

REALIZATION THEORY METHODS FOR THE STABILITY INVESTIGATION OF NONLINEAR INFINITE-DIMENSIONAL INPUT-OUTPUT SYSTEMS

Volker Reitmann, St. Petersburg

(Received October 15, 2009)

Abstract. Realization theory for linear input-output operators and frequency-domain methods for the solvability of Riccati operator equations are used for the stability and instability investigation of a class of nonlinear Volterra integral equations in a Hilbert space. The key idea is to consider, similar to the Volterra equation, a time-invariant control system generated by an abstract ODE in a weighted Sobolev space, which has the same stability properties as the Volterra equation.

Keywords: infinite dimensional Volterra integral equation, realization theory, absolute instability, frequency-domain method

MSC 2010: 34B16, 34C25

1. Introduction

The first step in the derivation of equations describing the dynamic behavior of observations is very often a Volterra integral equation, which represents causal or input-output properties of such observations or time-series. Stability, oscillating behavior, and other qualitative properties from a Volterra integral equation can be observed directly by frequency-domain methods developed in [9]. However, for other types of dynamic behavior such as instability and dichotomy it is useful to consider together with the given Volterra integral equation an associated realization as an evolution equation in some function spaces. In the present paper we consider infinite-dimensional Volterra equations. A state space realization of such equations is shown in Section 2. In Section 3 we discuss a theorem about the solvability of Riccati operator equations for the realization of Volterra integral equations in a Hilbert space.

Supported by DAAD (German Academic Exchange Service)

This result is close to a theorem proved by V. A. Brusin [3] for the finite-dimensional case. It is used in Section 4 for the derivation of frequency-domain conditions for stability and instability of Volterra integral equations in a Hilbert space.

2. Realization of infinite-dimensional Volterra equations as time invariant control systems in weighted Hilbert spaces

Suppose that Y and U are Hilbert spaces and introduce the Fréchet spaces $L^2_{\mathrm{loc}}(\mathbb{R};Y)$ and $L^2_{\mathrm{loc}}(\mathbb{R};U)$. Assume that

$$\phi\colon\ L^2_{\mathrm{loc}}(\mathbb{R}_+;Y)\times\mathbb{R}_+\times L^2_{\mathrm{loc}}(\mathbb{R}_+;Y)\to L^2_{\mathrm{loc}}(\mathbb{R}_+;Y) \tag{2.1}$$

is a nonlinear operator generating the *Volterra functional equation*

$$y=\phi(y,t,h). \tag{2.2}$$

Assume also that there are a continuous linear operator

$$\mathcal{T}\colon\ L^2_{\mathrm{loc}}(\mathbb{R};U)\to L^2_{\mathrm{loc}}(\mathbb{R};Y) \tag{2.3}$$

and a nonlinear operator

$$\varphi\colon\ L^2_{\mathrm{loc}}(\mathbb{R}_+;Y)\times\mathbb{R}_+\to L^2_{\mathrm{loc}}(\mathbb{R}_+;U) \tag{2.4}$$

such that the operator (2.1) can be written as

$$\phi(y,t,h)=\mathcal{T}\varphi(y,t)+h(t), \tag{2.5}$$

where $h\in L^2_{\mathrm{loc}}(\mathbb{R}_+;Y)$ is considered as a *perturbation* or a *forcing function.* Thus the Volterra functional equation has the form

$$y=\mathcal{T}u+h, \tag{2.6a}$$

$$u=\varphi(y,t). \tag{2.6b}$$

We call (2.6a) the *linear part* and (2.6b) the *nonlinear part* of (2.5). A function $y\in L^2_{\mathrm{loc}}(\mathbb{R}_+;Y)$ satisfying (2.6a), (2.6b) for a.a. $t\in\mathbb{R}_+$ is called a *solution.* Any pair (y,u), where y is a solution of (2.6a), (2.6b) and $u=\varphi(y,t)$ is said to be a *process* generated by (2.6a), (2.6b).

For any interval $\mathcal{J} \subset \mathbb{R}$, a Hilbert space Z and any $s \in \mathbb{R}$ denote by τ^s the *shift operator* acting on functions $f\colon \mathcal{J} \to Z$ by

$$\tau^s f(t) := \begin{cases} f(t+s) & \text{if } \ t+s \in \mathcal{J}, \\ 0 & \text{if } \ t+s \notin \mathcal{J}. \end{cases}$$

The input-output operator (2.3) is called *time invariant* if $\tau^t \mathcal{T} = \mathcal{T}\tau^t$ for every $t \in \mathbb{R}$ and is called *causal* if for all $t \geqslant 0$

$$u(t) = 0, \ \forall\, t \leqslant T \quad \Rightarrow \quad \mathcal{T}u(t) = 0, \ \forall\, t \leqslant T.$$

This implies that $\mathcal{T}$ in (2.3) is defined by its restriction

$$\mathcal{T}\colon\ L^2_{\mathrm{loc}}(\mathbb{R}_+; U) \to L^2_{\mathrm{loc}}(\mathbb{R}_+; Y). \tag{2.7}$$

For any interval $\mathcal{J} \subset \mathbb{R}$, a Hilbert space Z and a parameter $\varrho \in \mathbb{R}$ we introduce the weighted spaces $L^2_\varrho(\mathcal{J}; Z)$ and $W^{1,2}_\varrho(\mathcal{J}; Z)$ by

$$L^2_\varrho(\mathcal{J}; Z) := \left\{ f \in L^2_{\mathrm{loc}}(\mathcal{J}; Z)\colon \ \int_{\mathcal{J}} \mathrm{e}^{-2\varrho t} |f(t)|^2_Z \,\mathrm{d}t < \infty \right\}$$

and

$$W^{1,2}_\varrho(\mathcal{J}; Z) := \{ f \in L^2_\varrho(\mathcal{J}; Z)\colon\ \dot{f} \in L^2_\varrho(\mathcal{J}; Z) \}.$$

($\dot{f}$ denotes the distribution derivative.) Let us assume that $\mathcal{T}$ from (2.3) can be considered for some $\varrho \in \mathbb{R}$ as a bounded linear operator

$$\mathcal{T}\colon\ L^2_\varrho(\mathbb{R}; U) \to L^2_\varrho(\mathbb{R}; Y). \tag{2.8}$$

If the property (2.8) is satisfied the input-output operator can be realized as a time-invariant control system in weighted Hilbert spaces. The key information for this is given by the *Hankel operator* $\mathcal{H}$ associated with the input-output operator $\mathcal{T}$, i.e.

$$\mathcal{H}\colon\ L^2_0(\mathbb{R}_-; U) \to L^2(\mathbb{R}_+; Y)$$

given by $\mathcal{H} = \mathbf{P}_+ \mathcal{T} \mathbf{P}_-$, where $\mathbf{P}_+ := \mathbf{P}_{\mathbb{R}_+}$, $\mathbf{P}_- := \mathbf{P}_{\mathbb{R}_-}$ and $\mathbf{P}_E u := \zeta_E u$, where ζ_E is the characteristic function of $E \subset \mathbb{R}$. The space $L^2_0(\mathbb{R}_-; Z)$ is the space of compactly supported square integrable functions which is dual to the space $L^2_{\mathrm{loc}}(\mathbb{R}_+; Z)$ via the pairing $(\psi, \varphi) := \int_{-\infty}^{+\infty} (\psi(-t), \varphi(t))_Z \,\mathrm{d}t$. According to [7] we can describe a state space description whose input-output behaviour is given by $\mathcal{T}$ as

$$z(t; z_0, u) = \tau^t z_0 + \tau^t \mathcal{T}(\zeta_{[0,t]} u), \tag{2.9a}$$

$$y(t; z_0, u) = z_0(t) + (\mathcal{T}u)(t), \ \ t \geqslant 0 \tag{2.9b}$$

for $z_0 \in Z_0 := L^2_\varrho(\mathbb{R}_+;Y)$ and $u \in L^2_{\text{loc}}(\mathbb{R}_+;U)$. It is clear that (2.9a), (2.9b) also has the *time invariance property*, i.e.

$$\begin{aligned} z(t+s;z_0,u) &= z(t;z(s;z_0,u),\tau^s u), \\ y(t+s;z_0,u) &= y(t;z(s;z_0,u),\tau^s u), \end{aligned}$$

for $t,s \in \mathbb{R}_+$, for $z_0 \in Z_0$, for $u \in L^2_{\text{loc}}(\mathbb{R}_+;U)$. The state space realization (2.9a), (2.9b) is generated by the time-invariant control evolution system

$$\dot{z} = Az + Bu, \tag{2.10a}$$

$$y = C(z - (\lambda I - A)^{-1}Bu) + \chi(\lambda)u, \tag{2.10b}$$

defined in the rigged Hilbert space structure ([1]) (or Gelfand triple [8]) $Z_1 \subset Z_0 \subset Z_{-1}$ with Z_0 as above and $Z_1 := W^{1,2}_\varrho(\mathbb{R}_+;Y)$ and the linear operators $A \in \mathcal{L}(Z_1,Z_0) \cap \mathcal{L}(Z_0,Z_{-1}), B \in \mathcal{L}(U,Z_{-1})$ and $C \in \mathcal{L}(Z_1,Y)$ given by

$$\begin{aligned} A\xi &:= \dot{\xi},\ \ \xi \in D(A), \\ B^*\eta &:= (\mathcal{H}\eta)(0),\ \ \eta \in Z^*_{-1} := \{\eta \in W^{1,2}_\varrho(\mathbb{R}_-;Y)\colon\ \eta(0) = 0\}, \\ Cz &:= z(0),\ \ z \in Z_1. \end{aligned}$$

In (2.10b) $\lambda \notin \sigma(A)$ is an arbitrary value. For these values and any other value $\mu \notin \sigma(A)$ the operator $\chi(\lambda) \in \mathcal{L}(U,Y)$ is defined by the identity

$$\chi(\lambda) - \chi(\mu) = (\mu - \lambda)C(\lambda I - A)^{-1}(\mu I - A)^{-1}B. \tag{2.11}$$

If we have the additional properties $B \in \mathcal{L}(U,Z_0)$ or $C \in \mathcal{L}(Z_0,Y)$ the usual transfer operator $C(\lambda I - A)^{-1}B$ makes sense. In this case it follows from (2.11) that $\chi(\lambda) = C(\lambda I - A)^{-1}B$ and (2.10b) goes over to the usual output equation

$$y = Cz. \tag{2.12}$$

Assume for this that the input-output operator $\mathcal{T}$ from (2.8) can be represented as a convolution operator

$$(\mathcal{T}u)(t) := \int_0^t K(t-s)u(s)\,\mathrm{d}s, \tag{2.13}$$

where $K(\cdot)$ is a certain kernel called the *weighting pattern* of $\mathcal{T}$.

Assume that the map $t \in \mathbb{R}_+ \mapsto K(t) \in \mathcal{L}(U,Y)$ is twice piecewise-differentiable and satisfies the following condition: There exist a $\varrho_0 > 0$ and a constant $\gamma > 0$ such that

$$\|K(t)\|_{\mathcal{L}(U,Y)} \leqslant \gamma \mathrm{e}^{-\varrho_0 t}, \quad \forall\, t > 0, \qquad \text{and} \tag{2.14}$$

$$\int_0^\infty [\|\dot{K}(t)\|^2_{\mathcal{L}(U,Y)} + \|\ddot{K}(t)\|^2_{\mathcal{L}(U,Y)}]\mathrm{e}^{2\varrho_0 t}\,\mathrm{d}t < \infty. \tag{2.15}$$

Under these conditions we can choose the following state space realization of (2.13):

$$\begin{aligned}
&Z_0 := W^{1,2}_{-\varrho}(0,\infty;Y), \quad \text{where } 0 < \varrho < \varrho_0 \text{ is arbitrary},\\
&D(A) := \left\{\xi(s) \in W^{1,2}_{-\varrho}(0,\infty;Y)\colon \int_0^\infty \mathrm{e}^{2\varrho s}\|\ddot{\xi}(s)\|^2_Y\,\mathrm{d}s < \infty\right\},\\
&(A\xi)(s) := \dot{\xi}(s), \qquad \forall\, \xi \in D(A),\\
&(B\eta)(s) := K(s)\eta, \quad \forall\, \eta \in U,\\
&(Cz)(s) := z(0), \qquad \forall\, z \in Z_0.
\end{aligned}$$

Thus we have defined a time-invariant control system

$$\dot{z} = Az + Bu, \tag{2.21a}$$

$$y = Cz, \tag{2.21b}$$

where A from (2.18) is a closed linear operator that acts in Z_0 given in (2.16) and has the dense domain of definition $D(A)$ from (2.17). It is clear that A is the generator of some C_0 semigroup $\{S(t)\}_{t\geqslant 0}$. The map (2.19) defines a linear bounded operator $B\colon U \to Z_0$. If $z_0 \in D(A)$ the generalized solution $z(\cdot, z_0)$ of (2.21a) starting in Z_0 is a continuous function $t \mapsto z(t, z_0) \in Z_0$, which can be represented in integral form as

$$z(t,z_0) = S(t)z_0 + \int_0^t S(t-\tau)Bu(\tau)\,\mathrm{d}\tau \qquad \text{with} \tag{2.22}$$

$$\|S(t)\| \leqslant \alpha \mathrm{e}^{-\varkappa t}, \qquad \forall\, t > 0, \tag{2.23}$$

where α and $\varkappa$ are positive numbers, and which satisfies for any $u \in L^2(\mathbb{R}_+;U)$ the output relation

$$C\int_0^t S(t-\tau)Bu(\tau)\,\mathrm{d}\tau = \int_0^t K(t-\tau)u(\tau)\,\mathrm{d}\tau. \tag{2.24}$$

Note that if $z(t,z_0) \in D(A)$ for $t > 0$ then $z(\cdot, z_0)$ is an ordinary strong solution of (2.21a).

3. Solvability of the Riccati operator equation for the realizations of a class of Volterra equations

Let us assume that $F_1 = F_1^* \in \mathcal{L}(Y,Y), F_2 \in \mathcal{L}(U,Y)$ and $F_3 = F_3^* \in \mathcal{L}(U,U)$ are bounded linear operators and introduce the bilinear form

$$j(x,y;u,v) := (F_1 x, y)_Y + (F_2 u, x)_Y + (F_2 v, y)_Y + (F_3 u, u)_U, \qquad \forall\, x,y \in Y,\ \forall\, u,v \in U. \tag{3.1}$$

A direct calculation shows that

$$j(x,y;u,v) = j(y,x;v,u), \quad \forall\, x,y \in Y,\ \forall\, u,v \in U. \tag{3.2}$$

Introduce the linear operator

$$\mathcal{K}(u,h)(t) := (\mathcal{T}u)(t) + h(t), \quad u \in L^2(\mathbb{R}_+;U),\ h \in W^{1,2}_{-\varrho}(\mathbb{R}_+;Y),$$

and consider for $T > 0$ and a parameter ν, $|\nu| \leqslant \nu_0$, the bilinear functional

$$J^T_\nu(u,h) := \int_0^T [j(\mathcal{K}(u,h),\mathcal{K}(u,h);u,u) - \nu \|z(t,h,u)\|^2_{W^{1,2}_{-\varrho}(\mathbb{R}_+;Y)}]\,\mathrm{d}t, \tag{3.3}$$

which is for $\varrho \in (0,\varrho_0)$ a continuous map $L^2(0,T;U) \times W^{1,2}_{-\varrho}(0,+\infty;Y) \to \mathbb{R}$.

The next theorem (for a proof see [6]) contains a version of the operator Riccati equation. For the case $U = Y = \mathbb{R}^n$ this theorem was proved in [3].

Theorem 3.1. *Let $\chi(p)$ be the Laplace transform of the absolutely continuous function $\mathbf{P}_\infty K$ (where $\mathbf{P}_\infty K(t) = K(t)$ if $t \geqslant 0$ and 0 if $t < 0$). Suppose that $F_1 = F_1^* \geqslant 0$, $F_3 = F_3^* > 0$, F_3^{-1} exists,*

$$\chi(p) \in \mathcal{L}(U,Y), \quad \forall\, p \in \mathbb{C}, \qquad \text{and}$$

$$\Pi(\mathrm{i}\omega) := \chi^*(\mathrm{i}\omega) F_1 \chi(\mathrm{i}\omega) + 2\,\mathrm{Re}(F_2^* \chi(\mathrm{i}\omega)) + F_3 > 0, \qquad \forall\, \omega \in \mathbb{R}. \tag{3.4}$$

Then there exists a sufficiently small $\nu_0 > 0$ such that for any $\nu \in [0,\nu_0]$ we have (the index ν is omitted):

1) *For any $h \in W^{1,2}_{-\varrho}(0,+\infty;Y)$ there exists a $\tilde{u}(h) \in L^2(0,+\infty;U)$ such that*

$$J^T_0(\tilde{u}(h),h) < J^T_0(u,h), \quad \forall\, u \in L^2(0,T;U),\ \|u - \tilde{u}(h)\|_{L^2(0,T;U)} > 0.$$

2) *There exists a bounded self-adjoint operator*

$$M_T = M_T^* \colon W^{1,2}_{-\varrho}(0,+\infty;Y) \to W^{1,2}_{-\varrho}(0,+\infty;Y)$$
$$(M_T h, h)_{W^{1,2}_{-\varrho}(0,+\infty;Y)} = J^T_0(\tilde{u}(h),h), \quad \forall\, h \in W^{1,2}_{-\varrho}(0,+\infty;Y).$$

3) *In the case* $T = \infty$ *the operator* $M := M_\infty$ *satisfies the Riccati operator equation*

$$
\begin{aligned}
S(h,g) &:= (Ah, Mg)_{W^{1,2}_{-\varrho}(0,+\infty;Y)} + (Mh, Ag)_{W^{1,2}_{-\varrho}(0,+\infty;Y)} - (L^*h, L^*g)_U \\
&\quad + (F_1Ch, Cg)_Y - \nu(h,g)_{W^{1,2}_{-\varrho}(0,+\infty;Y)}, \quad \forall\, h, g \in D(A),
\end{aligned} \tag{3.5}
$$

where $N := \sqrt{F_3}$, $L := (MB + C^*F_2)N^{-1} \in \mathcal{L}(U, W^{1,2}_{-\varrho}(0,+\infty;Y))$ *and* $(L^*h, v)_U = (h, Lv)_{W^{1,2}_{-\varrho}(0,+\infty;Y)}$, $\forall\, h \in W^{1,2}_{-\varrho}(0,+\infty;Y)$, $\forall\, v \in U$.

Corollary 3.2. *Suppose that for* (2.13) *and the associated state space realization* (2.21a), (2.21b) *the conditions of Theorem* 3.1 *are satisfied. Then for sufficiently small* $\nu > 0$, *any* $h \in D(A)$ *and any continuous function* $u \in L^2(0,\infty;U)$ *the pair* $(z(\cdot), y(\cdot))$, *where* $z(\cdot) \equiv z(\cdot, h, u)$ *is the solution of* (2.21a) *with* $z(0,h,u) = h$ *and* $y(\cdot) = Cz(\cdot)$, *satisfies for* $t > 0$ *the relation*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Mz(t), z(t))_{W^{1,2}_{-\varrho}(0,+\infty;Y)} &= \|L^*z(t) + Nu(t)\|^2_U \\
&\quad - [(F_1y(t), y(t))_Y + 2(F_2u(t), y(t))_Y + (F_3u(t), u(t))_U] \\
&\quad + \nu\|z(t)\|^2_{W^{1,2}_{-\varrho}(0,+\infty;Y)}.
\end{aligned} \tag{3.6}
$$

Here $M = M^*$, L *and* N *are the operators from part* 3) *of Theorem* 3.1.

In the next section Theorem 3.1 will be used to derive frequency-domain conditions for stability and instability properties of Volterra integral equations.

4. Stability and instability of infinite-dimensional Volterra equations by their state space realizations

Consider the Volterra integral equation

$$
y(t) = h(t) + \int_0^t K(t-\tau)\varphi(y(\tau),\tau)\,\mathrm{d}\tau, \tag{4.1}
$$

where $K(t) \in \mathcal{L}(U,Y)$ (U, Y Hilbert spaces) is twice piecewise-differentiable, satisfies (2.14) and (2.15), and has therefore a state space realization (2.16)–(2.21b). Suppose that

$$
\varphi\colon\ Y \times \mathbb{R}_+ \to U \tag{4.2}
$$

is a continuous function. Instead of one fixed nonlinearity φ we consider a family $\mathcal{N}$ of continuous maps (4.2) such that for any $\varphi \in \mathcal{N}$ and any $h \in D(A)$ with $D(A)$

from (2.17) the nonlinear integral equation (4.1) has a unique solution $y(\cdot,h,\varphi)$ and this solution is continuous. Suppose also that there are linear bounded operators $G_1=G_1^*\in\mathcal{L}(Y,Y)$, $G_1\leqslant 0$, $G_2\in\mathcal{L}(U,Y)$ and $G_3=G_3^*\in\mathcal{L}(U,U)$, $G_3<0$, G_3^{-1} such that for any $\varphi\in\mathcal{N}$ we have

$$(G_1y,y)_Y+2(G_2\varphi(y,t),y)_Y+(G_3\varphi(y,t),\varphi(y,t))_U\geqslant 0,\quad \forall\, t\geqslant 0,\ \forall\, y\in Y. \tag{4.3}$$

A broad discussion of this condition one can find in [2], [3], [9]. Now we consider together with the state space equation (2.21a), (2.21b) and the nonlinearity $\varphi\in\mathcal{N}$ the nonlinear evolution system

$$\dot z=Az+B\varphi(y,t),\quad y=Cz. \tag{4.4}$$

Theorem 4.1. *Suppose that the following conditions are satisfied:*

a) *Let $\chi(\cdot)$ be the Laplace transform of $\mathbf{P}_\infty K$ and let with the operators $F_1=-G_1$, $F_2=-G_2$ and $F_3=-G_3$ from* (4.3) *the frequency-domain condition* (3.4) *be true.*

b) *For any $h\in W^{1,2}_{-\varrho}(0,+\infty;Y)$ and $\varphi\in\mathcal{N}$ the solution $y(\cdot)=y(\cdot,h,\varphi)$ of* (4.1) *exists and is continuous.*

Then for any $h\in W^{1,2}_{-\varrho}(0,+\infty;Y)$ and $\varphi\in\mathcal{N}$ the solution $z(\cdot)=z(\cdot,h,\varphi)$ of (4.4) *with the initial condition $z(0)=h$ exists and there are a bounded linear self-adjoint operator $P\colon W^{1,2}_{-\varrho}(0,+\infty;Y)\to W^{1,2}_{-\varrho}(0,+\infty;Y)$ and a constant $\delta>0$ such that for any $t_1,t_2\geqslant 0$, $t_1<t_2$, we have*

$$\begin{aligned}(Pz(t,h,\varphi),z(t,h,\varphi))_{W^{1,2}_{-\varrho}(0,+\infty;Y)}\Big|_{t_1}^{t_2}\\ \leqslant -\delta\int_{t_1}^{t_2}[\|\varphi(y(t,h,\varphi),t)\|_U^2+\|z(t,h,\varphi)\|^2_{W^{1,2}_{-\varrho}(0,+\infty;Y)}]\,\mathrm{d}t,\\ \forall\,\varphi\in\mathcal{N},\ \forall\, h\in D(A).\end{aligned} \tag{4.5}$$

P r o o f. Let us assume for a moment that the kernel $K(\cdot)$ satisfies the conditions from Section 3 and that $h\in D(A)$. Introduce the continuous function $u(t):=\varphi(y(t,h,\varphi),t)$ for $t>0$. With this function we can apply Corollary 3.2. Integration of (3.6) on an arbitrary time interval $0\leqslant t_1<t_2$ with $P:=-M$ and $F_i:=-G_i$, $i=1,2,3$, gives

$$\begin{aligned}(Pz(t,h,\varphi),z(t,h,\varphi))_{W^{1,2}_{-\varrho}(0,+\infty;Y)}\Big|_{t_1}^{t_2}\\ \leqslant\int_{t_1}^{t_2}[(G_1y(t,h,\varphi),y(t,h,\varphi))_Y+2(G_2u(t),y(t,h,\varphi))_Y\\ +(G_3u(t),u(t))_U]\,\mathrm{d}t-\nu\int_{t_1}^{t_2}\|z(t,h,\varphi)\|^2_{W^{1,2}_{-\varrho}(0,+\infty;Y)}\,\mathrm{d}t.\end{aligned} \tag{4.6}$$

It follows from (2.23) and the boundedness of the operator P that the left- and right-hand sides of (4.6) depend continuously on $h \in W^{1,2}_{-\varrho}(0,+\infty;Y)$ and $K(t) \in \mathcal{L}(U,Y)$.

This and the density of $D(A)$ in $W^{1,2}_{-\varrho}(0,+\infty;Y)$ implies (see also [3]) that the inequality (4.6) can be continued for functions $h \in W^{1,2}_{-\varrho}(0,+\infty;Y)$.

Since the inequality (3.4) is strict we can get a similar inequality (4.6) with $\widetilde{G}_3 = G_3 - \delta_1 I$ where I is the unit operator and $\delta_1 > 0$ is sufficiently small. This modified inequality (4.6) and (4.3) immediately give (4.5). □

The next theorem can be derived in a similar way.

Theorem 4.2. *Suppose that $\chi(\cdot)$ is the Laplace transform of $\mathbf{P}_\infty K$, the operator function $(I-\chi(p)R)^{-1}$ has poles in the right half-plane and the frequency-domain condition (3.4) is satisfied with $F_i = -G_i$, $i=1,2,3$. Then there exists a bounded linear self-adjoint operator*

$$\begin{aligned} &P\colon\ W^{1,2}_{-\varrho}(0,+\infty;Y) \to W^{1,2}_{-\varrho}(0,+\infty;Y) \quad \text{such that}\\ &\mathcal{C} := \{h \in W^{1,2}_{-\varrho}(0,+\infty;Y)\colon\ (Ph,h)_{W^{1,2}_{-\varrho}(0,+\infty;Y)} < 0\}\end{aligned}$$

is a quadratic cone $\mathcal{C} \neq \emptyset$ in $W^{1,2}_{-\varrho}(0,+\infty;Y)$ with the following properties:

a) *There exists a constant $\beta > 0$ such that for any $h \in \mathcal{C}$ and any $\varphi \in \mathcal{N}$*

$$\lim_{t\to\infty} \mathrm{e}^{-\beta t} \int_0^t \|\varphi(y(s,h,\varphi),s)\|_U^2 \,\mathrm{d}s = \infty. \tag{4.7}$$

b) *Any solution $y(\cdot,h,\varphi)$ of (4.1) which does not satisfy (4.7) has the property $\int_0^\infty \|\varphi(y(s,h,\varphi),s)\|_U^2 \,\mathrm{d}s < \infty$ and, consequently,*

$$\varphi(y(\cdot,h,\varphi),\cdot) \in L^2(0,\infty;U) \quad \text{and} \quad y(\cdot,h,\varphi) \in L^2(0,\infty;Y). \tag{4.8}$$

The abstract stability theory for Volterra integral equations, developed in Section 4, can be used to show that different realizations of a given Volterra equation, such as a PDE with boundary control or an ODE with delay, can have the same stability properties as the Volterra equation. For the equation describing a fluid conveying tube this was done in [6].

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Author's address: *Volker Reitmann*, Saint-Petersburg State University, Department of Mathematics and Mechanics, St. Petersburg, Russia, e-mail: `vreitmann@aol.com`.