Mathematica Bohemica

Vladimir B. Răsvan

Stability and sliding modes for a class of nonlinear time delay systems

Mathematica Bohemica, Vol. 136 (2011), No. 2, 155–164

Persistent URL: `http://dml.cz/dmlcz/141578`

STABILITY AND SLIDING MODES FOR A CLASS OF NONLINEAR TIME DELAY SYSTEMS

Vladimir Răsvan, Craiova

(Received October 15, 2009)

Abstract. The following time delay system

$$\dot{x} = Ax(t) + \sum_{1}^{r} bq_i^* x(t - \tau_i) - b\varphi(c^* x(t))$$

is considered, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ may have discontinuities, in particular at the origin. The solution is defined using the "redefined nonlinearity" concept. For such systems sliding modes are discussed and a frequency domain inequality for global asymptotic stability is given.

Keywords: time lag, extended nonlinearity, absolute stability

MSC 2010: 34A36, 34D20, 34K20, 93C23, 93D10

1. Motivation and problem statement

The present paper starts from two phenomena that have as a consequence instabilities in feedback control systems.

A. Since the very beginning of the automatic control, the actuator dynamics has been a permanent challenge in control engineering, due to such nonlinearities as saturation, dead zone or dry friction; an "ever green" echo of these aspects is the problem of the PIO—Pilot-In the loop-Oscillations. Here Category II PIO is defined by the presence of *series rate or position limits*, i.e. of saturation nonlinear functions. Additional phenomena displayed by various mechanical hardware such as dry friction lead to discontinuous nonlinear functions. Discontinuous nonlinear functions

This work has been supported by the Research Project ID-95 financed by the National Council of Higher Education Scientific Research of Romania (CNCSIS).

transform the mathematical description of the nonlinear systems in differential equations with discontinuous right hand side; defining the solutions of such equations is quite a problem. A well known phenomenon associated with discontinuous nonlinear control systems is that of the sliding modes which are associated also with variable structure systems, hybrid systems, in general to all systems whose behavior implies some commutation.

B. Another phenomenon connected with control systems dynamics is supplied by time lags. A combination, i.e. simultaneous presence of time lags and discontinuous nonlinearities is not impossible, on the contrary.

Consequently, the purpose of this paper is to consider time lag systems with discontinuous nonlinearities from the point of view of stability and sliding modes. The problem has not been without interest up to now [10] but it is felt that the way of tackling it here is somehow new. Accordingly, the paper is organized as follows: first an interesting special structure of time delay systems is introduced; then the problem of defining the solution for the discontinuous nonlinearity case is considered, together with the problem of the sliding modes. Finally, global asymptotic stability is discussed and Popov-like frequency domain inequalities are proposed, some usual assumptions being relaxed.

2. A class of time delay systems

In the following we shall consider the time delay system

$$\dot{x}(t) = Ax(t) + \sum_{1}^{r} bq_i^* x(t-\tau_i) - b\varphi(c^* x(t)) \tag{2.1}$$

which is obviously a special case of

$$\dot{x}(t) = Ax(t) + \sum_{1}^{r} B_i x(t-\tau_i) - b\varphi(c^* x(t)) \tag{2.2}$$

with $B_i = bq_i^*$ being special dyadic matrices. If we consider (2.1) or (2.2) as generated by the feedback connection of the nonlinear block $\mu(t) = -\varphi(c^* x(t))$ with the linear block with time delay

$$\dot{x}(t) = Ax(t) + \sum_{1}^{r} B_i x(t-\tau_i) + b\mu(t), \quad \sigma = c^* x \tag{2.3}$$

and take $B_i = bq_i^*$, we obtain the class of linear systems discussed in [6] for the case $r = 1$ (single delay case). This class is not as narrow as one might think since it covers the higher order time delay equations

$$y^{(n)}(t) + \sum_1^{n-1} \alpha_k y^{(k)}(t) + \sum_1^r \sum_1^{n-1} \beta_{ki} y^{(k)}(t-\tau_i) = \mu(t) \tag{2.4}$$

with the corresponding choices of the vectors and matrices.

The converse is also true [6]: if (A, b) is a controllable pair, then (2.3) and (2.4) are linearly equivalent; this equivalence is concerned only with states. More precisely, if (A, b) is controllable, there exists T, $\det T \neq 0$, such that $A_c = TAT^{-1}$, $b_c = TB$. If we take in (2.3) $x_c = Tx$ we find

$$\dot{x}_c(t) = A_c x_c(t) + \sum_1^r b_c q_i^* x_c(t-\tau_i) + b_c\mu(t), \quad \sigma = c^* T^{-1} x_c. \tag{2.5}$$

We may denote $q_i^* T^{-1} = (q_i^c)^*$ but $c^* T^{-1}$ may not have the form corresponding to the equation unless the transfer function of (2.3), which is invariant with respect to coordinate changes, is also *all-pole*. The transfer function is

$$\chi(s) = c^* \left(sI - A - b \sum_1^r q_i^* \mathrm{e}^{-s\tau_i} \right)^{-1} b = \frac{c^*(sI-A)^{-1}b}{1 - \sum_1^r \mathrm{e}^{-s\tau_i} q_i^* (sI-A)^{-1} b} \tag{2.6}$$

and if the rational function $c^*(sI - A)^{-1}b$ is all-pole, then $\chi(s)$ is such. Other properties of this class of systems may be found in [6].

3. Extended nonlinearities. Basic theory and sliding modes

We shall refer here to system (2.2); we shall consider that $\varphi(\sigma)$ may have finite discontinuities. If we start from the standard point of view of the theory of absolute stability, system (2.2) describes a feedback structure where the nonlinear block incorporates the discontinuities while the linear time invariant one incorporates the constant delays. This structure shows that, from the physical point of view, the differential inclusion approach should allow a correct definition of $\mu(t) = -\varphi(c^* x(t))$ along the solution of the system. It appears now quite natural to introduce

Definition 3.1. By a solution of (2.2) we understand a pair of functions $(x(t), \mu(t))$ defined on a segment of non-zero measure $[t_0, t_1]$ and such that $x(t)$ is absolutely continuous, $\mu(t)$ is integrable and

$$\dot{x}(t) = Ax(t) + \sum_1^r B_i x(t-\tau_i) + b\mu(t), \quad \sigma(t) = c^* x(t); \ -\mu(t) \in \varphi(\sigma(t)) \tag{3.1}$$

holds a.e. on $[t_0, t_1]$. The function $\mu(t)$ is called an extended nonlinearity $\varphi(\sigma(t))$.

As shown in [4] system (2.2) may be written as a differential inclusion where the multi-valued functional $f\colon\ \mathbb{R}\times X\to\mathbb{R}^n$ reads

$$f(t,\phi)=A\phi(0)+\sum_1^r B_i\phi(-\tau_i)-b\varphi(c^*\phi(0)). \tag{3.2}$$

The properties of this functional imply that the differential inclusion has a solution satisfying some initial condition defined on some function space, e.g. $\mathcal{C}(-\tau,0;\mathbb{R}^n)$, $\tau=\max\limits_i \tau_i$. If $b^*b\neq 0$ then $\mu(t)$ is deduced from

$$-\mu(t)=(b^*b)^{-1}b^*\left(\dot{x}(t)-Ax(t)-\sum_1^r B_i x(t-\tau_i)\right)$$

and clearly $-\mu(t)\in\varphi(\sigma(t))$. In this way we introduce the so-called *extended nonlinearity.* We may now define the sliding modes.

Definition 3.2. Let $\sigma(t)=\sigma_0$, where σ_0 is a discontinuity point of $\varphi(\sigma)$, for $t\in\mathcal{I}\subseteq[t_0,t_1]$ Then the solution $(x(t),\mu(t))$, $t\in\mathcal{I}$, of (3.1) is said to be in a sliding mode.

We shall not reproduce here the construction of [4] concerning sliding modes, nor shall we give its extension to time delay systems (which is not difficult due to the fact that the right hand side is finite dimensional valued). The following lemma which is a straightforward extension of Lemma 1.3 of [4] is useful.

Lemma 3.1. *Let $(x(t),\mu(t))$ satisfy*

$$\dot{x}(t)=Ax(t)+b\sum_1^r q_i^* x(t-\tau_i)+b\mu(t),\quad \sigma_0=c^*x+h_0\mu \tag{3.3}$$

with A, q_i, b, c, h_0, σ_0 being constant. Assume the transfer function from the input $\mu(t)$ to the output $\sigma(t)=c^*x(t)+h_0\mu(t)$

$$\chi(s)=h_0+c^*\left(sI-A-b\sum_1^r q_i^*\mathrm{e}^{-s\tau_i}\right)^{-1}b=h_0+\frac{c^*(sI-A)^{-1}b}{1-\sum_1^r \mathrm{e}^{-s\tau_i}q_i^*(sI-A)^{-1}b} \tag{3.4}$$

satisfies $\chi(s)\not\equiv 0$ and either $h_0=0$ or $\sigma_0=0$. Then $x(t)$ has its dynamics determined by the transmission (invariant) zeros of the transfer function $\chi(s)$ or it is confined to an invariant manifold where its dynamics is again determined by the transmission (invariant) zeros.

P r o o f. If $h_0 \neq 0$ then $\sigma_0 = 0$ according to Lemma's assumptions. We deduce $\mu(t) = -(1/h_0)c^* x(t)$, hence $x(t)$ is a solution of

$$\dot{x} = \Big(A_0 - \frac{1}{h_0} bc^*\Big)x(t) + \sum_1^r b q_k^* x(t - \tau_k)$$

whose characteristic equation reads

$$\det\left(sI - A_0 - \sum_1^r A_k \mathrm{e}^{-s\tau_k} + \frac{1}{h_0} bc^*\right) = \frac{1}{h_0}\chi(s)\det\left(sI - A_0 - \sum_1^r A_k \mathrm{e}^{-s\tau_k}\right)$$

and the first case is proved without using the special forms of the matrices A_k.

Let now $h_0 = 0$ and $\sigma_0 \neq 0$. Since $c^* x(t) \equiv \sigma_0 = \text{const}$ we have $c^* \dot{x}(t) = 0$. Therefore

$$c^* A_0 x(t) + (c^* b)\sum_1^r q_k^* x(t - \tau_k) + (c^* b)\mu(t) = 0.$$

Assume for a while that $(c^* b) \neq 0$. Then $\mu(t)$ is obtained from the above equation and substituted in the equation for x to obtain

$$\dot{x} = (I - (c^* b)^{-1} bc^*)\left(A_0 x(t) + \sum_1^r A_k x(t - \tau_k)\right).$$

Using the same approach as previously we obtain the characteristic equation

$$\det\left(sI - (I - (c^* b)^{-1} bc^*)\left(A_0 + \sum_1^r A_k \mathrm{e}^{-s\tau_k}\right)\right)$$
$$= (c^* b)^{-1} s\chi(s)\det\left(sI - A_0 - \sum_1^r A_k \mathrm{e}^{-s\tau_k}\right)$$

and this case is also proved without making use of the dyadic form of the matrices A_k.

If $c^* b = 0$ then $c^* \dot{x}(t) = 0$ reads $c^* A_0 x(t) = 0$: here the special form $A_k = b q_k^*$ was used; we differentiate once more to obtain

$$c^* A_0^2 x(t) + (c^* A_0 b)\sum_1^r q_k^* x(t - \tau_k) + (c^* A_0 b)\mu(t) = 0.$$

If $c^* A_0 b \neq 0$ we may obtain $\mu(t)$ and substitute it in the differential equation for x; if $c^* A_0 b = 0$ we obtain $c^* A_0^2 x(t) \equiv 0$ and differentiate once more; this process goes on up to the moment when $c^* A_0^k b \neq 0$, $0 \leqslant k \leqslant n-1$; the existence of such a k follows from the fact that $\chi(s) \not\equiv 0$ and, since $h_0 = 0$, this implies $c^*(sI - A_0)^{-1}b \not\equiv 0$. It follows that $x(t)$ is confined to a $(n-k-1)$-dimensional linear manifold defined by

$$c^* x = \sigma_0, \quad c^* A_0 x = 0, \quad \ldots, \quad c^* A_0^k x = 0$$

and satisfies

$$\dot{x} = (I - (c^* A_0^k b)^{-1} b c^* A_0^k)\Big(A_0 x(t) + b \sum_1^r q_k^* x(t - \tau_k)\Big)$$

whose characteristic equation is obtained after some computation to be

$$\det\Big(sI - (I - (c^* A_0^k b)^{-1} b c^* A_0^k)\Big(A_0 + b \sum_1^r q_k^* \mathrm{e}^{-s\tau_k}\Big)\Big) = (c^* A_0^k b)^{-1} s^k \chi(s) \det\Big(sI - A_0 - b \sum_1^r q_k^* \mathrm{e}^{-s\tau_k}\Big).$$

The proof is thus complete. The basic things are now at hand to discuss stability results. □

4. Stability results

A. Our main mathematical object to analyze will be here system (2.1) whose nonlinear function is supposed to satisfy the conditions of [2], [3], [4]: i) it is piecewise continuous with finite (first kind) discontinuities; ii) it is bounded, i.e. $|\varphi(\sigma)| \leqslant m$, $\forall \sigma \in \mathbb{R}$; iii) it is subject to the pseudo-sector condition

$$\varphi(\sigma)\sigma - \varepsilon\sigma^2 - \frac{\varphi^2(\sigma)}{k} > 0, \quad 0 < k \leqslant +\infty \tag{4.1}$$

for some $\varepsilon \in [0, k/4)$, $0 < |\sigma| \leqslant m\gamma_0$ where $\gamma_0 > 0$ is defined by the conditions of the problem.

Remark that (4.1) with finite k will require $\varphi(\sigma)$ being continuous at $\sigma = 0$. In order to state and prove the main stability result, we recall the basic fact that, for time delay and distributed parameter systems, the frequency domain inequalities are obtained using the integral form of the equations and the corresponding result for nonlinear integral equations. Due to this fact it is useful to reproduce here, without proofs, the following basic theorem [3].

Theorem 4.1. *Consider a system described by the nonlinear integral equation*

$$\sigma(t) = \varrho(t) - \int_0^t \kappa(t-\tau)\varphi(\sigma(\tau))\,\mathrm{d}\tau \tag{4.2}$$

under the following assumptions: a) $\varrho, \kappa \in \mathcal{L}^1(0,\infty) \cap \mathcal{L}^2(0,\infty)$*;* b) $\dot\varrho, \dot\kappa \in \mathcal{L}^1(0,\infty)$*;* c) $\int_t^\infty |\kappa(\lambda)|\,\mathrm{d}\lambda \in \mathcal{L}^2(0,\infty)$*;* d) $\varphi\colon\ \mathbb{R}\to\mathbb{R}$ *is subject to conditions* i)–iii) *stated above, with γ_0 being the $\mathcal{L}^1$ norm of $\kappa(t)$. If there exists a real ϑ such that the frequency domain inequality*

$$\frac{1}{k} + \varepsilon|\chi(\mathrm{i}\omega)|^2 + \mathrm{Re}\,(1+\mathrm{i}\omega\vartheta)\chi(\mathrm{i}\omega) \geqslant 0, \quad \forall \omega \in \mathbb{R}_+ \tag{4.3}$$

holds then $\lim\limits_{t\to\infty}\sigma(t) = 0$. Here $\chi(s)$ is the Laplace transform of $\kappa(t)$.

The integral equation (4.2) contains a discontinuous nonlinear function; therefore, in order to define the solution we take the approach of [4]. Since the extended nonlinearity is obviously integrable, the solution of (4.2) may be considered as the solution of

$$\sigma(t) = \varrho(t) + \int_0^t \kappa(t-\tau)\xi(\tau)\,\mathrm{d}\tau, \quad \xi(t) = -\varphi(\sigma(t)) \tag{4.4}$$

with the extended nonlinearity as previously. To end this section, if $\varphi(\sigma)$ is discontinuous at $\sigma = 0$ then $k = \infty$ in (4.1) and (4.3). However, the frequency domain condition is still improved with respect to the standard Popov inequality, due to the positive term $\varepsilon|\chi(\mathrm{i}\omega)|^2$; the same improvement *might* be obtained *via* a circle criterion; however, we have here a significant detail—(4.1) holds on a finite interval only, unlike the standard cases (including that of the circle criterion).

B. We shall turn now to the case of system (2.1) having in mind the line of [3] where the system is without delays, i.e. $q_i = 0$, $i = \overline{1,r}$. The main problem here is to obtain $\lim\limits_{t\to\infty} x(t) = 0$. If $\varphi(\sigma)$ is continuous at 0 it is a standard way that gives not only the asymptotic behavior (attractiveness of 0) but also Liapunov stability, hence global asymptotic stability. If $\varphi(\sigma)$ is discontinuous at 0 then it is still possible to obtain for (2.1) a result which is analogous to Theorem 3 of [3].

Theorem 4.2. *Consider system* (2.1) *under the following assumptions:* a) *the characteristic equation*

$$\det\left(\lambda I - A - \sum_1^r b q_i^* \mathrm{e}^{-\lambda\tau_i}\right) = 0 \tag{4.5}$$

has all its roots in the left half plane of $\mathbb{C}$, (c^*, A) is an observable pair and there exists k, $0 \leqslant k \leqslant n-1$ such that $c^*A^kb \neq 0$; b) $\varphi\colon\ \mathbb{R}\to\mathbb{R}$ *is such that $|\varphi(\sigma)| \leqslant m$,*

φ has a discontinuity at $\sigma = 0$ and $|\varphi(\sigma)/\sigma| > \varepsilon \geqslant 0$ for $0 < |\sigma| \leqslant m\gamma_0$ where γ_0 is the $\mathcal{L}^1$ norm of $\kappa(t)$ defined previously. Assume also that there exists a real ϑ such that

$$\varepsilon|\chi(\mathrm{i}\omega)|^2 + \mathrm{Re}\,(1 + \mathrm{i}\omega\vartheta)\chi(\mathrm{i}\omega) \geqslant 0,\ \forall\omega \in \mathbb{R}_+ \tag{4.6}$$

with $\chi(s) = c^*\Big(sI - A - b\sum\limits_1^r q_i^* \mathrm{e}^{-s\tau_i}\Big)^{-1} b$ the transfer function of the linear part of (2.1), *such that all its transmission (invariant) zeros are outside the imaginary axis* $\mathrm{i}\mathbb{R}$. *Then $\lim\limits_{t\to\infty} x(t) = 0$ for any initial condition $(x_0, \psi(\cdot))$ where ψ is an $\mathbb{R}^n$-valued function defined on* $[-\tau, 0)$, $\tau = \max\{\tau_1, \ldots, \tau_r\}$.

P r o o f. Let us mention first that the proof of this theorem, given in [9], makes largely use of a canonical change of coordinates used in [2], [3] but which goes back to [1]. Here we shall take an approach that strongly relies on Lemma 3.1 just proved above. First we use the standard Cauchy formula for time delay systems [5], [8] to obtain the integral form (4.4). Due to the assumption on the roots of (4.5), both $\varrho(t)$ and $\kappa(t)$ satisfy exponential estimates with strictly negative exponent; therefore all assumptions of Theorem 4.1 concerning ϱ, κ, their derivatives and integrals are fulfilled; $\chi(s)$ of (4.3) is exactly the Laplace transform of Theorem 4.1. Application of Theorem 4.1 will give $\lim\limits_{t\to\infty} \sigma(t) = \lim\limits_{t\to\infty} c^*x(t) = 0$.

Should $\varphi(\cdot)$ be continuous at 0, $\sigma(t) \to 0$ would imply $\varphi(\sigma(t)) \to 0$ and a standard application of Lemma 18.2 of [8] would give $\lim\limits_{t\to\infty} x(t) = 0$ along with Liapunov stability. But since $\varphi(\cdot)$ is discontinuous at 0, the trajectory of the system approaches asymptotically the dynamics of (3.1) with $c^*x = 0$. We may clearly apply Lemma 3.1 for the case $h_0 = 0$ (the fact that here $\sigma_0 = 0$ is irrelevant) to find that the dynamics of $x(t)$ as confined to the $(n - k - 1)$-dimensional linear manifold—also called "of the sliding modes"—is given by the system's invariant zeros which are outside the imaginary axis $\mathrm{i}\mathbb{R}$. Since $x(t)$ is *a priori* bounded due to the assumption on the roots of (4.5) and to the boundedness of $\varphi(\sigma)$, a simple application of Lemma 22.3 of [7] will give $\lim\limits_{t\to\infty} x(t) = 0$. This ends the proof which is new as compared to that of [9].

Remark also that, in comparison with the standard results, here the use of Lemma 22.3 of [7] combined with the boundedness of the solution allowed to relax the minimal phase assumption which is standard for the sliding mode system by replacing it with the hyperbolicity of the transmission zeros.

C. Following [3] we may give an extension of the previous theorem which is also an extension of Theorem 4 there.

Theorem 4.3. *Assume all conditions of Theorem* 4.2 *being fulfilled, except that a simple zero of* $\chi(s)$ *is allowed. Then* $x(t)$ *approaches asymptotically the stationary set*

$$\mathcal{S} = \left\{ x \in \mathbb{R}^n \colon \ x = \left(A + b \sum_1^r q_i \right)^{-1} b\xi, \ \xi \in [\varphi(0_-), \varphi(0_+)] \right\}. \tag{4.7}$$

P r o o f. Since all assumptions of Theorem 4.2 are valid, except that about the transmission zeros, application of this theorem gives boundedness of all state variables and $\sigma(t) \equiv c^* x(t) \to 0$. Let $k \in [0, n-1]$ be the positive integer such that $c^* A^k b \neq 0$. Following [1], [2], [3] we perform two successive changes of coordinates aiming at having σ, $\sigma', \dots, \sigma^{(k-1)}$ as the first k state variables ξ_i, $i = 1, \dots, k$; ξ_{k+1} is the only state variable whose equation contains both the nonlinearity and time delay terms and the remaining variables account for the dynamics defined by the transmission zeros. Denote by v_1 the vector formed by $\xi_1 = \sigma$, …, $\xi_k = \sigma^{(k-1)}$ and by v_2 the vector formed by $\xi_{k+2}, \dots, \xi_n$ whose dynamics is given by the transmission zeros and is without delays. Let Q and T be the nonsingular matrices of the two successive changes of coordinates and denote by $\hat{x}$ the resulting state vector whose components are v_1, ξ_{k+1}, v_2. We shall obtain finally the following system

$$\begin{aligned} &\dot{\xi}_i = \xi_{i+1}, \quad i = 1, \dots, k, \\ &\dot{\xi}_{k+1} = \frac{c^* A^{k+1} b}{c^* A^k b} \xi_{k+1}(t) + \xi_{k+2}(t) - (c^* A^k b) \left(\varphi(\xi_1(t)) - \sum_1^r q_j^* (TQ)^{-1} \hat{x}(t - \tau_j) \right), \\ &\dot{v}_2 = A_{21} v_1(t) + \varrho_{k+2} \xi_{k+1}(t) + A_{22} v_2 \end{aligned} \tag{4.8}$$

where $v_1(t) \to 0$, $\xi_{k+1}(t) \to 0$ either from Lemma 3.1 or by applying successively the lemma of Barbălat [7]. We have thus the system

$$\dot{v}_2 = A_{22} v_2 + f_2(t), \quad f_2(t) \to 0$$

where A_{22} has a simple zero eigenvalue. A change of coordinates is performed in this system by $z_2 = T_2 v_2$ with $\det T_2 \neq 0$ to obtain the structure

$$\dot{\zeta}_{k+2} = \psi_{k+2}(t), \quad \dot{z}_3 = A_3 z_3 + f_3(t)$$

where the scalar ζ_{k+2} and the vector z_3 are the components of z_2; also the scalar function $\psi_{k+2}(t)$ and the vector function $f_3(t)$ are the components of the vector function $T f_2(t)$ that approaches asymptotically 0 since $f_2(t)$ is such. Also A_3 has its eigenvalues outside the imaginary axis, hence by the same Lemma 22.3 of [7] $z_3(t) \to 0$, being bounded as all state variables. For the bounded state variable ζ_{k+2} we are in position to apply Lemma 22.4 of [7] to obtain that for every pair of numbers $\varepsilon > 0$ (arbitrarily small) and $T_0 > 0$ (arbitrarily large) there exists $T_1 > 0$ (sufficiently large) such that

$$|\zeta_{k+2}(t+\vartheta) - \zeta_{k+2}(t)| \leqslant \varepsilon, \quad \forall t > T_1,\ 0 \leqslant \vartheta \leqslant T_0$$

which shows that $\zeta_{k+2}(t)$ approaches asymptotically a constant value.

Summarizing, it follows that $x(t)$ approaches asymptotically for $t \to \infty$ some constant vector. But the constant solutions of the system (2.1) correspond, according to the definition of the extended nonlinearity, to the constant solutions of (3.1) associated with the discontinuity at 0 and this gives exactly the set (4.7). This completes the proof.

5. Conclusions

In this paper we have shown a simple way of extending the idea of the solution for quasi-linear or almost linear systems with discontinuous nonlinearity to time delay systems. This is based mainly on the feedback structure where the "good" subsystem is linear and time invariant (with lumped or distributed parameters) while the "unpleasant" one is nonlinear and possibly discontinuous. Using these results the extension of some stability results to the class of systems considered is obtained.

References

[1] *D. V. Anosov*: About the stability of equilibria of the relay systems. Avtomat. i Telemekhanika *20* (1959), 135–149. (In Russian.)

[2] *A. Kh. Gelig*: Stability analysis of nonlinear discontinuous control systems with non-unique equilibrium state. Avtomat. i Telemekhanika *25* (1964), 153–160. (In Russian.)

[3] *A. Kh. Gelig*: Stability of controlled systems with bounded nonlinearities. Avtomat. i Telemekhanika *29* (1969), 15–22. (In Russian.)

[4] *A. Kh. Gelig, G. A. Leonov, V. A. Yakubovich*: Stability of Nonlinear Systems with Non-unique Equilibrium State. Nauka, Moskva, 1978 (In Russian.); World Scientific, Singapore, 2004. (In English.)

[5] *A. Halanay*: Differential Equations. Stability. Oscillations. Time Lags. Academic Press, New York, 1966.

[6] *A. Halanay*: On the controllability of linear difference-differential systems. Math. Systems Theory Econom. *2* (1969), 329–336.

[7] *V. M. Popov*: Hyperstability of Control Systems. Editura Academiei, Bucharest & Springer, Berlin, 1973.

[8] *Vl. Răsvan*: Absolute Stability of Time Lag Control Systems. Editura Academiei, Bucharest, 1975, improved Russian version by Nauka, Moskva, 1983. (In Romanian.)

[9] *Vl. Răsvan, D. Danciu, D. Popescu*: Nonlinear and time delay systems for flight control. Math. Repts. *11* (2009), 359–367.

[10] *J. P. Richard, F. Gouaisbaut, W. Perruquetti*: Sliding mode control in the presence of delay. Kybernetika *37* (2001), 277–294.

Author's address: *Vl. Răsvan*, University of Craiova, Department of Automatic Control, A. I. Cuza str. 13, RO-200 585 Craiova, Romania, e-mail: `vrasvan@automation.ucv.ro`.