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PRESERVATION OF EXPONENTIAL STABILITY FOR EQUATIONS WITH SEVERAL DELAYS

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Abstract. We consider preservation of exponential stability for the scalar nonoscillatory linear equation with several delays

$$\dot{x}(t)+\sum_{k=1}^{m} a_k(t)x(h_k(t))=0, \quad a_k(t)\geqslant 0$$

under the addition of new terms and a delay perturbation. We assume that the original equation has a positive fundamental function; our method is based on Bohl-Perron type theorems. Explicit stability conditions are obtained.

Keywords: exponential stability, nonoscillation, explicit stability condition, perturbation

MSC 2010: 34K20

1. Introduction

In this paper we consider the scalar differential equation with several variable delays

$$\dot{x}(t)+\sum_{k=1}^{r} b_k(t)x(g_k(t))=0, \quad t\geqslant t_0, \tag{1.1}$$

as a perturbation of the equation

$$\dot{x}(t)+\sum_{k=1}^{m} a_k(t)x(h_k(t))=0, \quad t\geqslant t_0. \tag{1.2}$$

Equation (1.1) is considered for $t\geqslant t_0\geqslant 0$ with the initial conditions

$$x(t)=\varphi(t), \quad t<t_0,\ x(t_0)=x_0,\ t_0\geqslant 0 \tag{1.3}$$

under the following assumptions:

(a1) $a_k(t)$ are Lebesgue measurable essentially bounded on $[0,\infty)$ functions;

(a2) $h_k(t)$ are Lebesgue measurable functions,

$$h_k(t) \leqslant t, \ \sup_{t\geqslant 0}[t - h_k(t)] < \infty;$$

(a3) $\varphi\colon (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

We assume that conditions (a1)–(a3) hold for all equations throughout the paper.

Definition 1. A function $x\colon \mathbb{R} \to \mathbb{R}$ is called a *solution of the problem* (1.1), (1.3) if it is locally absolutely continuous on $[t_0,\infty)$, satisfies equation (1.1) for almost all $t \in [t_0,\infty)$ and the equalities (1.3) for $t \leqslant t_0$.

Definition 2. A solution $X(t,s)$ of the problem $\dot{x}(t) + \sum\limits_{k=1}^{m} a_k(t)x(h_k(t)) = 0$, $t \geqslant s \geqslant 0$, $x(t) = 0$, $t < s$, $x(s) = 1$, is called the *fundamental function* of (1.1).

In several publications perturbations of delays have been studied. For example, in [1] the equation with variable delays

$$\dot{x}(t) = \sum_{k=1}^{m} a_k x(t - \tau_k - \nu_k(t)) \tag{1.4}$$

was treated as a perturbation of the autonomous delay equation

$$\dot{y}(t) = \sum_{k=1}^{m} a_k y(t - \tau_k). \tag{1.5}$$

Lemma 1.1 [1]. *Assume that equation* (1.5) *is asymptotically stable and*

$$\sum_{k=1}^{m} |a_k| \limsup_{t\to\infty} |\nu_k(t)| < \frac{1}{\sum\limits_{k=1}^{m} |a_k| \int_0^\infty |v(s)|\,\mathrm{d}s}, \tag{1.6}$$

where $v(t)$ *is the fundamental solution* ($v(t) = 0$, $t < 0$, $v(0) = 1$) *of equation* (1.5). *Then equation* (1.4) *is asymptotically stable.*

If $a_k \leqslant 0$ and the fundamental solution $v(t)$ is positive, then $\sum\limits_{k=1}^{m} |a_k| \int_0^\infty |v(s)|\,\mathrm{d}s = 1$ and thus Lemma 1.1 gives an explicit stability condition for the perturbed equation.

Our main method is based on the Bohl-Perron theorem. Previously it was applied to perturbation problems for impulsive delay differential equations in [2].

Below we present a solution representation formula for nonhomogeneous equation (1.1) with Lebesgue measurable right-hand side $f(t)$:

$$\dot{x}(t)+\sum_{k=1}^{m} a_k(t)x(h_k(t))=f(t). \tag{1.7}$$

Lemma 1.2 [3], [4]. *Suppose conditions* (a1)–(a3) *hold. Then the solution of* (1.7), (1.3) *has the form*

$$x(t)=X(t,t_0)x_0-\int_{t_0}^{t} X(t,s)\sum_{k=1}^{m} a_k(s)\varphi(h_k(s))\,\mathrm{d}s+\int_{t_0}^{t} X(t,s)f(s)\,\mathrm{d}s, \tag{1.8}$$

where $\varphi(t)=0,\ t\geqslant t_0$.

Definition 3. Equation (1.1) is (*uniformly*) *exponentially stable*, if there exist $K>0$, $\lambda>0$, such that the fundamental function $X(t,s)$ of (1.1) has the estimate $|X(t,s)|\leqslant K\,\mathrm{e}^{-\lambda(t-s)}$ for $t\geqslant s\geqslant 0$.

For linear equations this definition is equivalent to the uniform asymptotic stability [3]. Under our assumptions the exponential stability does not depend on the values of the parameters of the equation on any finite interval. Thus all our conditions should only be satisfied for sufficiently large t.

Denote by $L_\infty[t_0,\infty)$ the space of all measurable essentially bounded on $[t_0,\infty)$ functions with the sup-norm.

The following result is a Bohl-Perron-like Theorem.

Lemma 1.3 [5], [6]. *Suppose for any* $f\in L_\infty[t_0,\infty)$ *the solution of the problem*

$$\dot{x}(t)+\sum_{k=1}^{r} b_k(t)x(g_k(t))=f(t),\quad t\geqslant t_0, \qquad x(t)=0,\quad t\leqslant t_0 \tag{1.9}$$

is bounded on $[t_0,\infty)$. *Then equation* (1.1) *is exponentially stable.*

We will need the following auxiliary results concerned with nonoscillatory equations.

Lemma 1.4 [7]**.** *Assume that* $a_k(t) \geqslant 0$, $\sum\limits_{k=1}^{m} a_k(t) \geqslant a_0 > 0$ *and the fundamental function* $X(t,s)$ *of equation* (1.2) *is positive. Then equation* (1.2) *is exponentially stable. Moreover, there exists* $t_0 \geqslant 0$ *such that*

$$0 \leqslant \int_{t_0}^{t} X(t,s) \sum_{k=1}^{m} a_k(s)\,\mathrm{d}s \leqslant 1.$$

The fundamental function is positive if and only if there exists an eventually positive solution of equation (1.2).

Lemma 1.5 [8]**.** *The fundamental function* $X(t,s)$ *of equation* (1.2) *is positive for* $t \geqslant t_0$ *if* $a_k(t) \geqslant 0$ *and*

$$\int_{\min\limits_k h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)\,\mathrm{d}s \leqslant \frac{1}{\mathrm{e}}, \quad t \geqslant t_0.$$

2. Explicit stability conditions

Theorem 2.1. *Assume that* $a_k(t) \geqslant 0$, $\sum\limits_{k=1}^{m} a_k(t) \geqslant a_0 > 0$ *and the fundamental function* $X(t,s)$ *of equation* (1.2) *is positive. Assume in addition that* $r \geqslant m$ *and*

$$\limsup_{t\to\infty} \frac{1}{\sum\limits_{k=1}^{m} a_k(t)} \left[\sum_{k=1}^{m} |a_k(t)-b_k(t)| + \sum_{k=1}^{m} |b_k(t)| \left| \int_{g_k(t)}^{h_k(t)} \sum_{i=1}^{r} |b_i(s)|\,\mathrm{d}s \right| + \sum_{k=m+1}^{r} |b_k(t)| \right]$$

is less than one. Then equation (1.1) *is exponentially stable.*

P r o o f. Without loss of generality we can assume that for $t \geqslant t_0$ and some $\mu \in (0,1)$ we have

$$\frac{1}{\sum\limits_{k=1}^{m} a_k(t)} \left[\sum_{k=1}^{m} |a_k(t) - b_k(t)| + \sum_{k=1}^{m} |b_k(t)| \left| \int_{g_k(t)}^{h_k(t)} \sum_{i=1}^{r} |b_i(s)|\,\mathrm{d}s \right| + \sum_{k=m+1}^{r} |b_k(t)| \right] \leqslant \mu.$$

Let us demonstrate that the solution of the non-homogeneous equation with the zero initial conditions is bounded on $[t_0, \infty)$ for any $f \in L_\infty[t_0, \infty)$.

After transformations equation (1.9) has the form

$$\dot{x}(t)+\sum_{k=1}^{m} a_k(t)x(h_k(t))-\sum_{k=1}^{m}(a_k(t)-b_k(t))x(h_k(t)) \\ -\sum_{k=1}^{m} b_k(t)\int_{g_k(t)}^{h_k(t)} \dot{x}(s)\,\mathrm{d}s+\sum_{k=m+1}^{r} b_k(t)x(g_k(t))=f(t).$$

After substituting $\dot{x}(t)$ from (1.9) we have

$$\dot{x}(t)+\sum_{k=1}^{m} a_k(t)x(h_k(t))-\sum_{k=1}^{m}(a_k(t)-b_k(t))x(h_k(t)) \\ +\sum_{k=1}^{m} b_k(t)\int_{g_k(t)}^{h_k(t)} \sum_{i=1}^{r} b_i(s)x(g_i(s))\,\mathrm{d}s+\sum_{k=m+1}^{r} b_k(t)x(g_k(t))=g(t),$$

where $g(t):=f(t)+\sum\limits_{k=1}^{m} b_k(t)\int_{h_k(t)}^{g_k(t)} f(s)\,\mathrm{d}s$. Evidently $g\in L_\infty[t_0,\infty)$.

Hence a solution of equation (1.9) is also a solution of the equation $x+Hx=r$, where

$$(Hx)(t)=\int_{t_0}^{t} X(t,s)\Big[-\sum_{k=1}^{m}(a_k(s)-b_k(s))x(h_k(s)) \\ +\sum_{k=1}^{m} b_k(s)\int_{g_k(s)}^{h_k(s)}\sum_{i=1}^{r} b_i(\tau)x(g_i(\tau))\,\mathrm{d}\tau+\sum_{k=m+1}^{r} b_k(s)x(g_k(s))\Big]\,\mathrm{d}s,$$

$$r(t):=\int_{t_0}^{t} X(t,s)g(s)\,\mathrm{d}s\in L_\infty[t_0,\infty).$$

By Lemma 1.4 we have

$$|(Hx)(t)|\leqslant\int_{t_0}^{t} X(t,s)\Big[\sum_{k=1}^{m}\Big(|a_k(s)-b_k(s)| \\ +\sum_{k=1}^{m}|b_k(s)|\Big|\int_{g_k(s)}^{h_k(s)}\sum_{i=1}^{r}|b_i(\tau)|\,\mathrm{d}\tau\Big|+\sum_{k=m+1}^{r}|b_k(s)|\Big)\Big]\,\mathrm{d}s\|x\|_{L_\infty[t_0,\infty)} \\ \leqslant\sup_{t\geqslant t_0}\int_{t_0}^{t} X(t,s)\sum_{k=1}^{m} a_k(s)\mu\,\mathrm{d}s\|x\|_{L_\infty[t_0,\infty)}\leqslant\mu\|x\|_{L_\infty[t_0,\infty)}.$$

Thus the norm of H in $L_\infty[t_0,\infty)$ does not exceed $\mu<1$, so the inverse $(I+H)^{-1}$ is a bounded operator and the function $x=(I+H)^{-1}r$ is bounded. By Lemma 1.3 equation (1.1) is exponentially stable. □

Corollary 1. *Assume that* $a_k(t) \geqslant 0$, $\sum\limits_{k=1}^{m} a_k(t) \geqslant a_0 > 0$ *and the fundamental function* $X(t,s)$ *of equation* (1.2) *is positive. If*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum\limits_{i=1}^{m} a_i(t)} \left| \int_{g_k(t)}^{h_k(t)} \sum_{i=1}^{m} a_i(s)\,\mathrm{d}s \right| < 1, \tag{2.1}$$

then the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = 0 \tag{2.2}$$

is exponentially stable.

Corollary 2. *Assume that* $a_k(t) \geqslant 0$, $\sum\limits_{k=1}^{m} a_k(t) \geqslant a_0 > 0$. *If* (2.1) *holds, where*

$$h_k(t) \geqslant h(t) = t - \frac{1}{\mathrm{e} \sup\limits_{t\geqslant t_0} \sum\limits_{i=1}^{m} a_i(t)},$$

then equation (2.2) *is exponentially stable.*

P r o o f. For $t \geqslant t_0$ we have

$$\begin{aligned} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)\,\mathrm{d}s &\leqslant \sup_{t\geqslant t_0} \sum_{i=1}^{m} a_i(t) \sup_{t\geqslant t_0} (t - h_k(t)) \\ &\leqslant \sup_{t\geqslant t_0} \sum_{i=1}^{m} a_i(t) \frac{1}{\mathrm{e} \sup\limits_{t\geqslant t_0} \sum\limits_{i=1}^{m} a_i(t)} = \frac{1}{\mathrm{e}}. \end{aligned}$$

By Lemma 1.5 the fundamental function $X(t,s)$ of equation (1.2) is positive for $t \geqslant t_0$. Corollary 1 implies this corollary. □

R e m a r k 1. If $a_k(t) \equiv a_k > 0$ then equality (2.1) has the form

$$\limsup_{t\to\infty} \sum_{k=1}^{m} a_k \left(t - g_k(t)\right) < 1 + \frac{1}{\mathrm{e}}.$$

This stability condition was obtained in [1] for piecewise continuous delays.

Corollary 3. *Assume that* $a_k(t) \geqslant 0$, $\sum_{k=1}^{m} a_k(t) \geqslant a_0 > 0$ *and the fundamental function* $X(t,s)$ *of equation* (1.2) *is positive. If* $\limsup_{t\to\infty} \sum_{k=1}^{m} |a_k(t) - b_k(t)| \Big/ \sum_{i=1}^{m} a_i(t) < 1$, *then the equation* $\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0$ *is exponentially stable.*

Corollary 4. *Suppose that there exists a set of indices* $I \subset \{1,\ldots,m\}$ *such that* $a_k(t) \geqslant 0$, $k \in I$, $\sum_{k\in I} a_k(t) \geqslant a_0 > 0$, *and the equation* $\dot{x}(t) + \sum_{k\in I} a_k(t)x(h_k(t)) = 0$ *has a positive fundamental function. If* $\limsup_{t\to\infty} \sum_{k\notin I} |a_k(t)| \Big/ \sum_{k\in I} a_k(t) < 1$, *then equation* (1.2) *is exponentially stable.*

Consider now two autonomous equations

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t-\delta_k) = 0, \tag{2.3}$$

$$\dot{x}(t) + \sum_{k=1}^{r} b_k x(t-\sigma_k) = 0. \tag{2.4}$$

Corollary 5. *Suppose that* $a_k > 0$, $k = 1,\ldots,m$, $r \geqslant m$, *and the characteristic equation of* (2.3)

$$\lambda = \sum_{k=1}^{m} a_k \mathrm{e}^{\lambda\delta_k} \tag{2.5}$$

has a positive root. If in addition

$$\sum_{k=1}^{m} |a_k - b_k| + \left(\sum_{k=1}^{m} |b_k(\delta_k - \sigma_k)|\right)\left(\sum_{k=1}^{r} |b_k|\right) + \sum_{k=m+1}^{r} |b_k| < \sum_{k=1}^{m} a_k,$$

then equation (2.4) *is exponentially stable.*

P r o o f. Suppose $\lambda_0 > 0$ is a positive root of equation (2.5). Then equation (2.3) has a positive solution $x(t) = \mathrm{e}^{-\lambda_0 t}$ and the fundamental function of equation (2.3) is positive. Theorem 2.1 implies this corollary. □

Corollary 6. *Suppose that* $a_k > 0$, $k = 1,\ldots,m$ *and the characteristic equation* (2.5) *has a positive root. If* $\sum_{k=1}^{m} a_k|\delta_k - \sigma_k| < 1$, *then the equation*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t-\sigma_k) = 0 \tag{2.6}$$

is exponentially stable.

Corollary 7. *Suppose that* $a_k > 0$, $k = 1, \dots, m$. *If* $\sum\limits_{k=1}^{m} a_k \left| \sigma_k - \left(\mathrm{e} \sum\limits_{i=1}^{m} a_i \right)^{-1} \right| < 1$, *then equation* (2.6) *is exponentially stable.*

Corollary 8. *Suppose that* $a_k > 0$ *and the characteristic equation* (2.5) *has a positive root. If* $\sum\limits_{k=1}^{m} |a_k - b_k| < \sum\limits_{k=1}^{m} a_k$, *then the equation* $\dot{x}(t) + \sum\limits_{k=1}^{m} b_k x(t - \delta_k) = 0$ *is exponentially stable.*

Theorem 2.2. *Assume that* $a_k(t) \geqslant 0$, $\sum\limits_{k=1}^{m} a_k(t) \geqslant a_0 > 0$ *and the fundamental function* $X(t,s)$ *of equation* (1.2) *is positive. Assume in addition that* $r \leqslant m$ *and*

$$\limsup_{t\to\infty} \frac{1}{\sum\limits_{k=1}^{m} a_k(t)} \left[\sum_{k=1}^{r} |a_k(t) - b_k(t)| + \sum_{k=1}^{r} |b_k(t)| \left| \int_{g_k(t)}^{h_k(t)} \sum_{i=1}^{r} |b_i(s)| \,\mathrm{d}s \right| + \sum_{k=r+1}^{m} |a_k(t)| \right]$$

is less than one. Then equation (1.1) *is exponentially stable.*

P r o o f. The proof is similar to the proof of Theorem 2.1 after we rewrite (1.9) as

$$\begin{aligned} \dot{x}(t) &+ \sum_{k=1}^{m} a_k(t) x(h_k(t)) - \sum_{k=1}^{r} (a_k(t) - b_k(t)) x(h_k(t)) \\ &- \sum_{k=1}^{r} b_k(t) \int_{g_k(t)}^{h_k(t)} \dot{x}(s) \,\mathrm{d}s - \sum_{k=r+1}^{m} a_k(t) x(h_k(t)) = f(t). \end{aligned}$$

□

3. Discussion and open problems

Unlike most papers on this topic, all results of the present paper are obtained under the assumption that coefficients and delays are measurable and solutions are absolutely continuous functions.

Let us compare Corollary 1 and Lemma 1.1 from [1] for constant coefficients satisfying $\sum\limits_{k=1}^{m} a_k = -1$, $a_k \leqslant 0$, $k = 0, \dots, m$. If the comparison autonomous equation (1.5) has a positive fundamental function, then we obtain sufficient stability conditions for (1.4)

$$\limsup_{t\to\infty} \sum_{k=1}^{m} |a_k| |\nu_k(t)| < 1 \text{ and } \sum_{k=1}^{m} |a_k| \limsup_{t\to\infty} |\nu_k(t)| < 1, \tag{3.1}$$

respectively. Evidently the first condition is better than the second inequality. We treat equations with variable coefficients and measurable parameters as perturbations of equations with positive coefficients and positive fundamental functions, while in [1] vector equations are considered as perturbations of arbitrary stable autonomous delay equations.

Overall, all stability preservation results of the present paper assume that the non-perturbed equation has a positive fundamental function. In other words, they answer the following question: how much can we perturb a nonoscillatory equation with positive coefficients so that its stability property be preserved? The signs of the perturbed coefficients can be arbitrary.

Finally, let us list some relevant open problems.

(A) Extend the results to the case when the coefficients of the non-perturbed equation (1.2) may be positive and negative, and also oscillating.

(B) Consider perturbations of the exponentially stable system of delay equations with variable coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x(h_k(t)) = 0. \tag{3.2}$$

Is it possible to obtain explicit stability conditions for the perturbed equation (in terms of delays and coefficients, not the fundamental function of (3.2))?

(C) Consider more general delays, for example, study the equation with a distributed delay

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(g_k(t)) + \int_{h(t)}^{t} K(t,s)x(s)\,\mathrm{d}s = 0 \tag{3.3}$$

as a perturbation of either (1.1) or $\dot{x}(t) + \int_{g(t)}^{t} M(t,s)x(s)\,\mathrm{d}s = 0$ and obtain explicit conditions under which stability is preserved.

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